A Kallman-Rota Inequality for Evolution Semigroups

This is the Published version of the following publication


The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository  https://vuir.vu.edu.au/17708/
A KALLMAN-ROTA INEQUALITY FOR EVOLUTION SEMIGROUPS

C. BUSÈ AND S.S. DRAGOMIR

Abstract. A Kallman-Rota type inequality for evolution semigroups and applications for real valued functions are given.

1. Introduction

Let $X$ be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on $X$. The norms in $X$ and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$.

Let $\mathbb{R}_+$ the set of all non-negative real numbers and $J \in \{\mathbb{R}_+, \mathbb{R}\}$. The set $\{ (t, s) : t \geq s \in J \}$ will be denoted by $\Delta_J$. A family $U_J = \{ U(t, s) : (t, s) \in \Delta_J \} \subset \mathcal{L}(X)$ is called an evolution family of bounded linear operators on $X$ if $U(t, t) = I$ (the identity operator on $X$) and $U(t, s)U(s, r) = U(t, r)$ for all $t \geq s \geq r \in J$. Such a family is said to be strongly continuous if for each $x \in X$, the maps $(t, s) \mapsto U(t, s)x : \Delta_J \to X$ are continuous. A strongly continuous evolution family is said to be exponentially bounded if there exist $\omega \in \mathbb{R}$ and $K_\omega \geq 1$ such that

$$\|U(t, s)\| \leq K_\omega e^{\omega(t-s)}$$

for all $(t, s) \in \Delta_J$ and uniformly stable if there exists $M \in \mathbb{R}_+$ such that

$$\sup_{(t,s) \in \Delta_J} \|U(t, s)\| \leq M < \infty.$$  

We remind that a family $T = \{ T(t) : t \geq 0 \} \subset \mathcal{L}(X)$ is called one-parameter semigroup if $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t \geq s \geq 0$. An one-parameter semigroup is called strongly continuous or $C_0$-semigroup if for each $x \in X$ the maps $t \mapsto T(t)x$ are continuous on $\mathbb{R}_+$. For a $C_0$-semigroup $T$, its infinitesimal generator $A$ with the domain $D(A)$ is defined by

$$D(A) := \left\{ x \in X : \text{there exists in } X, \lim_{t \to 0} \frac{T(t)x - x}{t} =: Ax \right\}.$$ 

It is easy to see that if $T = \{ T(t) : t \geq 0 \}$ is a strongly continuous semigroup then the family $U_J = \{ U(t, s) := T(t-s) : (t, s) \in \Delta_J \}$ is a strongly continuous and exponentially bounded evolution family. Conversely, if $U_J$ is a strongly continuous evolution family and $U(t, s) = U(t-s, 0)$ for all $(t, s) \in \Delta_J$ then the family $T := \{ T(t) = U(t, 0) : t \geq 0 \}$ is a strongly continuous one-parameter semigroup. For
more details about the strongly continuous semigroups and evolution families we refer to [3].

**Lemma 1.** Let $T := \{T(t) : t \geq 0\}$ be a strongly continuous one-parameter semigroup and $A : D(A) \subset X \to X$ its infinitesimal generator. If $T$ is uniformly stable, that is, there is a positive constant $M$ such that $\sup_{t \geq 0} \|T(t)\| \leq M$, then

$$
\|Ax\|^2 \leq 4M^2 \|A^2x\| \|x\|, \quad \text{for all } x \in D(A^2).
$$

**Proof.** See [4].

We are recalling the notion of evolution semigroup. For more details we refer to [1], [2] and references therein. We will consider the both cases, i.e., the evolution semigroups for evolution families on $\Delta_{\mathbb{R}^+}$ and on $\Delta_{\mathbb{R}}$.

Let $U_{\mathbb{R}^+}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on $X$. Let us consider the following spaces:

- $C_{00}(\mathbb{R}^+, X)$ is the space consisting by all $X$-valued, continuous functions on $\mathbb{R}^+$, such that $f(0) = \lim_{t \to \infty} f(t) = 0$, endowed with the sup-norm.
- $L^p(\mathbb{R}^+, X)$, $1 \leq p < \infty$ is the usual Lebesgue-Bochner space of all measurable functions $f : \mathbb{R}^+ \to X$, identifying functions which are equal almost everywhere, such that $||f||_p := \left( \int_0^{\infty} ||f(s)||^p ds \right)^{1/p} < \infty$.

Let $X$ be either $C_{00}(\mathbb{R}^+, X)$ or $L^p(\mathbb{R}^+, X)$ and $f \in X$.

