Ostrowski Type Inequalities for Functions whose Modulus of the Derivatives are Convex and Applications

OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE MODULUS OF THE DERIVATIVES ARE CONVEX AND APPLICATIONS

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ABSTRACT. Some inequalities of the Ostrowski type for functions whose modulus of derivatives are convex and applications for special means and to the f and $HH-$divergences in Information Theory are given.

1. INTRODUCTION

The following Ostrowski type inequalities for absolutely continuous functions are known (see [2], [3] and [4]).

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.1}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right](b-a)^{\frac{1}{p}}\|f'\|_q & \\ \qquad \text{if } f' \in L_q[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1 & \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|.$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ are sharp in the sense that they cannot be replaced by smaller constants.

The above inequalities may also be obtained from Fink's result in [5] on choosing $n = 1$ and performing some appropriate computations.

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The main aim of this paper is to point out some similar results in the case when the modulus of the derivative f' is a convex function on (a,b). Applications for special means and f and $HH-$divergence in Information Theory are also provided.

2. The Results

We start with the following lemma which is of intrinsic interest (see also [1]).

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, then, for any $x\in[a,b]$,*

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt \tag{2.1}$$

Proof. For any $x,t\in[a,b], x\neq t$, one has

$$\frac{f(x)-f(t)}{x-t}=\frac{1}{x-t}\int_t^x f'(u)du=\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda$$

showing that

$$f(x)=f(t)+(x-t)\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda \text{ for any } x,t\in[a,b]. \tag{2.2}$$

Integrating (2.2) over t on $[a,b]$ and dividing the result by $(b-a)$, gives the desired identity (2.1). ∎

Using the above lemma the following result can be pointed out improving Ostrowski's inequality.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is convex on (a,b). If $f'\in L_\infty[a,b]$, then for any $x\in[a,b]$,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right]. \tag{2.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Using (2.1) and taking the modulus, we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &= \frac{1}{b-a}\left|\int_a^b\int_0^1 (x-t)f'\left[(1-\lambda)x+\lambda t\right]d\lambda dt\right| \\ &\leq \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\left|f'\left[(1-\lambda)x+\lambda t\right]\right|d\lambda dt\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\left[(1-\lambda)\left|f'(x)\right|+\lambda\left|f'(t)\right|\right]d\lambda dt \\
&\quad \text{(by convexity of } |f'|\text{)} \\
&= \frac{1}{b-a}\int_a^b |x-t|\left[\left|f'(x)\right|\int_0^1(1-\lambda)\,d\lambda+\left|f'(t)\right|\int_0^1\lambda d\lambda\right]dt \\
&= \frac{1}{b-a}\int_a^b |x-t|\left[\frac{\left|f'(x)\right|+\left|f'(t)\right|}{2}\right]dt := M(x) \\
&\leq \frac{1}{2}\frac{1}{b-a}ess.\sup_{t\in[a,b]}\left[\left|f'(x)\right|+\left|f'(t)\right|\right]\int_a^b|x-t|\,dt \\
&= \frac{1}{2}\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[\left|f'(x)\right|+\left\|f'\right\|_\infty\right] \\
&= \frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[\left|f'(x)\right|+\left\|f'\right\|_\infty\right],
\end{aligned}
$$

and the inequality (2.3) is proved.
Assume that (2.3) holds with a constant $C>0$, that is,

$$
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq C\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[\left|f'(x)\right|+\left\|f'\right\|_\infty\right] \tag{2.4}
$$

for any $x\in[a,b]$ with f as in the hypothesis of the theorem.
Consider the function

$$
f_0:[a,b]\to\mathbb{R},\ f_0(t)=k\left|t-\frac{a+b}{2}\right|,\ k>0,t\in[a,b].
$$

Since $|f_0'(t)|=k$, for any $t\in[a,b]$ and

$$
\frac{1}{b-a}\int_a^b f_0(t)dt=\frac{k}{4}(b-a),\ \left\|f_0'\right\|_\infty=k
$$

then choosing $f=f_0$ and $x=\frac{a+b}{2}$ in (2.4), we get

$$
\frac{k}{4}(b-a)\leq\frac{Ck(b-a)}{2}
$$

giving $C\geq\frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. ∎

The following particular case is interesting.

