Superadditivity of Some Functionals Associated to Jensen's Inequality for Convex Functions on Linear Spaces with Applications

SUPERADDITIVITY OF SOME FUNCTIONALS ASSOCIATED TO JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS ON LINEAR SPACES WITH APPLICATIONS

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ABSTRACT. Some new results related to Jensen's celebrated inequality for convex functions defined on convex sets in linear spaces are given. Applications for norm inequalities in normed linear spaces and f-divergences in Information Theory are provided as well.

1. INTRODUCTION

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the generalised triangle inequality, the arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C, p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$, then

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{1.1}$$

is well known in the literature as *Jensen's inequality*.

We introduce the following notations (see also [16]):

$$\begin{aligned} F(C,\mathbb{R}) &: = \text{the linear space of all real functions on } C, \\ F^{+}(C,\mathbb{R}) &: = \{f \in F(C,\mathbb{R}) : f(x) > 0 \text{ for all } x \in C\}, \\ P_f(\mathbb{N}) &: = \{I \subset \mathbb{N}\text{: } I \text{ is finite}\}, \\ J(\mathbb{R}) &: = \left\{p = \{p_i\}_{i\in\mathbb{N}}, p_i \in \mathbb{R} \text{ are such that } P_I \neq 0 \text{ for all } I \in P_f(\mathbb{N})\right\}, \end{aligned}$$

and

$$\begin{aligned} J^{+}(\mathbb{R}) &: = \{p \in J(\mathbb{R}) : p_i \geq 0 \text{ for all } i \in \mathbb{N}\}, \\ J_*(C) &: = \left\{x = \{x_i\}_{i\in\mathbb{N}} : x_i \in C \text{ for all } i \in \mathbb{N}\right\} \end{aligned}$$

and

$$Conv(C,\mathbb{R}) := \text{ the cone of all convex functions defined on } C,$$

respectively.

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In [16] the authors considered the following functional associated with the Jensen inequality:

$$J(f,I,p,x) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \tag{1.2}$$

where $f \in F(C,\mathbb{R}), I \in P_f(\mathbb{N}), p \in J^+(\mathbb{R}), x \in J_*(C)$. They established some quasi-linearity and monotonicity properties and applied the obtained results for norm and means inequalities.

The following result concerning the properties of the functional $J(f,I,\cdot,x)$ as a *function of weights* holds (see [16, Theorem 2.4]):

Theorem 1. *Let $f \in Conv(C,\mathbb{R}), I \in P_f(\mathbb{N})$ and $x \in J_*(C)$.*

(i) If $p,q \in J^+(\mathbb{R})$ then

$$J(f,I,p+q,x) \ge J(f,I,p,x) + J(f,I,q,x)\ (\ge 0) \tag{1.3}$$

i.e., $J(f,I,\cdot,x)$ is superadditive on $J^+(\mathbb{R})$;

(ii) If $p,q \in J^+(\mathbb{R})$ with $p \ge q$, meaning that $p_i \ge q_i$ for each $i \in \mathbb{N}$, then

$$J(f,I,p,x) \ge J(f,I,q,x)\ (\ge 0) \tag{1.4}$$

i.e., $J(f,I,\cdot,x)$ is monotonic nondecreasing on $J^+(\mathbb{R})$.

The behavior of this functional as an *index set function* is incorporated in the following (see [16, Theorem 2.1]):

Theorem 2. *Let $f \in Conv(C,\mathbb{R}), p \in J^+(\mathbb{R})$ and $x \in J_*(C)$.*

(i) If $I,H \in P_f(\mathbb{N})$ with $I \cap H = \varnothing$, then

$$J(f,I\cup H,p,x) \ge J(f,I,p,x) + J(f,H,p,x)\ (\ge 0) \tag{1.5}$$

i.e., $J(f,\cdot,p,x)$ is superadditive as an index set function on $P_f(\mathbb{N})$;

(ii) If $I,H \in P_f(\mathbb{N})$ with $H \subset I$, then

$$J(f,I,p,x) \ge J(f,H,p,x)\ (\ge 0) \tag{1.6}$$

i.e., $J(f,\cdot,p,x)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

As pointed out in [16], the above Theorem 2 is a generalisation of the Vasić-Mijalković result for convex functions of a real variable obtained in [26] and therefore creates the possibility to obtain vectorial inequalities as well.

For applications of the above results to logarithmic convex functions, to norm inequalities, in relation with the arithmetic mean-geometric mean inequality and with other classical results, see [16].

Motivated by the above results, we introduce in the present paper a more general functional, establish its main properties and use it for some particular cases that provide inequalities of interest. Applications for norm inequalities in normed linear spaces and f-divergences in Information Theory are provided as well.