It is easy to see that for each $t \geq 0$, the function $T(t)f$ given by

$$
(T(t)f)(s) := \begin{cases} 
U(s, s-t)f(s-t), & s \geq t \\
0, & 0 \leq s < t
\end{cases}
$$

belongs to $X$, and the family $T = \{T(t) : t \geq 0\}$ is an one-parameter semigroup of bounded linear operators acting on $X$. Moreover, the following result, holds:

**Lemma 2.** The semigroup $T$ defined in (1.3) is strongly continuous. If $(A, D(A))$ is the generator of $T$ with its domain then for every $u, f$ in $X$ the following statements are equivalent:

- (i) $u \in D(A)$ and $Au = -f$;
- (ii) $u(t) = \int_0^t U(t, s)f(s)ds$;

**Proof.** See [7].

The strongly continuous semigroup $T$ defined in (1.3) is called evolution semigroup associated to $U_{\mathbb{R}^+}$ on the space $X$.

We will state here our first result.

**Theorem 1.** Let $U_{\mathbb{R}^+}$ be a strongly continuous uniformly stable evolution family of bounded linear operators acting on $X$, and let $g \in X$. Suppose that the following conditions are fulfilled:

- (i) $\int_0^t U(t, s)g(s)ds$ belongs to $X$;
- (ii) $\int_0^{t-s} U(t, s)g(s)ds$ belongs to $X$.
Then the following inequality holds:

\[(1.4) \quad \left\| \int_0^\cdot U(\cdot, s)g(s)ds \right\|_{\mathcal{X}}^2 \leq 4M^2 \left\| g \right\|_{\mathcal{X}} \times \left\| \int_0^\cdot (-s)U(\cdot, s)g(s)ds \right\|_{\mathcal{X}},\]

where \(M\) is the constant from the estimation (1.1).

\(\text{BUC}(\mathbb{R}, \mathcal{X})\) is the space of all \(\mathcal{X}\)-valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm. The following three spaces are closed subspaces of \(\text{BUC}(\mathbb{R}, \mathcal{X})\):

- \(C_0(\mathbb{R}, \mathcal{X})\) is the space of all \(\mathcal{X}\)-valued, continuous functions on \(\mathbb{R}\) such that \(\lim_{t \to \infty} f(t) = 0\).
- \(\text{AP}(\mathbb{R}, \mathcal{X})\) is the space of all almost periodic functions, that is, the smallest closed subspace of \(\text{BUC}(\mathbb{R}, \mathcal{X})\) containing the functions of the form \(t \mapsto e^{i\mu t} x, \mu \in \mathbb{R}\) and \(x \in \mathcal{X}\), see e.g. \([6]\).
- \(\text{AAP}(\mathbb{R}, \mathcal{X})\) is the space of all \(\mathcal{X}\)-valued asymptotically almost periodic functions on \(\mathbb{R}\), i.e., the space consisting in all functions \(f\) for which there exist \(g \in C_0(\mathbb{R}, \mathcal{X})\) and \(h \in \text{AP}(\mathbb{R}, \mathcal{X})\) such that \(f = g + h\).

Let \(Y\) one of the spaces described before and \(f \in Y\). If \(U_R\) satisfies certain conditions, which will be outlined in Lemma 3 below, then for each \(t \geq 0\) the function given by

\[(1.5) \quad s \mapsto (T(t)f)(s) := U(s, s-t)f(s-t) : \mathbb{R} \to \mathcal{X}\]

belongs to \(Y\), and the family \(T := \{T(t) : t \geq 0\}\) is an one-parameter semigroup of bounded linear operators on \(Y\). The semigroup \(T\) can be not strongly continuous. However, in certain cases, this semigroup is strongly continuous, and is called evolution semigroup associated to \(U_R\) on the space \(Y\).

**Lemma 3.** Let \(U_R\) be a strongly continuous evolution family of bounded linear operators on \(\mathcal{X}\), and \(q\) be a fixed positive real number.

(i) If \(Y = C_0(\mathbb{R}, \mathcal{X})\), and \(U_R\) is exponentially bounded, then the semigroup associated to \(U_R\), defined in (1.5), is a strongly continuous one-parameter semigroup of bounded linear operators on \(Y\);

(ii) If \(Y\) is either the spaces \(\text{AP}(\mathbb{R}, \mathcal{X})\) or \(\text{AAP}(\mathbb{R}, \mathcal{X})\) and \(U_R\) is \(q\)-periodic, that is, \(U(t+q, s+q) = U(t, s)\) for all \((t, s) \in \Delta_R\), then the semigroup given in (1.5), is a strongly continuous semigroup on \(Y\).