Corollary 1. *With the assumptions of Theorem 3, we have the inequality*

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{1}{8}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left\|f'\right\|_\infty\right] \tag{2.5}
$$

and the constant $\frac{1}{8}$ *is the best possible.*

The following result in terms of the p-norms also holds:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be as in Theorem 3. If $f'\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p \tag{2.6}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. According to the proof of Theorem 2, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{1}{b-a}\int_a^b|x-t|\left[\frac{|f'(x)|+|f'(t)|}{2}\right]dt:=M(x).$$

Using Hölder's integral inequality for $p>1, \frac{1}{p}+\frac{1}{q}=1$, we get that

$$\begin{aligned} M(x) &\le \frac{1}{2(b-a)}\left(\int_a^b|x-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b(|f'(x)|+|f'(t)|)^p\,dt\right)^{\frac{1}{p}} \\ &= \frac{1}{2(b-a)}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p \end{aligned}$$

and the inequality (2.6) is proved.
Reconsider the function utilised in Theorem 2,

$$f_0:[a,b]\to\mathbb{R},\ f_0(t)=k\left|t-\frac{a+b}{2}\right|,\ k>0,\ t\in[a,b]$$

which has $|f_0'(t)|\ (=k)$ convex in $[a,b]$. If we assume that (2.6) holds with a constant $D>0$ instead of $\frac{1}{2}$, so that

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{D}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p,$$

then taking $f=f_0$ over $x=\frac{a+b}{2}$, we get,

$$\frac{k}{4}(b-a)\le\frac{D}{(q+1)^{\frac{1}{q}}}\left(\frac{1}{2^q}\right)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}k(b-a)^{\frac{1}{p}},q>1,p>1,\frac{1}{p}+\frac{1}{q}=1$$

giving, on simplification,

$$D\ge\frac{1}{2}(q+1)^{\frac{1}{q}},q>1.$$

Taking the limit as $q\to\infty$ and since,

$$\lim_{q\to\infty}(q+1)^{\frac{1}{q}}=\exp\left\{\lim_{q\to\infty}\left[\frac{\ln(1+q)}{q}\right]\right\}=\exp 0=1,$$

we deduce that $D\ge\frac{1}{2}$, which proves the sharpness of the constant. ∎

A particular case is the following mid-point inequality:

Corollary 2. *With the assumptions of Theorem 3, we have,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{2.7}$$

$$\leq \frac{1}{4(q-a)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\left(\int_a^b\left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(t)|\right]^p dt\right)^{\frac{1}{p}} \quad \left(p>1,\frac{1}{p}+\frac{1}{q}=1\right)$$

The constant $\frac{1}{4}$ *is sharp in the previous sense.*

Finally, the case involving the 1-norm is embodied in the following theorem:

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be as in Theorem 2. If* $f'\in L_1[a,b]$*, then, for any* $x\in[a,b]$*,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)|+\|f'\|_1\right]. \tag{2.8}$$

Proof. We have, from the proof of Theorem 2, that

$$\begin{aligned} M(x) &\leq \sup_{t\in[a,b]}|x-t|\frac{1}{b-a}\int_a^b\left[\frac{|f'(x)|+|f'(t)|}{2}\right]dt \\ &= \frac{1}{2(b-a)}\max(x-a,b-x)\left[(b-a)|f'(x)|+\int_a^b|f'(t)|dt\right] \\ &= \frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)|+\|f'\|_1\right] \end{aligned}$$

and the inequality (2.8) is proved. ∎

In particular, we have the mid-point inequality:

Corollary 3. *Assume that* f *is as in Theorem 4. Then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{1}{4}\left[(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|+\int_a^b|f'(t)|dt\right]. \tag{2.9}$$

