



VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

A Kallman-Rota Inequality for Evolution Semigroups

This is the Published version of the following publication

Buse, Constantin and Dragomir, Sever S (2002) A Kallman-Rota Inequality for Evolution Semigroups. RGMIA research report collection, 5 (2).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/17708/>

A KALLMAN-ROTA INEQUALITY FOR EVOLUTION SEMIGROUPS

C. BUŞE AND S.S. DRAGOMIR

ABSTRACT. A Kallman-Rota type inequality for evolution semigroups and applications for real valued functions are given.

1. INTRODUCTION

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X . The norms in X and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$.

Let \mathbb{R}_+ the set of all non-negative real numbers and $\mathbf{J} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set $\{(t, s) : t \geq s \in \mathbf{J}\}$ will be denoted by $\Delta_{\mathbf{J}}$. A family

$$\mathcal{U}_{\mathbf{J}} = \{U(t, s) : (t, s) \in \Delta_{\mathbf{J}}\} \subset \mathcal{L}(X)$$

is called an *evolution family* of bounded linear operators on X if $U(t, t) = I$ (the identity operator on X) and $U(t, s)U(s, r) = U(t, r)$ for all $t \geq s \geq r \in \mathbf{J}$. Such a family is said to be *strongly continuous* if for each $x \in X$, the maps

$$(t, s) \mapsto U(t, s)x : \Delta_{\mathbf{J}} \rightarrow X$$

are continuous. A strongly continuous evolution family is said to be *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $K_{\omega} \geq 1$ such that

$$\|U(t, s)\| \leq K_{\omega} e^{\omega(t-s)} \text{ for all } (t, s) \in \Delta_{\mathbf{J}}$$

and *uniformly stable* if there exists $M \in \mathbb{R}_+$ such that

$$(1.1) \quad \sup_{(t,s) \in \Delta_{\mathbf{J}}} \|U(t, s)\| \leq M < \infty.$$

We remind that a family $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ is called *one-parameter semigroup* if $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t \geq s \geq 0$. An one-parameter semigroup is called *strongly continuous* or C_0 -semigroup if for each $x \in X$ the maps $t \mapsto T(t)x$ are continuous on \mathbb{R}_+ . For a C_0 -semigroup \mathbf{T} , its infinitesimal generator A with the domain $D(A)$ is defined by

$$D(A) := \left\{ x \in X : \text{there exists in } X, \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} =: Ax \right\}.$$

It is easy to see that if $\mathbf{T} = \{T(t) : t \geq 0\}$ is a strongly continuous semigroup then the family $\mathcal{U}_{\mathbf{J}} = \{U(t, s) := T(t-s) : (t, s) \in \Delta_{\mathbf{J}}\}$ is a strongly continuous and exponentially bounded evolution family. Conversely, if $\mathcal{U}_{\mathbf{J}}$ is a strongly continuous evolution family and $U(t, s) = U(t-s, 0)$ for all $(t, s) \in \Delta_{\mathbf{J}}$ then the family $\mathbf{T} := \{T(t) = U(t, 0) : t \geq 0\}$ is a strongly continuous one-parameter semigroup. For

1991 *Mathematics Subject Classification.* 47A30.

Key words and phrases. Kallman-Rota Inequality, Evolution Semigroups.

more details about the strongly continuous semigroups and evolution families we refer to [3].

Lemma 1. *Let $\mathbf{T} := \{T(t) : t \geq 0\}$ be a strongly continuous one-parameter semigroup and $A : D(A) \subset X \rightarrow X$ its infinitesimal generator. If \mathbf{T} is uniformly stable, that is, there is a positive constant M such that $\sup_{t \geq 0} \|T(t)\| \leq M$, then*

$$(1.2) \quad \|Ax\|^2 \leq 4M^2 \|A^2x\| \|x\|, \quad \text{for all } x \in D(A^2).$$

Proof. See [4]. ■

We are recalling the notion of evolution semigroup. For more details we refer to [1], [2] and references therein. We will consider the both cases, i.e., the evolution semigroups for evolution families on $\Delta_{\mathbb{R}_+}$ and on $\Delta_{\mathbb{R}}$.

