Some New Iyengar Type Inequalities

SOME NEW IYENGAR TYPE INEQUALITIES

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Abstract. Some new Iyengar type inequalities for an integral are obtained by using the generalised Taylor formula with integral remainder.

1. Introduction

Let $f(x)$ be a differentiable function on a closed interval $[a,b]$ such that $|f'(x)| \leq M$, then

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{(b-a)[f(a)+f(b)]}{2}\right| \leq \frac{(b-a)^2 M}{4} - \frac{[f(b)-f(a)]^2}{4M}. \tag{1}$$

In 1938, K. S. K. Iyengar [15] established inequality (1) by using a geometric approach. So, we call (1) the Iyengar inequality.

Using the Rolle and Lagarange mean value theorems, the following inequalities were obtained naturally and simply in [24], producing a refinement of the Iyengar inequality (1).

Theorem A. *Let $f(x)$ be continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b), and $m \leq f'(x) \leq M$ for $x \in (a,b)$. If $f(x)$ is not a constant, then we have*

$$\begin{aligned}\frac{mM(b-a)^2 + 2(b-a)\left(Mf(a)-mf(b)\right) + \left(f(a)-f(b)\right)^2}{2\left(M-m\right)} &\leq \int_a^b f(x)\,dx \\ &\leq -\frac{mM(b-a)^2 + 2(b-a)\left(mf(a)-Mf(b)\right) + \left(f(a)-f(b)\right)^2}{2\left(M-m\right)}. \end{aligned} \tag{2}$$

For $f'(x)$ integrable on $[a,b]$, the inequalities in (2) were obtained independently by R.P. Agarwal and S.S. Dragomir in [2] using the Hayashi inequality. They

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expressed the result (2) as

$$\left|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}\right| \leq \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)}. \quad (3)$$

In [3], the inequalities in (2) and in (3) were derived in a different manner and rearranged as

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{2}[f(a)+f(b)]\right| \leq \frac{(b-a)^2}{2(M-m)}(S-m)(M-S) \quad (4)$$

$$\leq \frac{M-m}{2}\left(\frac{b-a}{2}\right)^2, \quad (5)$$

where $S=\frac{f(b)-f(a)}{b-a}$.

In the papers [25, 26], using the Taylor formula for functions with a single variable or several variables, the Iyengar inequality (1) was generalized as follows.

Theorem B. *Let $f(x)$ be a differentiable function of $C^n([a,b])$ satisfying $N\leq f^{(n+1)}(x)\leq M$ for $x\in(a,b)$. Denote*

$$S_n(u,v,w)=\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}u^k f^{(k-1)}(v)+(-1)^n\frac{w}{n!}u^n \quad (6)$$

$$\text{and}\quad \frac{\partial^k S_n}{\partial u^k}=S_n^{(k)}(u,v,w), \quad (7)$$

then, for any $t\in[a,b]$, we have when n is an odd,

$$\sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,N)-S_{n+2}^{(i)}(b,b,N)\right)t^i\leq\int_a^b f(x)\,x \leq \sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,M)-S_{n+2}^{(i)}(b,b,M)\right)t^i; \quad (8)$$

and when n is even

$$\sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,N)-S_{n+2}^{(i)}(b,b,M)\right)t^i\leq\int_a^b f(x)\,x \leq \sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,M)-S_{n+2}^{(i)}(b,b,N)\right)t^i. \quad (9)$$

We may deduce both Theorem A and the following result as specialisations of Theorem B.