Another way to estimate the difference

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

is presented in the following theorem.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is convex on (a,b). Then, for any $x\in[a,b]$,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\left\{\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|f'(x)|(b-a) + \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p\right\}, \tag{2.10}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. With the notation of Theorem 2, we have,

$$\begin{aligned}
M(x) &= \frac{1}{2(b-a)}\left[|f'(x)|\int_a^b|x-t|\,dt+\int_a^b|x-t|\,|f'(t)|\,dt\right]\\
&= \frac{1}{2(b-a)}\left[|f'(x)|\frac{(x-a)^2+(b-x)^2}{2}+\int_a^b|x-t|\,|f'(t)|\,dt\right]\\
&= \frac{1}{2}\left[|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)+\frac{1}{b-a}\int_a^b|x-t|\,|f'(t)|\,dt\right].
\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b|x-t|\,|f'(t)|\,dt\\
\leq\; &\frac{1}{b-a}\left(\int_a^b|x-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b|f'(t)|^p\,dt\right)^{\frac{1}{p}}\\
=\; &\frac{1}{b-a}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|f'\|_p\\
=\; &\frac{1}{(q+1)\frac{1}{q}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p,
\end{aligned}$$

and the theorem is proved. ∎

The following particular corollary is of interest providing a bound for the mid-point.

Corollary 4. *Let f be as in the previous theorem. Then one has the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{4}\left\{\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p\right\} \tag{2.11}$$

3. Applications for Special Means

In the applications below we consider the following definitions of some special means:

– Arithmetic mean,

$$A = A(a,b) = \frac{a+b}{2};\ a,b > 0.$$

– Goemetric mean,

$$G = G\left(a,b\right) = \sqrt{ab};\ \ a,b > 0.$$

– Logarithmic mean,

$$L = L\left(a,b\right) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b > 0, \\ a, & a = b. \end{cases}$$

– p-Logarithmic mean,

$$L_p(a,b) = \begin{cases} a & \text{if } a = b \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases} \quad \text{for} \quad p \in \mathbb{R} \backslash \left\{0, -1\right\}.$$

– Identric mean,

$$I = I(a,b) = \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}.$$

The well known fact that $G < L < I < A$ will be used in the following.

1. Consider the function f with domain $[a,b] \subset (0,\infty)$, $f(x) = x^p$ and $p \in \mathbb{R}$, $p \geq 2$, which is absolutely continuous, and whose modulus of the first derivative is a convex function.

1.1 If we use this function in Corollary 1, we get that

$$\left|\left(\frac{a+b}{2}\right)^p - \frac{1}{b-a}\int_a^b t^p dt\right| \leq \frac{1}{8}(b-a)\left[\left|p\left(\frac{a+b}{2}\right)^{p-1}\right| + pb^{p-1}\right]$$

so that

$$\left|A^p(a,b) - L_p^p(a,b)\right| \leq \frac{p}{8}(b-a)[A^{p-1}(a,b) + b^{p-1}]$$

or equivalently

$$0 \leq L_p^p(a,b) - A^p(a,b) \leq \frac{p}{8}(b-a)[A^{p-1}(a,b) + b^{p-1}].$$

1.2 For the same function, we get from Corollary 3 that

$$\begin{aligned} \left|A^p(a,b) - L_p^p(a,b)\right| &\leq \frac{1}{4}\left[(b-a)\left|p\left(\frac{a+b}{2}\right)^{p-1}\right| + \int_a^b \left|pt^{p-1}\right| dt\right] \\ &= \frac{p}{4}\left[(b-a)A^{p-1}(a,b) + \frac{b^p - a^p}{p}\right] \\ &= \frac{p}{4}(b-a)\left[A^{p-1}(a,b) + L_{p-1}^{p-1}(a,b)\right]. \end{aligned}$$

That is,

$$0 \leq L_p^p(a,b) - A^p(a,b) \leq \frac{p}{4}(b-a)\left[A^{p-1}(a,b) + L_{p-1}^{p-1}(a,b)\right].$$