Let \((B, D(B))\) the generator of the evolution semigroup given in (1.5). If \(u\) and \(g\) belongs to \(Y\) then the following statements are equivalent:

(iii) \(u \in D(B)\) and \(Bu = -g\);

(iv)

\[(1.6) \quad u(t) = U(t, s)u(s) + \int_s^t U(t, s)g(s)ds, \quad \text{for all } t \geq s.\]

**Proof.** See \([5]\), \([9]\) for evolution semigroups defined on \(C_0(\mathbb{R}, \mathcal{X})\) and \([8]\) for evolution semigroups on \(\text{AP}(\mathbb{R}, \mathcal{X})\) or \(\text{AAP}(\mathbb{R}, \mathcal{X})\).
Let $\mathcal{Y}$ be one of the spaces $C_0(\mathbb{R}, X), AP(\mathbb{R}, X), AAP(\mathbb{R}, X)$ and let $\mathcal{Y}_0$ be the set of all functions $f \in \mathcal{Y}$ such that $\lim_{t \to -\infty} f(t) = 0$. It is clearly that $\mathcal{Y}_0$ is a closed subspace of $\mathcal{Y}$.

We may now state our second result.

**Theorem 2.** Let $\mathcal{U}_k$ be a strongly continuous uniformly stable evolution family of bounded linear operators on $X$ and $q > 0$, fixed. The following statements hold:

(i) If $\mathcal{Y} = C_0(\mathbb{R}, X)$, then the evolution semigroup given in (1.5) is defined on $\mathcal{Y}_0$.

(ii) If $\mathcal{Y}$ is one of the both spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_k$ is $q$-periodic then the evolution semigroup given in (1.4) is defined on $\mathcal{Y}_0$.

If $(C, D(C))$ is the generator of the evolution semigroup on $\mathcal{Y}_0$, given in (1.5), and $v, h$ belongs to $\mathcal{Y}_0$, then the following statements are equivalent:

(iii) $v \in D(C)$ and $Cv = -h$;

(iv) \begin{equation}
    v(t) = \int_{-\infty}^{t} U(t, s)h(s)ds,
\end{equation}

for every real number $t$. Moreover, the following inequality holds:

\begin{equation}
    \left\| \int_{-\infty}^{t} U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}^2 \leq 4M^2 \|h\|_{\mathcal{Y}} \times \left\| \int_{-\infty}^{t} (\cdot - s)U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}.
\end{equation}

2. Proofs

**Proof of Theorem 1.** Let $T$ be the evolution semigroup associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space $\mathcal{X}$ and $(A, D(A))$ its infinitesimal generator. From Lemma 2 it follows that the function $t \mapsto u(t) := \int_{0}^{t} U(t, s)g(s)ds$ belongs to $D(A)$ and $Au = -g$. The function $t \mapsto v(t) := \int_{0}^{t} U(t, r)u(r)dr$ belongs to $\mathcal{X}$. Indeed, using the Fubini Theorem, we have:

\[ v(t) = \int_{0}^{t} \left[ \int_{0}^{r} U(t, s)g(s)ds \right] dr \]
\[ = \int_{0}^{t} \left[ \int_{0}^{r} U(t, s)g(s)ds \right] dr \]
\[ = \int_{0}^{t} \left[ \int_{0}^{r} 1_{[0, r]}(s)U(t, s)g(s)ds \right] dr \]
\[ = \int_{0}^{t} \left[ \int_{s}^{t} U(t, s)g(s)ds \right] dr \]
\[ = \int_{0}^{t} (t - s)U(t, s)g(s)ds, \]

where $1_{[0, r]}$ is the characteristic function of the interval $[0, r]$. Using again Lemma 2 follows that $v \in D(A^2)$ and $A^2v = A(Av) = -Au = g$.

Now the inequality (1.4) follows by Lemma 1, if we replace $x$ with $v$ in (1.2).