Theorem C. *Let $f(x)$ be differentiable on the closed interval $[a,b]$, and $N \leq f''(x) \leq M$ for $x \in (a,b)$, then*

$$\begin{aligned}
&\frac{N(b^3-a^3)}{6} + \frac{\left\{f(a)-f(b)+bf'(b)-af'(a)+\dfrac{N(a^2-b^2)}{2}\right\}^2}{2\left[(a-b)N-f'(a)+f'(b)\right]} \\
&\leq \int_a^b f(x)\,x - bf(b) + af(a) + \frac{b^2f'(b)-a^2f'(a)}{2} \qquad (10)\\
&\leq \frac{M(b^3-a^3)}{6} + \frac{\left\{f(a)-f(b)+bf'(b)-af'(a)+\dfrac{M(a^2-b^2)}{2}\right\}^2}{2\left[(a-b)M-f'(a)+f'(b)\right]}.
\end{aligned}$$

In recent years, R.P. Agarwal, P. Cerone, S.S. Dragomir and S. Wang, utilizing Hayashi inequality or Steffensen inequality, generalized Iyengar' inequality to those involving higher derivatives, weighted integral, and so on. See [2]–[6], and [10].

Meanwhile, F. Qi, B.-N. Guo, Y.-J. Zhang and L.-H. Cui, proceeded in an alternate direction. Using the mean value theorems for functions with a single variable or several variables, generalized Iyengar inequality to involve bounds of higher order derivatives, with norm bounds, for weighted multiple integral, and the like. See [8, 12, 14] and [24]–[29].

The Iyengar inequality (1) has been researched by many mathematicians, and there is much literature devoted to it. Please refer to references in this paper. Before elaborating on the contributions of the current article, it is useful to present some background notation and definitions.

Definition 1. A sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ is called harmonic if it satisfies the following Appell condition

$$P_i'(t) \triangleq \frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \triangleq P_{i-1}(t) \qquad (11)$$

and $P_0(t,x)=1$ for all defined (t,x) and $i \in \mathbb{N}$.

It is well-known that Bernoulli polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x-1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!}x^i, \quad |x|<2\pi, \quad t \in \mathbb{R}, \qquad (12)$$

and are uniquely determined by the following formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t)=1; \qquad (13)$$

$$\text{and} \quad B_i(t+1)-B_i(t) = it^{i-1}. \qquad (14)$$

Similarly, Euler polynomials can be defined by

$$\frac{2e^{tx}}{e^x+1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!}x^i, \quad |x|<\pi, \quad t \in \mathbb{R}, \qquad (15)$$

and are uniquely determined by the following properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t)=1; \qquad (16)$$

$$E_i(t+1)+E_i(t) = 2t^i. \qquad (17)$$

For further details about Bernoulli polynomials and Euler polynomials, please refer to [1, 23.1.5 and 23.1.6]. Moreover, some new generalizations of Bernoulli numbers and polynomials can be found in [13, 17, 27].

There are many examples of harmonic sequences of polynomials. For instance ([3, 7, 18]), for i a nonegative integer, $t, \tau, \theta \in \mathbb{R}$ and $\tau \neq \theta$,

$$P_{i,\lambda}(t) \triangleq P_{i,\lambda}(t;\tau;\theta) = \frac{[t-(\lambda\theta+(1-\lambda)\tau)]^i}{i!}, \tag{18}$$

$$P_{i,B}(t) \triangleq P_{i,B}(t;\tau;\theta) = \frac{(\tau-\theta)^i}{i!} B_i\left(\frac{t-\theta}{\tau-\theta}\right), \tag{19}$$

$$P_{i,E}(t) \triangleq P_{i,E}(t;\tau;\theta) = \frac{(\tau-\theta)^i}{i!} E_i\left(\frac{t-\theta}{\tau-\theta}\right). \tag{20}$$

As usual, let $B_i = B_i(0)$, $i \in \mathbb{N}$, denote Bernoulli numbers. From properties (13) and (14), (16) and (17) of Bernoulli and Euler polynomials respectively, we can obtain easily that, for $i \geq 1$,

$$B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \tag{21}$$

and, for $j \in \mathbb{N}$,

$$E_j(0) = -E_j(1) = -\frac{2}{j+1}(2^{j+1}-1)B_{j+1}. \tag{22}$$

It is also a well known fact that $B_{2i+1} = 0$ for all $i \in \mathbb{N}$.

In [18], the following generalized Taylor formula was established.