2. Now, consider the function f with domain $[a,b] \subset (0,\infty)$, $f(x) = \ln(x)$. The function is absolutely continuous, and the modulus of the first derivative is convex.
2.1 From Corollary 1, we obtain,

$$\begin{aligned} 0 &\leq \left|\ln\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \ln(t)dt\right| = \left|\ln\left(\frac{a+b}{2}\right) - \ln I(a,b)\right| \\ &= \ln\frac{A(a,b)}{I(a,b)} \\ &\leq \frac{1}{8}(b-a)\left[\left|\frac{2}{a+b}\right| + \frac{1}{a}\right] \\ &= \frac{1}{8}(b-a)\left[A^{-1}(a,b) + \frac{1}{a}\right] \end{aligned}$$

and so

$$0 \leq \ln\frac{A(a,b)}{I(a,b)} \leq \frac{b-a}{8}\left[A^{-1}(a,b) + a^{-1}\right]$$

or, equivalently,

$$1 \leq \frac{A(a,b)}{I(a,b)} \leq \exp\left[\frac{b-a}{8}\left[A^{-1}(a,b) + a^{-1}\right]\right].$$

2.2 From Corollary 3, we get that

$$\begin{aligned} 0 &\leq |\ln A(a,b) - \ln I(a,b)| = \ln\frac{A(a,b)}{I(a,b)} \\ &\leq \frac{1}{4}\left[(b-a)\left|\frac{2}{a+b}\right| + \int_a^b \left|\frac{1}{t}\right| dt\right]. \end{aligned}$$

That is,

$$0 \leq \ln\frac{A(a,b)}{I(a,b)} \leq \frac{b-a}{4}A^{-1}(a,b) + \frac{1}{4}\ln\frac{b}{a}$$

or, equivalently,

$$1 \leq \frac{A(a,b)}{I(a,b)} \leq \left(\frac{b}{a}\right)^{\frac{1}{4}} \exp\frac{b-a}{4}\left[A^{-1}(a,b)\right].$$

2.3 Taking $f(x) = \ln x$ in Corollary 4, gives

$$1 \leq \frac{A(a,b)}{I(a,b)} \leq \frac{b-a}{4}\left[\frac{A^{-1}(a,b)}{2} + \frac{1}{(q+1)^{\frac{1}{q}}} L_{-p}^{-1}(a,b)\right].$$

3. Now, consider the function $f(x) = \frac{1}{x}$ which has domain $[a,b] \subset (0,\infty)$. This function is absolutely continuous and the modulus of the first derivative is convex.
3.1 From Corollary 1, we have,

$$\left|\frac{2}{a+b} - L^{-1}(a,b)\right| \leq \frac{1}{8}(b-a)\left[\left|\frac{1}{\left(\frac{a+b}{2}\right)^2}\right| + \frac{1}{a^2}\right]$$

giving

$$\left|A^{-1}(a,b)-L^{-1}(a,b)\right|\leq\frac{1}{8}(b-a)\left[A^{-2}(a,b)+a^{-2}\right]$$

or equivalently

$$0\leq L^{-1}(a,b)-A^{-1}(a,b)\leq\frac{1}{8}(b-a)\left[A^{-2}(a,b)+a^{-2}\right],\quad(\text{since}\quad A(a,b)\geq L(a,b))$$

which may further be represented as

$$0\leq A(a,b)-L(a,b)\leq\frac{1}{8}(b-a)A(a,b)L(a,b)\left[A^{-2}(a,b)+a^{-2}\right].$$