Let $\mathcal{U}_{\mathbb{R}_+}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on X . Let us consider the following spaces:

- $C_{00}(\mathbb{R}_+, X)$ is the space consisting by all X -valued, continuous functions on \mathbb{R}_+ , such that

$$f(0) = \lim_{t \rightarrow \infty} f(t) = 0,$$

endowed with the sup-norm.

- $L_p(\mathbb{R}_+, X)$, $1 \leq p < \infty$ is the usual Lebesgue-Bochner space of all measurable functions $f : \mathbb{R}_+ \rightarrow X$, identifying functions which are equal almost everywhere, such that

$$\|f\|_p := \left(\int_0^\infty \|f(s)\|^p ds \right)^{\frac{1}{p}} < \infty.$$

Let \mathcal{X} be either $C_{00}(\mathbb{R}_+, X)$ or $L_p(\mathbb{R}_+, X)$ and $f \in \mathcal{X}$.

It is easy to see that for each $t \geq 0$, the function $T(t)f$ given by

$$(1.3) \quad (T(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ 0, & 0 \leq s < t \end{cases}$$

belongs to \mathcal{X} , and the family $\mathbf{T} = \{T(t) : t \geq 0\}$ is an one-parameter semigroup of bounded linear operators acting on \mathcal{X} . Moreover, the following result, holds:

Lemma 2. *The semigroup \mathbf{T} defined in (1.3) is strongly continuous. If $(A, D(A))$ is the generator of \mathbf{T} with its domain then for every u, f in \mathcal{X} the following statements are equivalent:*

- (i) $u \in D(A)$ and $Au = -f$;
- (ii) $u(t) = \int_0^t U(t, s)f(s)ds$;

Proof. See [7]. ■

The strongly continuous semigroup \mathbf{T} defined in (1.3) is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space \mathcal{X} .

We will state here our first result.

Theorem 1. *Let $\mathcal{U}_{\mathbb{R}_+}$ be a strongly continuous uniformly stable evolution family of bounded linear operators acting on X , and let $g \in \mathcal{X}$. Suppose that the following conditions are fulfilled:*

- (i) $\int_0^\cdot U(\cdot, s)g(s)ds$ belongs to \mathcal{X} ;
- (ii) $\int_0^\cdot (\cdot - s)U(\cdot, s)g(s)ds$ belongs to \mathcal{X} .

Then the following inequality holds:

$$(1.4) \quad \left\| \int_0^\cdot U(\cdot, s)g(s)ds \right\|_{\mathcal{X}}^2 \leq 4M^2 \|g\|_{\mathcal{X}} \times \left\| \int_0^\cdot (\cdot - s)U(\cdot, s)g(s)ds \right\|_{\mathcal{X}},$$

where M is the constant from the estimation (1.1).

$BUC(\mathbb{R}, X)$ is the space of all X -valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm. The following three spaces are closed subspaces of $BUC(\mathbb{R}, X)$:

- $C_0(\mathbb{R}, X)$ is the space of all X -valued, continuous functions on \mathbb{R} such that $\lim_{t \rightarrow \infty} f(t) = 0$.
- $AP(\mathbb{R}, X)$ is the space of all almost periodic functions, that is, the smallest closed subspace of $BUC(\mathbb{R}, X)$ containing the functions of the form

$$t \mapsto e^{i\mu t}x, \quad \mu \in \mathbb{R} \text{ and } x \in X,$$

see e.g. [6].

- $AAP(\mathbb{R}, X)$ is the space of all X -valued asymptotically almost periodic functions on \mathbb{R} , i.e., the space consisting in all functions f for which there exist $g \in C_0(\mathbb{R}, X)$ and $h \in AP(\mathbb{R}, X)$ such that $f = g + h$.