Theorem D. *Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then, for any $x \in I$, we have*

$$f(x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\right] + R_n(f;a,x), \tag{23}$$

where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{24}$$

If we set in (23) and (24) $P_n(t) = \frac{(t-x)^n}{n!}$, then we obtain directly the classical Taylor formula with remainder of integral form. Generalised Taylor formulae from product Appel polynomials was considered in [4] which involves the product of polynomials satisfying the Appel condition (1).

In this article, using the generalized Taylor formula (23), we will derive some new Iyengar type inequalities for an integral of functions with single variable, which generalize some related known results obtained in [8, 12, 14, 15, 24, 25, 28], for example.

2. Some integral identities

In this section, we establish two identities involving integrals which form the basis for the procurement of our main results.

Theorem 1. *Let $\{P_i(x)\}_{i=0}^{\infty}$ and $\{Q_i(x)\}_{i=0}^{\infty}$ be harmonic sequences of polynomials. If $f:[a,b]\to\mathbb{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n\in\mathbb{N}$, then we have the following generalized Taylor identity*

$$
\begin{aligned}
(n+1)\int_a^b f(x)\,dx &= b\sum_{k=0}^{n} q(k,k;b) - a\sum_{k=0}^{n} p(k,k;a) \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k}\big[p(i,i-1,a) - q(i,i-1;b)\big] \\
&\quad + t\sum_{k=0}^{n}\big[p(k,k;a) - q(k,k;b)\big] + \sum_{k=1}^{n}\sum_{i=1}^{k}\big[q(i,i-1,t) - p(i,i-1;t)\big] \\
&\quad + \int_a^t (t-s)p(n,n+1;s)\,ds + \int_t^b (t-s)q(n,n+1;s)\,ds, \qquad (25)
\end{aligned}
$$

where $t\in[a,b]$ and

$$p(\ell,m;x) = (-1)^{\ell}P_\ell(x)f^{(m)}(x), \qquad (26)$$

$$q(\ell,m;x) = (-1)^{\ell}Q_\ell(x)f^{(m)}(x) \qquad (27)$$

for any nonnegative integers $0\le \ell\le n$ and $0\le m\le n+1$ and $x\in[a,b]$.

Proof. Let t be a parameter such that $a\le t\le b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^t f(x)\,\mathrm{d}x + \int_t^b f(x)\,\mathrm{d}x. \qquad (28)$$

From the generalized Taylor formula (23), for $x\in[a,b]$, it follows that

$$f(x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\big[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\big] + R_{n,P}(f;a,x), \qquad (29)$$

$$f(x) = f(b) + \sum_{k=1}^{n}(-1)^{k+1}\big[Q_k(x)f^{(k)}(x) - Q_k(b)f^{(k)}(b)\big] + R_{n,Q}(f;b,x). \qquad (30)$$

By integration by parts we have

$$
\begin{aligned}
&\int_a^t P_k(x)f^{(k)}(x)\,\mathrm{d}x \\
&\qquad = \big[P_k(t)f^{(k-1)}(t) - P_k(a)f^{(k-1)}(a)\big] - \int_a^t P_{k-1}(x)f^{(k-1)}(x)\,\mathrm{d}x.
\end{aligned}
$$

Clearly, we can apply the same procedure to the term $\int_a^t P_{k-1}(x)f^{(k-1)}(x)\,\mathrm{d}x$. So, by successive integration by parts we obtain

$$
\begin{aligned}
&(-1)^k\int_a^t P_k(x)f^{(k)}(x)\,\mathrm{d}x \\
&\qquad = \sum_{i=1}^{k}(-1)^i\big[P_i(t)f^{(i-1)}(t) - P_i(a)f^{(i-1)}(a)\big] + \int_a^t f(x)\,\mathrm{d}x. \qquad (31)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(-1)^k & \int_t^b Q_k(x) f^{(k)}(x)\,\mathrm{d}x \\
&= \sum_{i=1}^{k} (-1)^i \big[Q_i(b) f^{(i-1)}(b) - Q_i(t) f^{(i-1)}(t)\big] + \int_t^b f(x)\,\mathrm{d}x. \quad (32)
\end{aligned}
$$