3.2 From Corollary 3, we get,

$$\left|A^{-1}(a,b)-L^{-1}(a,b)\right|\leq\frac{1}{4}\left(b-a\right)\left[A^{-2}(a,b)+G^{-2}(a,b)\right]$$

or equivalently

$$0\leq L^{-1}(a,b)-A^{-1}(a,b)\leq\frac{1}{4}\left(b-a\right)\left[A^{-2}(a,b)+G^{-2}(a,b)\right]$$

or still further

$$0\leq A(a,b)-L(a,b)\leq\frac{1}{4}\left(b-a\right)A(a,b)L(a,b)\left[A^{-2}(a,b)+G^{-2}(a,b)\right].$$

3.3 Taking $f(x)=\frac{1}{x}$ in Corollary 4, produces

$$\left|A^{-1}(a,b)-L^{-1}(a,b)\right|\leq\frac{b-a}{4}\left[\frac{A^{-2}(a,b)}{2}+\frac{1}{(q+1)^{\frac{1}{q}}}L_{-2p}^{-2}(a,b)\right]$$

or

$$0\leq L^{-1}(a,b)-A^{-1}(a,b)\leq\frac{b-a}{4}\left[\frac{A^{-2}(a,b)}{2}+\frac{1}{(q+1)^{\frac{1}{q}}}L_{-2p}^{-2}(a,b)\right]$$

which may be further expressed as

$$0\leq A(a,b)-L(a,b)\leq\frac{b-a}{4}A(a,b)L(a,b)\left[\frac{A^{-2}(a,b)}{2}+\frac{1}{(q+1)^{\frac{1}{q}}}L_{-2p}^{-2}(a,b)\right].$$

4. Applications for f and $HH-$Divergence Measures in Information Theory

Assume that a set χ and the $\sigma-$finite measure $\mu:\chi\to\bar{\mathbb{R}}$ are given. Consider the set of all probability densities on μ to be

$$\Omega:=\left\{p|p:\chi\to\mathbb{R},\,p\left(x\right)\geq0,\,\int_{\chi}p\left(x\right)d\mu\left(x\right)=1\right\}. \tag{4.1}$$

The $f-$divergence on Ω is defined as follows

$$D_{f}\left(p,q\right):=\int_{\chi}p\left(x\right)f\left[\frac{q\left(x\right)}{p\left(x\right)}\right]d\mu\left(x\right),\quad p,q\in\Omega, \tag{4.2}$$

where f is convex on $(0,\infty)$. It is also assumed that $f\left(u\right)$ is zero and strictly convex at $u=1$.

By appropriately defining this convex function, various divergences such as the Kullback-Leibler divergence D_{KL}, variation distance D_v, Hellinger distance D_H,

χ^2-divergence D_{χ^2}, Jeffrey's distance D_J, triangular discrimination D_Δ, etc. may be obtained. They are defined as follows:

$$D_v(p,q) := \int_\chi |p(x) - q(x)|\, d\mu(x), \quad p,q \in \Omega; \tag{4.3}$$

$$D_H(p,q) := \int_\chi \left|\sqrt{p(x)} - \sqrt{q(x)}\right| d\mu(x), \quad p,q \in \Omega; \tag{4.4}$$

$$D_{\chi^2}(p,q) := \int_\chi p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \Omega; \tag{4.5}$$

$$D_J(p,q) := \int_\chi [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega; \tag{4.6}$$

$$D_\Delta(p,q) := \int_\chi \frac{[p(x) - q(x)]^2}{p(x) + q(x)} d\mu(x), \quad p,q \in \Omega. \tag{4.7}$$

In [6], Shioya and Da-te introduced the generalised Ling-Wong $f-$divergence $D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$ and the Hermite-Hadamard $(HH)-$divergence

$$D_{HH}^f(p,q) := \int_\chi \frac{p^2(x)}{q(x) - p(x)}\left(\int_1^{\frac{q(x)}{p(x)}} f(t)\, dt\right) d\mu(x), \quad p,q \in \Omega. \tag{4.8}$$

They proved, by the use of the Hermite-Hadamard inequality for convex functions,

$$D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \le D_{HH}^f(p,q) \le \frac{1}{2} D_f(p,q), \tag{4.9}$$

provided that f is convex and normalised, i.e., $f(1) = 0$.