Let \mathcal{Y} one of the spaces described before and $f \in \mathcal{Y}$. If $\mathcal{U}_{\mathbb{R}}$ satisfies certain conditions, which will be outlined in Lemma 3 below, then for each $t \geq 0$ the function given by

$$(1.5) \quad s \mapsto (T(t)f)(s) := U(s, s-t)f(s-t) : \mathbb{R} \rightarrow X$$

belongs to \mathcal{Y} , and the family $\mathbf{T} := \{T(t) : t \geq 0\}$ is an one-parameter semigroup of bounded linear operators on \mathcal{Y} . The semigroup \mathbf{T} can be not strongly continuous. However, in certain cases, this semigroup is strongly continuous, and is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}}$ on the space \mathcal{Y} .

Lemma 3. *Let $\mathcal{U}_{\mathbb{R}}$ be a strongly continuous evolution family of bounded linear operators on X , and q be a fixed positive real number.*

- (i) *If $\mathcal{Y} = C_0(\mathbb{R}, X)$, and $\mathcal{U}_{\mathbb{R}}$ is exponentially bounded, then the semigroup associated to $\mathcal{U}_{\mathbb{R}}$, defined in (1.5), is a strongly continuous one-parameter semigroup of bounded linear operators on \mathcal{Y} ;*
- (ii) *If \mathcal{Y} is either the spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_{\mathbb{R}}$ is q -periodic, that is, $U(t+q, s+q) = U(t, s)$ for all $(t, s) \in \Delta_{\mathbb{R}}$, then the semigroup given in (1.5), is a strongly continuous semigroup on \mathcal{Y} .*

Let $(B, D(B))$ the generator of the evolution semigroup given in (1.5). If u and g belongs to \mathcal{Y} then the following statements are equivalent:

- (iii) $u \in D(B)$ and $Bu = -g$;
- (iv)

$$(1.6) \quad u(t) = U(t, s)u(s) + \int_s^t U(t, s)g(s)ds,$$

for all $t \geq s$.

Proof. See [5], [9] for evolution semigroups defined on $C_0(\mathbb{R}, X)$ and [8] for evolution semigroups on $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$. ■

Let \mathcal{Y} be one of the spaces $C_0(\mathbb{R}, X)$, $AP(\mathbb{R}, X)$, $AAP(\mathbb{R}, X)$ and let \mathcal{Y}_0 be the set of all functions $f \in \mathcal{Y}$ such that $\lim_{t \rightarrow (-\infty)} f(t) = 0$. It is clearly that \mathcal{Y}_0 is a closed subspace of \mathcal{Y} .

We may now state our second result.

Theorem 2. *Let $\mathcal{U}_{\mathbb{R}}$ be a strongly continuous uniformly stable evolution family of bounded linear operators on X and $q > 0$, fixed. The following statements hold:*

- (j) *If $\mathcal{Y} = C_0(\mathbb{R}, X)$, then the evolution semigroup given in (1.5) is defined on \mathcal{Y}_0 ;*
- (jj) *If \mathcal{Y} is one of the both spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_{\mathbb{R}}$ is q -periodic then the evolution semigroup given in (1.4) is defined on \mathcal{Y}_0 .*

If $(C, D(C))$ is the generator of the evolution semigroup on \mathcal{Y}_0 , given in (1.5), and v, h belongs to \mathcal{Y}_0 , then the following statements are equivalent:

- (jjj) *$v \in D(C)$ and $Cv = -h$;*
- (jv)

$$(1.7) \quad v(t) = \int_{-\infty}^t U(t, s)h(s)ds,$$

for every real number t . Moreover, the following inequality holds:

$$(1.8) \quad \left\| \int_{-\infty}^{\cdot} U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}^2 \leq 4M^2 \|h\|_{\mathcal{Y}} \times \left\| \int_{-\infty}^{\cdot} (\cdot - s)U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}.$$