2. Some Superadditivity Properties for the Weights

We consider the more general functional

$$D(f,I,p,x;\Phi) := P_I \Phi\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right], \tag{2.1}$$

where $f \in Conv(C,\mathbb{R}), I \in P_f(\mathbb{N}), p \in J^+(\mathbb{R}), x \in J_*(C)$ and $\Phi : [0,\infty) \to \mathbb{R}$ is a function whose properties will determine the behavior of the functional D as follows. Obviously, for $\Phi(t) = t$ we recapture from D the functional J considered in [16].

First of all we observe that, by Jensen's inequality, the functional D is well defined and *positive homogeneous* in the third variable, i.e.,

$$D(f, I, \alpha p, x; \Phi) = \alpha D(f, I, p, x; \Phi),$$

for any $\alpha > 0$ and $p \in J^+(\mathbb{R})$.

The following result concerning the superadditivity and monotonicity of the functional D as a function of weights holds:

Theorem 3. *Let $f \in Conv(C,\mathbb{R}), I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0,\infty) \to \mathbb{R}$ is monotonic nondecreasing and concave where is defined.*

(i) If $p, q \in J^+(\mathbb{R})$ then

$$D(f, I, p+q, x; \Phi) \geq D(f, I, p, x; \Phi) + D(f, I, q, x; \Phi) \tag{2.2}$$

i.e., D is superadditive as a function of weights;

(ii) If $p, q \in J^+(\mathbb{R})$ with $p \geq q$, meaning that $p_i \geq q_i$ for each $i \in \mathbb{N}$ and $\Phi : [0,\infty) \to [0,\infty)$ then

$$D(f, I, p, x; \Phi) \geq D(f, I, q, x; \Phi) \, (\geq 0) \tag{2.3}$$

i.e., D is monotonic nondecreasing as a function of weights.

Proof. (i). Let $p, q \in J^+(\mathbb{R})$. By the convexity of the function f on C we have

$$\begin{aligned}
&\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) f(x_i) - f\left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \\
&= \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) \right)}{P_I + Q_I} \\
&- f\left(\frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right)}{P_I + Q_I} \right) \\
&\geq \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) \right)}{P_I + Q_I} \\
&- \frac{P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right)}{P_I + Q_I} \\
&= \frac{P_I \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{P_I + Q_I} \\
&+ \frac{Q_I \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]}{P_I + Q_I}.
\end{aligned} \tag{2.4}$$

Since Φ is monotonic nondecreasing and concave, then by (2.4) we have

$$\begin{aligned} & \Phi\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right] \\ \geq \quad & \frac{P_I\Phi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I} \\ & +\frac{Q_I\Phi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}, \end{aligned}$$

which, by multiplication with $P_I+Q_I>0$ produces the desired result (2.2).

(ii). If $p\geq q$, then by (i) we have

$$\begin{aligned} D(f,I,p,x;\Phi) &= D(f,I,(p-q)+q,x;\Phi) \\ &\geq D(f,I,p-q,x;\Phi)+D(f,I,p,x;\Phi) \\ &\geq D(f,I,p,x;\Phi) \end{aligned}$$

since $D(f,I,p-q,x;\Phi)\geq 0$. □

Corollary 1. *Let $f\in Conv(C,\mathbb{R}), I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined.*

If there exists the numbers $M\geq m\geq 0$ such that $Mq\geq p\geq mq$, then we have

$$\begin{aligned} & MQ_I\Phi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right] \\ & \geq P_I\Phi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right] \\ & \geq mQ_I\Phi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]. \end{aligned} \tag{2.5}$$

In particular

$$\begin{aligned} & \frac{M}{m}\Phi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right] \\ & \geq \Phi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right] \\ & \geq \frac{m}{M}\Phi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]. \end{aligned} \tag{2.6}$$

Now, if we denote by

$$S(\mathbf{1}):=\left\{p\in J^+(\mathbb{R}):p_i\leq 1 \text{ for all } i\in\mathbb{N}\right\},$$

then we can state the following result as well:

Corollary 2. *Let $f\in Conv(C,\mathbb{R}), I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined.*

Then we have the bound

$$\sup_{p\in S(\mathbf{1})}\left\{P_I\Phi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]\right\} = card(I)\Phi\left[\frac{1}{card(I)}\sum_{i\in I}f(x_i)-f\left(\frac{1}{card(I)}\sum_{i\in I}x_i\right)\right], \tag{2.7}$$

where $card(I)$ denotes the cardinal of the finite set I.