We will illustrate the approach to developing bounds and expressions involving various divergence measures from the inequalities developed in Section 2.

We will use the inequality (2.5), namely

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \le \frac{1}{8}|b-a|\left[\left|f'\left(\frac{a+b}{2}\right)\right| + \|f'\|_\infty\right], \tag{4.10}$$

where $a, b \in \mathring{I}$, $a \neq b$ and $f : \mathring{I} \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function on the interior of I with $|f'| : \mathring{I} \to \mathbb{R}$ convex on $\mathring{I}$, to prove the following result.

Theorem 6. *Let r, R be such that $0 \le r \le 1 \le R \le \infty$ and $p, q \in \Omega$ with*

$$r \le \frac{q(x)}{p(x)} \le R, \quad \textit{for a.e. } x \in \chi. \tag{4.11}$$

If $f : [0,\infty) \to \mathbb{R}$ is differentiable on $(0,\infty)$ and $|f'|$ is convex on $[r,R]$ then,

$$\begin{aligned} &\left| D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) - D_{HH}^f(p,q)\right| \\ &\le \frac{1}{8}\left[\|f'\|_{[r,R],\infty} D_v(p,q) + D_{f^*}(p,q)\right], \end{aligned} \tag{4.12}$$

where $f^*(x) = |x-1|\left|f'\left(\frac{x+1}{2}\right)\right|$, $x \in [r,R]$ and $\|h\|_{[a,b],\infty} := ess \sup_{t\in[a,b]} |h(t)|$.

Proof. If in (4.10) we choose $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, then

$$\left| f\left(\frac{p(x)+q(x)}{2p(x)}\right) - \frac{p(x)}{q(x)-p(x)}\left(\int_1^{\frac{q(x)}{p(x)}} f(t)\,dt\right)\right| \leq \frac{1}{8}\cdot\frac{|q(x)-p(x)|}{p(x)}\left[\left|f'\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| + \|f'\|_{[r,R],\infty}\right]. \tag{4.13}$$

Multiplying (4.13) with $p(x) \geq 0$ and integrating on χ, we deduce the desired inequality (4.12). ∎

Another approach is embodied in the following theorem.

Theorem 7. *Let r, R be as in Theorem 6. If $f : [0,\infty) \to \mathbb{R}$ is twice differentiable on $(0,\infty)$ and $|f''|$ is convex on $[r,R]$, then,*

$$\left|D_f(p,q) - f(1) - D_{f^{\#}}(p,q)\right| \leq \frac{1}{8}\left[\|f'\|_{[r,R],\infty} D_{\chi^2}(p,q) + D_{f^{\dagger}}(p,q)\right], \tag{4.14}$$

where $f^{\#}(x) := (x-1)f'\left(\frac{1+x}{2}\right)$, and $f^{\dagger}(x) := (x-1)^2\left|f''\left(\frac{1+x}{2}\right)\right|$, $x \in [0,\infty)$.

Proof. Applying the inequality (4.10) for $a = 1$, $b = u$ and choosing instead of f, its derivative f', one may state the inequality

$$\left| f(u) - f(1) - (u-1)f'\left(\frac{u+1}{2}\right)\right| \leq \frac{1}{8}(u-1)^2\left[\left|f''\left(\frac{u+1}{2}\right)\right| + \|f'\|_{[r,R],\infty}\right].$$

If in this inequality we choose $u = \frac{q(x)}{p(x)}$, $x \in \chi$, then we get

$$\left| f\left(\frac{q(x)}{p(x)}\right) - f(1) - \left(\frac{q(x)}{p(x)} - 1\right)f'\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| \leq \frac{1}{8}\frac{(p(x)-q(x))^2}{p^2(x)}\left[\left|f''\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| + \|f'\|_{[r,R],\infty}\right]. \tag{4.15}$$

Multiplying (4.15) by $p(x) \geq 0$, $x \in \chi$ and then integrating on χ, we deduce the desired inequality (4.14). ∎

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