Integrating both sides of formula (29) over the interval $[a,t]$ and utilizing identity (31) yields

$$
\begin{aligned}
(n+1)\int_a^t f(x)\,\mathrm{d}x &= (t-a)\sum_{k=0}^{n} (-1)^k P_k(a) f^{(k)}(a) \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} (-1)^{i+1}\big[P_i(t) f^{(i-1)}(t) - P_i(a) f^{(i-1)}(a)\big] \qquad (33) \\
&\quad + (-1)^n \int_a^t \int_a^x P_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Similarly, integrating both sides of the result (30) over the interval $[t,b]$ and using the identity (32) gives

$$
\begin{aligned}
(n+1)\int_t^b f(x)\,\mathrm{d}x &= (b-t)\sum_{k=0}^{n} (-1)^k Q_k(b) f^{(k)}(b) \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} (-1)^{i+1}\big[Q_i(b) f^{(i-1)}(b) - Q_i(t) f^{(i-1)}(t)\big] \qquad (34) \\
&\quad + (-1)^n \int_t^b \int_b^x Q_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Now, combining (33) and (34) and utilising (28) produces

$$
\begin{aligned}
(n+1)\int_a^b f(x)\,\mathrm{d}x &= b\sum_{k=0}^{n} (-1)^k Q_k(b) f^{(k)}(b) - a\sum_{k=0}^{n} (-1)^k P_k(a) f^{(k)}(a) \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} (-1)^{i+1}\big[Q_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)\big] \\
&\quad + t\sum_{k=0}^{n} (-1)^{k+1}\big[Q_k(b) f^{(k)}(b) - P_k(a) f^{(k)}(a)\big] \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} (-1)^{i}\big[Q_i(t) - P_i(t)\big] f^{(i-1)}(t) \\
+ (-1)^n \bigg[\int_a^t \int_a^x & P_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x + \int_t^b \int_b^x Q_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x\bigg]. \quad (35)
\end{aligned}
$$

Further, interchanging the order of integration for the last two terms in (35) leads to

$$\int_a^t \int_a^x P_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x = \int_a^t (t-s) P_n(s) f^{(n+1)}(s)\,\mathrm{d}s, \tag{36}$$

$$\int_t^b \int_b^x Q_n(s) f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x = \int_t^b (t-s) Q_n(s) f^{(n+1)}(s)\,\mathrm{d}s. \tag{37}$$

Substituting (36) and (37) into (35), rearranging and introducing the notation in (26) and (27) produces the identity (25). The proof is complete. ∎

Remark 1. If we set

$$P_i(s) = Q_i(s) = \frac{(s-x)^i}{i!} \tag{38}$$

for i a nonnegative natural number in (29) and (30) from the very start, then following the same procedure as in the proof of Theorem 1, we would obtain

$$\int_a^t f(x)\,dx = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1} + \frac{1}{(n+1)!}\int_a^t (t-s)^{n+1} f^{(n+1)}(s)\,ds, \tag{39}$$

$$\int_t^b f(x)\,dx = -\sum_{i=0}^{n} \frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1} + \frac{1}{(n+1)!}\int_t^b (t-s)^{n+1} f^{(n+1)}(s)\,ds, \tag{40}$$

the combination of which, produces

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1} - \sum_{i=0}^{n} \frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1} + \frac{1}{(n+1)!}\int_a^b (t-s)^{n+1} f^{(n+1)}(s)\,ds. \tag{41}$$

The following result is a particularisation of Theorem 1, but because of its intrinsic importance is denoted as a theorem in its own right, rather than a corollary.