Remark 1. *If we consider the concave and monotonic increasing function $\Phi(t)=\ln t$ and assume that $f\in Conv(C,\mathbb{R})$ and $x\in J_*(C)$ are selected such that $\frac{1}{P_I}\sum_{i\in I}p_if(x_i)>f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)$ for any $I\in P_f(\mathbb{N})$ with $card(I)\geq 2$ and $p\in J^+(\mathbb{R})$ (notice that is enough to assume that f is strictly convex and x is not constant) then by the superadditivity of the functional*

$$\begin{aligned} D(f,I,p,x;\ln) \quad : \quad &= P_I\ln\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]\\ &= \ln K(f,I,p,x)\end{aligned}$$

where

$$K(f,I,p,x):=\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^{P_I} \tag{2.8}$$

we deduce that $K(f,I,\cdot,x)$ is supermultiplicative, i.e., it satisfies the property

$$K(f,I,p+q,x)\geq K(f,I,p,x)\,K(f,I,q,x) \tag{2.9}$$

for any $p,q\in J^+(\mathbb{R})$.

The proof is obvious by the monotonicity and the positive homogeneity of the functional $D(f,I,\cdot,x;\ln)$.

Notice that the inequality (2.9) has been obtain in a different way by Agarwal & Dragomir in [1].

Another important example of concave and monotonic increasing function is $\Phi(t)=t^s$ with $s\in(0,1]$. In this situation the functional

$$D_s(f,I,p,x):=\left[P_I^{s-1}\sum_{i\in I}p_if(x_i)-P_I^sf\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^s\geq 0 \tag{2.10}$$

is superadditive and monotonic nondecreasing as a function of the weights p.

It might be useful for applications to observe that the superadditivity property is translated into the following version of the Jensen's inequality

$$\begin{aligned}&\left[(P_I+Q_I)^{s-1}\sum_{i\in I}(p_i+q_i)f(x_i)-(P_I+Q_I)^sf\left(\frac{\sum_{i\in I}(p_i+q_i)x_i}{P_I+Q_I}\right)\right]^s\\ &\geq\left[P_I^{s-1}\sum_{i\in I}p_if(x_i)-P_I^sf\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^s\\ &\quad+\left[Q_I^{s-1}\sum_{i\in I}q_if(x_i)-Q_I^sf\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]^s(\geq 0),\end{aligned} \tag{2.11}$$

where $p,q\in J^{+}(\mathbb{R})$.

The property of monotonicity provides the following double inequality for $p,q\in J^{+}(\mathbb{R})$ *such that* $Mq\geq p\geq mq$ *and* $M\geq m\geq 0$:

$$
\begin{aligned}
(2.12)\qquad & M\left[Q_I^{s-1}\sum_{i\in I}q_if(x_i)-Q_I^sf\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]^s \\
&\geq\left[P_I^{s-1}\sum_{i\in I}p_if(x_i)-P_I^sf\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^s \\
&\geq m\left[Q_I^{s-1}\sum_{i\in I}q_if(x_i)-Q_I^sf\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]^s.
\end{aligned}
$$

This inequality has the following equivalent form

$$
\begin{aligned}
(2.13)\qquad & M^{1/s}\left[Q_I^{s-1}\sum_{i\in I}q_if(x_i)-Q_I^sf\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right] \\
&\geq P_I^{s-1}\sum_{i\in I}p_if(x_i)-P_I^sf\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right) \\
&\geq m^{1/s}\left[Q_I^{s-1}\sum_{i\in I}q_if(x_i)-Q_I^sf\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right].
\end{aligned}
$$

Finally, from the Corollary 2 we also have the bound

$$
\begin{aligned}
(2.14)\qquad & \sup_{p\in S(\mathbf{1})}\left\{P_I^{s-1}\sum_{i\in I}p_if(x_i)-P_I^sf\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right\} \\
&=[card(I)]^{s-1}\sum_{i\in I}f(x_i)-[card(I)]^sf\left(\frac{1}{card(I)}\sum_{i\in I}x_i\right).
\end{aligned}
$$

For a function $\Psi:(0,\infty)\to(0,\infty)$ we consider now the functional

$$
\begin{aligned}
(2.15)\qquad & D(f,I,p,x;\Phi,\Psi) \\
:= & \sum_{i\in I}\Psi(p_i)\,\Phi\left[\frac{1}{\sum_{i\in I}\Psi(p_i)}\sum_{i\in I}\Psi(p_i)f(x_i)-f\left(\frac{1}{\sum_{i\in I}\Psi(p_i)}\sum_{i\in I}\Psi(p_i)x_i\right)\right]
\end{aligned}
$$

where $f\in Conv(C,\mathbb{R}),I\in P_f(\mathbb{N}),p\in J^{+}(\mathbb{R}),x\in J_*(C)$. Now, if we denote by $\Psi(p)$ the sequence $\{\Psi(p_i)\}_{i\in\mathbb{N}}$, then we observe that

$$D(f,I,p,x;\Phi,\Psi)=D(f,I,\Psi(p),x;\Phi).$$

The following result may be stated:

Corollary 3. *Let* $f\in Conv(C,\mathbb{R}),I\in P_f(\mathbb{N})$ *and* $x\in J_*(C)$. *Assume that* $\Phi:[0,\infty)\to[0,\infty)$ *is monotonic nondecreasing and concave. If* $\Psi:(0,\infty)\to(0,\infty)$ *is concave, then* $D(f,I,\cdot,x;\Phi,\Psi)$ *is also concave on* $J^{+}(\mathbb{R})$.