**Proof of Theorem 2.** Firstly we prove that $\mathcal{Y}_0$ is an invariant subspace for each operator $T(t), t \geq 0$, given in (1.5). By Lemma 3 it suffices to prove that $\lim_{s \to -\infty} (T(t)f)(s) = \ldots$
0 for each \( t \geq 0 \) and every \( f \in \mathcal{Y}_0 \), and this fact is an easy consequence of the following estimations:

\[
\| (T(t)f)(s) \| \leq \| U(s, s-t) \| \| f(s-t) \| \leq M \| f(s-t) \| \rightarrow 0 \ 	ext{as} \ s \rightarrow (-\infty),
\]

where \( M \) is the positive constant from (1.1). Now, the implication \((jjj) \Rightarrow (jv)\) follows from Lemma 3, passing to the limit for \( s \rightarrow (-\infty) \). The converse implication \((jv) \Rightarrow (jjj)\) can be obtained on the following way.

Let \( v \) as in (1.7) and \( t > 0 \). Simple calculus gives

\[
\frac{T(t)v - v}{t} = -\frac{\int_{0}^{t} T(r)hdr}{t} \rightarrow -h \ 	ext{in} \ \mathcal{X}
\]

when \( t \rightarrow 0 \), that is \( v \in D(C) \) and \( Cv = -h \). Now the inequality (1.8), can be established as in the proof of Theorem 1 and we omit the details.

3. Applications

In this section some scalar inequalities are presented.

**Corollary 1.** Let \( g : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a continuous function such that \( g(0) = g(\infty) := \lim_{t \rightarrow \infty} g(t) = 0 \). Suppose that the functions:

\[
t \mapsto h(t) := \int_{0}^{t} g(s)ds \ \text{and} \ t \mapsto u(t) := \int_{0}^{t} (t-s) g(s)ds
\]

verifies the condition \( h(\infty) = u(\infty) = 0 \).

Then the following inequality holds:

\[
\sup_{t \geq 0} \left| \int_{0}^{t} g(s)ds \right|^2 \leq 4 \cdot \sup_{t \geq 0} |g(t)| \times \sup_{t \geq 0} \left| \int_{0}^{t} (t-s) g(s)ds \right|.
\]

**Proof.** We apply Theorem 1 for \( \mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R}) \) and for \( U(t, s)x = x \), where \( t \geq s \geq 0 \) and \( x \in \mathbb{R} \).

**Corollary 2.** Let \( g, h, u \) as in Corollary 1 and \( f \) be a continuous, positive and nondecreasing function on \( \mathbb{R}_+ \). The following inequality holds:

\[
\sup_{t \geq 0} \left[ \left| \int_{0}^{t} f(s)g(s)ds \right| \right]^2 \leq 4 \sup_{t \geq 0} |g(t)| \sup_{t \geq 0} \left[ \left| \int_{0}^{t} (t-s) f(s)g(s)ds \right| \right].
\]

**Proof.** Follows by Theorem 1 for \( \mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R}) \) and \( U(t, s) = \frac{f(s)}{f(t)} \).

**Corollary 3.** Let \( 1 \leq p < \infty \) and \( f \in L_p(\mathbb{R}_+, \mathbb{R}) \). If the functions

\[
t \mapsto g(t) := \int_{0}^{t} f(s)ds \ \text{and} \ t \mapsto h(t) := \int_{0}^{t} (t-s) f(s)ds
\]

belongs to \( L_p(\mathbb{R}_+, \mathbb{R}) \), then the following inequality holds:

\[
\| g \|_p^2 \leq 4 \| f \|_p \times \| h \|_p.
\]

**Proof.** Follows by Theorem 1 for \( \mathcal{X} = L_p(\mathbb{R}_+, \mathbb{R}) \) and for \( U(t, s)x = x \) where \( t \geq s \geq 0 \) and \( x \in \mathbb{R} \).
Corollary 4. Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be an almost periodic or asymptotically almost periodic function such that \( g(\infty) = 0 \). Then
\[
\sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{t} \frac{1 + \sin^2 s}{1 + \sin^2 t} g(s) ds \right|^2 \leq 16 \sup_{t \in \mathbb{R}} |g(t)| \times \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{t} \frac{1 + \sin^2 s}{1 + \sin^2 t} g(s) ds \right|.
\]

Proof. Follows by Theorem 2 for \( \mathcal{Y} = \text{AP}(\mathbb{R}, \mathbb{R}) \) or \( \mathcal{Y} = \text{AAP}(\mathbb{R}, \mathbb{R}) \) and \( U(t, s)x = \frac{1 + \sin^2 t}{1 + \sin^2 s} x \) where \( t \geq s \) and \( x \in \mathbb{R} \). It is clear that \( \mathcal{U} = \{ U(t, s); t \geq s \} \) is a \( \pi \)-periodic family consisting in operators acting on \( \mathbb{R} \), and \( \sup_{t \geq s} U(t, s) \leq 2 \).

References