Theorem 2. *Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. If $f : [a,b] \to \mathbb{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then we have the following identity*

$$\begin{aligned}(n+1)\int_a^b f(x)\,dx &= b\sum_{k=0}^{n} p(k,k;b) - a\sum_{k=0}^{n} p(k,k;a) \\ &\quad - \sum_{k=1}^{n}\sum_{i=1}^{k}\big[p(i,i-1;b) - p(i,i-1;a)\big] - t\sum_{k=0}^{n}\big[p(k,k;b) - p(k,k;a)\big] \\ &\quad + \int_a^b (t-s) p(n,n+1;s)\,ds,\end{aligned} \tag{42}$$

where $t \in [a,b]$ and $p(\ell,m;x) = (-1)^{\ell} P_{\ell}(x) f^{(m)}(x)$, $x \in [a,b]$, for any nonnegative integers $0 \le \ell \le n$ and $0 \le m \le n+1$.

Proof. This follows from taking $P_i(s) = Q_i(s)$ for $0 \le i \le \infty$ and $s \in [a,b]$ in Theorem 1. ∎

Remark 2. If we set

$$P_i(s) = \frac{[s-(\lambda b+(1-\lambda)a)]^i}{i!} \quad \text{and} \quad Q_i(s) = \frac{[s-(\mu b+(1-\mu)a)]^i}{i!} \tag{43}$$

in identity (25), then

$$\begin{aligned}
(n+1)\int_a^b f(x)\,\mathrm{d}x &= \sum_{k=0}^{n} \frac{(b-a)^k}{k!}\big[b(\mu-1)^k f^{(k)}(b) - a\lambda^k f^{(k)}(a)\big] \\
&\quad - \sum_{k=1}^{n}\sum_{i=1}^{k} \frac{(b-a)^i}{i!}\big[(\mu-1)^i f^{(i-1)}(b) - \lambda^i f^{(i-1)}(a)\big] \\
&\quad - t\sum_{k=0}^{n} \frac{(b-a)^k}{k!}\big[(\mu-1)^k f^{(k)}(b) - \lambda^k f^{(k)}(a)\big] \\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} \frac{(-1)^i}{i!}\big\{[t-(\mu b+(1-\mu)a)]^i - [t-(\lambda b+(1-\lambda)a)]^i\big\} f^{(i-1)}(t) \\
&\quad + \frac{(-1)^n}{n!}\bigg(\int_a^t (t-s)[s-(\lambda b+(1-\lambda)a)]^n f^{(n+1)}(s)\,ds \\
&\quad + \int_t^b (t-s)[s-(\mu b+(1-\mu)a)]^n f^{(n+1)}(s)\,ds\,dx\bigg).
\end{aligned} \tag{44}$$

If we set

$$P_i(s) = Q_i(s) = \frac{[s-(\lambda b+(1-\lambda)a)]^i}{i!} \tag{45}$$

in formula (42), then

$$\begin{aligned}
(n+1)\int_a^b f(x)\,\mathrm{d}x &= \sum_{k=0}^{n} \frac{(b-a)^k}{k!}\big[b(\lambda-1)^k f^{(k)}(b) - a\lambda^k f^{(k)}(a)\big] \\
&\quad - \sum_{k=1}^{n}\sum_{i=1}^{k} \frac{(b-a)^i}{i!}\big[(\lambda-1)^i f^{(i-1)}(b) - \lambda^i f^{(i-1)}(a)\big] \\
&\quad - t\sum_{k=0}^{n} \frac{(b-a)^k}{k!}\big[(\lambda-1)^k f^{(k)}(b) - \lambda^k f^{(k)}(a)\big] \\
&\quad + \frac{(-1)^n}{n!}\int_a^b (t-s)[s-(\lambda b+(1-\lambda)a)]^n f^{(n+1)}(s)\,ds.
\end{aligned} \tag{46}$$