2. PROOFS

Proof of Theorem 1. Let \mathbf{T} be the evolution semigroup associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space \mathcal{X} and $(A, D(A))$ its infinitesimal generator. From Lemma 2 it follows that the function $t \mapsto u(t) := \int_0^t U(t, s)g(s)ds$ belongs to $D(A)$ and $Au = -g$. The function $t \mapsto v(t) := \int_0^t U(t, r)u(r)dr$ belongs to \mathcal{X} . Indeed, using the Fubini Theorem, we have:

$$\begin{aligned} v(t) &= \int_0^t \left[U(t, r) \int_0^r U(r, s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^r U(t, s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^t 1_{[0, r]}(s)U(t, s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_s^t U(t, s)g(s)dr \right] ds \\ &= \int_0^t (t - s)U(t, s)g(s)ds, \end{aligned}$$

where $1_{[0, r]}$ is the characteristic function of the interval $[0, r]$. Using again Lemma 2 follows that $v \in D(A^2)$ and $A^2v = A(Av) = -Av = g$.

Now the inequality (1.4) follows by Lemma 1, if we replace x with v in (1.2). ■

Proof of Theorem 2. Firstly we prove that \mathcal{Y}_0 is an invariant subspace for each operator $T(t), t \geq 0$, given in (1.5). By Lemma 3 it suffices to prove that $\lim_{s \rightarrow (-\infty)} (T(t)f)(s) =$

0 for each $t \geq 0$ and every $f \in \mathcal{Y}_0$, and this fact is an easy consequence of the following estimations:

$$\|(T(t)f)(s)\| \leq \|U(s, s-t)\| \|f(s-t)\| \leq M \|f(s-t)\| \rightarrow 0 \text{ as } s \rightarrow (-\infty),$$

where M is the positive constant from (1.1). Now, the implication $(jjj) \Rightarrow (jv)$ follows from Lemma 3, passing to the limit for $s \rightarrow (-\infty)$. The converse implication $(jv) \Rightarrow (jjj)$ can be obtained on the following way.

Let v as in (1.7) and $t > 0$. Simple calculus gives

$$\frac{T(t)v - v}{t} = -\frac{\int_0^t T(r)h dr}{t} \rightarrow -h \text{ in } \mathcal{X}$$

when $t \rightarrow 0$, that is $v \in D(C)$ and $Cv = -h$. Now the inequality (1.8), can be established as in the proof of Theorem 1 and we omit the details. ■

3. APPLICATIONS

In this section some scalar inequalities are presented.

Corollary 1. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function such that $g(0) = g(\infty) := \lim_{t \rightarrow \infty} g(t) = 0$. Suppose that the functions:*

$$t \mapsto h(t) := \int_0^t g(s)ds \text{ and } t \mapsto u(t) := \int_0^t (t-s)g(s)ds$$

verifies the condition $h(\infty) = u(\infty) = 0$.

Then the following inequality holds:

$$\sup_{t \geq 0} \left| \int_0^t g(s)ds \right|^2 \leq 4 \cdot \sup_{t \geq 0} |g(t)| \times \sup_{t \geq 0} \left| \int_0^t (t-s)g(s)ds \right|.$$

Proof. We apply Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R})$ and for $U(t, s)x = x$, where $t \geq s \geq 0$ and $x \in \mathbb{R}$. ■

Corollary 2. *Let g, h, u as in Corollary 1 and f be a continuous, positive and nondecreasing function on \mathbb{R}_+ . The following inequality holds:*

$$\sup_{t \geq 0} \left[\frac{\left| \int_0^t f(s)g(s)ds \right|^2}{f(t)^2} \right] \leq 4 \sup_{t \geq 0} |g(t)| \sup_{t \geq 0} \left[\frac{\left| \int_0^t (t-s)f(s)g(s)ds \right|}{f(t)} \right].$$