Taking $\lambda = 0$ or $\lambda = 1$ in (46) yields expansions about b and a respectively. Namely,

$$\begin{aligned}
(n+1)\int_a^b f(x)\,dx &= b\sum_{k=0}^{n} \frac{(a-b)^k}{k!} f^{(k)}(b) - \sum_{k=1}^{n}\sum_{i=1}^{k} \frac{(a-b)^i}{i!} f^{(i-1)}(b) \\
&\quad - t\sum_{k=0}^{n} \frac{(a-b)^k}{k!} f^{(k)}(b) + \frac{1}{n!}\int_a^b (t-s)(a-s)^n f^{(n+1)}(s)\,ds.
\end{aligned} \tag{47}$$

or

$$(n+1)\int_a^b f(x)\,dx = \sum_{k=1}^{n}\sum_{i=1}^{k}\frac{(b-a)^i}{i!}f^{(i-1)}(a) - a\sum_{k=0}^{n}\frac{(b-a)^k}{k!}f^{(k)}(a)$$
$$+\,t\sum_{k=0}^{n}\frac{(b-a)^k}{k!}f^{(k)}(a) + \frac{1}{n!}\int_a^b (t-s)(b-s)^n f^{(n+1)}(s)\,ds. \quad (48)$$

Remark 3. If the assumption is made that,

$$P_i(s) = P_{i,B}(s;b;a) = \frac{(b-a)^i}{i!}B_i\left(\frac{s-a}{b-a}\right) \quad (49)$$

for nonnegative integer i, then from Theorem 2, straightforward calculations and utilising the properties (21), (22) together with the fact that $B_{2i+1} = 0$ for $i \in \mathbb{N}$, produces

$$(n+1)\int_a^b f(x)\,dx = \big[bf(b) - af(a)\big] - \frac{1}{2}(b-a)\big[af'(a) + bf'(b)\big]$$
$$+\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{(b-a)^{2k}}{(2k)!}B_{2k}\big[bf^{(2k)}(b) - af^{(2k)}(a)\big]$$
$$+\frac{n}{2}(b-a)\big[f(a)+f(b)\big] - \sum_{k=2}^{n}\sum_{i=1}^{\left[\frac{k}{2}\right]}\frac{(b-a)^{2i}}{(2i)!}B_{2i}\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]$$
$$-t\left\{\big[f(b)-f(a)\big] - \frac{1}{2}(b-a)\big[f'(a)+f'(b)\big] + \sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{(b-a)^{2k}}{(2k)!}B_{2k}\big[f^{(2k)}(b) - f^{(2k)}(a)\big]\right\}$$
$$+\frac{(a-b)^n}{n!}\int_a^b (t-s)B_n\left(\frac{s-a}{b-a}\right)f^{(n+1)}(s)\,ds, \quad (50)$$

where $[x]$ denotes the Gauss function, whose value is the largest integer not exceeding x.

Remark 4. Suppose that in Theorem 2

$$P_i(s) = P_{i,E}(s;a;b) = \frac{(a-b)^i}{i!}E_i\left(\frac{s-b}{a-b}\right) \quad (51)$$

for nonnegative integer i, then, by direct calculation and utilising (21), (22) and $B_{2i+1} = 0$ for $i \in \mathbb{N}$, we obtain

$$(n+1)\int_a^b f(x)\,dx = af(a) + bf(b)$$
$$+\sum_{k=1}^{\left[\frac{n+1}{2}\right]}\frac{2(1-4^k)(b-a)^{2k-1}}{(2k)!}B_{2k}\big[af^{(2k-1)}(a) + bf^{(2k-1)}(b)\big]$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\frac{2(4^i-1)(b-a)^{2i-1}}{(2i)!}B_{2i}\big[f^{(2(i-1))}(a) + f^{(2(i-1))}(b)\big]$$

$$- t\Biggl\{af(a) + bf(b) + \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{2(4^k-1)(b-a)^{2k-1}}{(2k)!} B_{2k}\bigl[f^{(2k-1)}(a) + f^{(2k-1)}(b)\bigr]\Biggr\} + \frac{(b-a)^n}{n!}\int_a^b (t-s)E_n\left(\frac{s-b}{a-b}\right) f^{(n+1)}(s)\,ds, \quad (52)$$

where $[x]$ also denotes the Gauss function as in Remark 3.