Proof. Follows by Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R})$ and $U(t, s) = \frac{f(s)}{f(t)}$. ■

Corollary 3. *Let $1 \leq p < \infty$ and $f \in L_p(\mathbb{R}_+, \mathbb{R})$. If the functions*

$$t \mapsto g(t) := \int_0^t f(s)ds \text{ and } t \mapsto h(t) := \int_0^t (t-s)f(s)ds$$

belongs to $L^p(\mathbb{R}_+, \mathbb{R})$, then the following inequality, holds:

$$\|g\|_p^2 \leq 4 \|f\|_p \times \|h\|_p.$$

Proof. Follows by Theorem 1 for $\mathcal{X} = L_p(\mathbb{R}_+, \mathbb{R})$ and for $U(t, s)x = x$ where $t \geq s \geq 0$ and $x \in \mathbb{R}$. ■

Corollary 4. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an almost periodic or asymptotically almost periodic function such that $g(-\infty) = 0$. Then*

$$\sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t \frac{1 + \sin^2 s}{1 + \sin^2 t} g(s) ds \right|^2 \leq 16 \sup_{t \in \mathbb{R}} |g(t)| \times \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t (t-s) \frac{1 + \sin^2 s}{1 + \sin^2 t} g(s) ds \right|.$$

Proof. Follows by Theorem 2 for $\mathcal{Y} = AP(\mathbb{R}, \mathbb{R})$ or $\mathcal{Y} = AAP(\mathbb{R}, \mathbb{R})$ and $U(t, s)x = \frac{1 + \sin^2 s}{1 + \sin^2 t} x$ where $t \geq s$ and $x \in \mathbb{R}$. It is clear that $\mathcal{U} = \{U(t, s); t \geq s\}$ is a π -periodic family consisting in operators acting on \mathbb{R} , and $\sup_{t \geq s} U(t, s) \leq 2$. ■

REFERENCES

- [1] C. Chicone, Yu. Latushkin, "Evolution Semigroups in Dynamical Systems and Differential Equations", *Amer. Math. Soc., Math. Surv. and Monographs*, **70**, 1999.
- [2] S. Clark, Yu Latushkin, S. Montgomery-Smith and T. Randolph, Stability radius and internal versus external stability in Banach spaces: An evolution semigroup approach, *SIAM J. Contr. and Optim.*, **38** (2000), 1757-1793.
- [3] K. Engel and R. Nagel, "One-parameter semigroups for linear evolution equations", Springer-Verlag, New-York, 2000.
- [4] R. R. Kallman and G. C. Rota, On the inequality $\|f'\| \leq 4\|f\|\|f''\|$, *Inequalities II*, O. Shisha, Ed., Academic Press, New-York, 1970, pp. 187-192.
- [5] Yu. Latushkin and S. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, *J. Func. Anal.*, **127**(1995), 173-197.
- [6] B. M. Levitan and V. V. Zhikov, "Almost Periodic Functions and Differential Equations", Moscow Univ. Publ. House, 1978. English translation by Cambridge Univ. Press, Cambridge U.K., 1982.
- [7] Nguyen Van Minh, Frank Răbiger and Roland Schnaubelt, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory*, **32**,(1998), 332-353.
- [8] T. Naito, Nguyen Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, *J. Differential Equations*, **152**(1999), 338-376.
- [9] R. Rau, Hyperbolic evolution semigroups on vector valued-functions, *Semigroup Forum*, **48** (1994), 107-118.

DEPARTMENT OF MATHEMATICS, WEST UNIVERSITY OF TIMISOARA, TIMISOARA, 1900, BD. V. PARVAN. NR. 4, ROMANIA

E-mail address: buse@tim1.uvt.ro

URL: <http://rgmia.vu.edu.au/BuseCVhtml/>

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, P.O. BOX 14428, MELBOURNE CITY MC, VICTORIA 8001,, AUSTRALIA.

E-mail address: sever.dragomir@vu.edu.au

URL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>