3. Some new Iyengar type inequalities

In the work that follows, we adopt the notation:

$$\begin{aligned}
\Gamma_P &= \sup_{x\in[a,b]} p(n,n+1;x) = \sup_{x\in[a,b]} \bigl\{(-1)^n P_n(x) f^{(n+1)}(x)\bigr\},\\
\gamma_P &= \inf_{x\in[a,b]} p(n,n+1;x) = \inf_{x\in[a,b]} \bigl\{(-1)^n P_n(x) f^{(n+1)}(x)\bigr\},\\
\Gamma_Q &= \sup_{x\in[a,b]} q(n,n+1;x) = \sup_{x\in[a,b]} \bigl\{(-1)^n Q_n(x) f^{(n+1)}(x)\bigr\},\\
\gamma_Q &= \inf_{x\in[a,b]} q(n,n+1;x) = \inf_{x\in[a,b]} \bigl\{(-1)^n Q_n(x) f^{(n+1)}(x)\bigr\},\\
\Gamma &= \sup_{x\in[a,b]} f^{(n+1)}(x), \quad \gamma = \inf_{x\in[a,b]} f^{(n+1)}(x),
\end{aligned} \quad (53)$$

where the definitions (26) and (27) have been used.

Suppose $f(x)$ is a n-times differentiable function on the closed interval $[a,b]$ and $f^{(n+1)}(x)$ exists on (a,b). Let

$$M = \Gamma = \sup_{s\in(a,b)} f^{(n+1)}(s), \quad N = \gamma = \inf_{s\in(a,b)} f^{(n+1)}(s), \quad (54)$$

then, from (41), we can deduce Theorem B and Theorem C, the results obtained in [25] and [26].

Bounds may also be obtained in terms of the traditional Lebesgue norms for $f^{(n+1)} \in L^p([a,b])$, $1 \le p < \infty$. Specifically, if $f^{(n)}(x)$ is absolutely continuous on the closed interval $[a,b]$, then, for any $t\in[a,b]$, it follows from (41), that

$$\left|\int_a^b f(x)\,dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}\right| \le \frac{(t-a)^{n+2}+(b-t)^{n+2}}{(n+2)!}\int_a^b \left|f^{(n+1)}(s)\right| ds. \quad (55)$$

Also, as a simple consequence of Hölder inequality, if $f^{(n+1)} \in L^p([a,b])$ for a positive number $p>1$, then, for any $t\in[a,b]$, we have

$$\left|\int_a^b f(x)\,dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}\right| \le \frac{(t-a)^{n+1+\frac{1}{q}}+(b-t)^{n+1+\frac{1}{q}}}{(n+1)!\sqrt[q]{nq+q+1}}\left(\int_a^b \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}, \quad (56)$$

where q satisfies $\frac{1}{p}+\frac{1}{q}=1$.

Note that inequality (56) has been proved in [12].

Let $\{P_i(x)\}_{i=0}^{\infty}$ and $\{Q_i(x)\}_{i=0}^{\infty}$ be two harmonic sequences of polynomials. If $f : [a, b] \to \mathbb{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then, from identity (25), it is easy to obtain the expression

$$\begin{aligned}
&\frac{1}{2}\big[(a-t)^2\gamma_{_Q} - (b-t)^2\Gamma_{_P}\big] \\
&\qquad \le (n+1)\int_a^b f(x)\,dx + a\sum_{k=0}^{n} p(k,k;a) - b\sum_{k=0}^{n} q(k,k;b) \\
&\qquad\qquad - \sum_{k=1}^{n}\sum_{i=1}^{k}\big[p(i,i-1,a) - q(i,i-1;b)\big] \\
&\quad - t\sum_{k=0}^{n}\big[p(k,k;a) - q(k,k;b)\big] - \sum_{k=1}^{n}\sum_{i=1}^{k}\big[q(i,i-1,t) - p(i,i-1;t)\big] \\
&\qquad\qquad\qquad \le \frac{1}{2}\big[(a-t)^2\Gamma_{_P} - (b-t)^2\gamma_{_Q}\big]. \qquad (57)
\end{aligned}$$

Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. If $f : [a, b] \to \mathbb{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then we have the following inequalities

$$\begin{aligned}
&\frac{1}{2}\big[a^2\gamma_{_P} - b^2\Gamma_{_P}\big] - \frac{\big\{b\Gamma_{_P} - a\gamma_{_P} - \sum_{k=0}^{n}\big[p(k,k;b) - p(k,k;a)\big]\big\}^2}{2\big(\gamma_{_P} - \Gamma_{_P}\big)} \\
&\qquad \le (n+1)\int_a^b f(x)\,dx - b\sum_{k=0}^{n} p(k,k;b) + a\sum_{k=0}^{n} p(k,k;a) \\
&\qquad\qquad + \sum_{k=1}^{n}\sum_{i=1}^{k}\big[p(i,i-1;b) - p(i,i-1;a)\big] \\
&\quad \le \frac{1}{2}\big[a^2\Gamma_{_P} - b^2\gamma_{_P}\big] - \frac{\big\{b\gamma_{_P} - a\Gamma_{_P} - \sum_{k=0}^{n}\big[p(k,k;b) - p(k,k;a)\big]\big\}^2}{2\big(\Gamma_{_P} - \gamma_{_P}\big)}. \qquad (58)
\end{aligned}$$

In fact, (58) follows from the identity (42) and maximizing or minimizing a quadratic polynomial in t.

Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. If $f : [a, b] \to \mathbb{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, and defining

$$\mathcal{U} := \max_{s\in[a,b]} |P_n(s)|\,, \quad \mathcal{V} := \max_{s\in[a,b]} \left|f^{(n+1)}(s)\right|, \quad \text{and} \quad \mathcal{W} := \max_{s\in[a,b]} \left|P_n(s) f^{(n+1)}(s)\right|,$$

then, from (42), maximizing or minimizing a quadratic polynomial in t yields

$$\begin{aligned}
&\frac{\big\{(a+b)\mathcal{U}\mathcal{V}-\sum_{k=0}^{n}\big[p(k,k;b)-p(k,k;a)\big]\big\}^2}{4\mathcal{U}\mathcal{V}}-\frac{a^2+b^2}{2}\mathcal{U}\mathcal{V}\\
&\quad\le\frac{\big\{(a+b)\mathcal{W}-\sum_{k=0}^{n}\big[p(k,k;b)-p(k,k;a)\big]\big\}^2}{4\mathcal{W}}-\frac{a^2+b^2}{2}\mathcal{W}\\
&\quad\le(n+1)\int_a^b f(x)\,dx-b\sum_{k=0}^{n}p(k,k;b)+a\sum_{k=0}^{n}p(k,k;a)\\
&\qquad+\sum_{k=1}^{n}\sum_{i=1}^{k}\big[p(i,i-1;b)-p(i,i-1;a)\big]\\
&\quad\le\frac{a^2+b^2}{2}\mathcal{W}-\frac{\big\{(a+b)\mathcal{W}+\sum_{k=0}^{n}\big[p(k,k;b)-p(k,k;a)\big]\big\}^2}{4\mathcal{W}}\\
&\quad\le\frac{a^2+b^2}{2}\mathcal{U}\mathcal{V}-\frac{\big\{(a+b)\mathcal{U}\mathcal{V}+\sum_{k=0}^{n}\big[p(k,k;b)-p(k,k;a)\big]\big\}^2}{4\mathcal{U}\mathcal{V}}. \qquad (59)
\end{aligned}$$

Remark 5. Now it is clear that, if estimating the following general integral remainders

$$\int_a^t (t-s)p(n,n+1;s)\,ds+\int_t^b (t-s)q(n,n+1;s)\,ds, \qquad (60)$$

$$\text{and} \quad \int_a^b (t-s)p(n,n+1;s)\,ds \qquad (61)$$

that appear in (25) and (42), then using better and more appropriate techniques than previously gives rise to more accurate Iyengar type inequalities.

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