Proof. Utilising the properties of monotonicity, superadditivity and positive homogeneity of the functional $D(f,I,\cdot,x;\Phi)$ we have successively

$$\begin{aligned} D(f,I,tp+(1-t)q,x;\Phi,\Psi) &= D(f,I,\Psi(tp+(1-t)q),x;\Phi) \\ &\geq D(f,I,t\Psi(p)+(1-t)\Psi(q),x;\Phi) \\ &\geq D(f,I,t\Psi(p),x;\Phi)+D(f,I,(1-t)\Psi(q),x;\Phi) \\ &= tD(f,I,\Psi(p),x;\Phi)+(1-t)D(f,I,\Psi(q),x;\Phi) \\ &= tD(f,I,p,x;\Phi,\Psi)+(1-t)D(f,I,p,x;\Phi,\Psi) \end{aligned}$$

for any $p,q\in J^{+}(\mathbb{R})$ and $t\in[0,1]$, which proves the statement. □

3. Some Superadditivity Properties for the Index

The following result concerning the superadditivity and monotonicity of the functional D as an index set function holds:

Theorem 4. *Let $f\in Conv(C,\mathbb{R}), p\in J^{+}(\mathbb{R})$ and $x\in J_{*}(C)$. Assume that $\Phi:[0,\infty)\to\mathbb{R}$ is monotonic nondecreasing and concave where is defined.*

(i) If $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$, then

$$D(f,I\cup H,p,x;\Phi)\geq D(f,I,p,x;\Phi)+D(f,H,p,x;\Phi) \tag{3.1}$$

i.e., $D(f,\cdot,p,x;\Phi)$ is superadditive as an index set function on $P_f(\mathbb{N})$;

(ii) If $I,H\in P_f(\mathbb{N})$ with $H\subset I$ and $\Phi:[0,\infty)\to[0,\infty)$, then

$$D(f,I,p,x;\Phi)\geq D(f,H,p,x;\Phi)\ (\geq 0) \tag{3.2}$$

i.e., $D(f,\cdot,p,x;\Phi)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

Proof. (i). Let $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$. By the convexity of the function f on C we have

$$\begin{aligned} &\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k f(x_k)-f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k x_k\right) \\ &=\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)\right)}{P_I+P_H} \\ &-f\left(\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j x_j\right)}{P_I+P_H}\right) \\ &\geq\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)\right)}{P_I+P_H} \\ &-\frac{P_I f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)+P_H f\left(\frac{1}{P_H}\sum_{j\in H}p_j x_j\right)}{P_I+P_H} \\ &=\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right]}{P_I+P_H} \\ &+\frac{P_H\left[\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_j x_j\right)\right]}{P_I+P_H}. \end{aligned} \tag{3.3}$$

Since Φ is monotonic nondecreasing and concave, then by (3.3) we have

$$\begin{aligned} & \Phi\left[\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k f(x_k)-f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k x_k\right)\right] \\ \geq\ & \frac{P_I\Phi\left[\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right]}{P_I+P_H} \\ & +\frac{P_H\Phi\left[\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_j x_j\right)\right]}{P_I+P_H}, \end{aligned}$$

which, by multiplication with $P_I+P_H>0$ produces the desired result (3.2).

(ii). If $I,H\in P_f(\mathbb{N})$ with $H\subset I$, then

$$\begin{aligned} D(f,I,p,x;\Phi) &= D(f,(I\setminus H)\cup H,p,x;\Phi) \\ &\geq D(f,I\setminus H,p,x;\Phi)+D(f,H,p,x;\Phi)\geq D(f,H,p,x;\Phi)\ (\geq 0) \end{aligned}$$

since $D(f,I\setminus H,p,x;\Phi)\geq 0$. □

For the special case $I=I_n:=\{1,...,n\}$ we write $D_n(f,p,x;\Phi)$ instead of $D(f,I_n,p,x;\Phi)$, i.e.,

$$D_n(f,p,x;\Phi)=P_n\Phi\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right)\right] \tag{3.4}$$

where $P_n=P_{I_n}=\sum_{i=1}^{n}p_i>0$.

The following particular case is of interest:

Corollary 4. *Let $f\in Conv(C,\mathbb{R}), p\in J^+(\mathbb{R})$ and $x\in J_*(C)$. Assume that $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined. Then*

$$\max_{I\sqsubseteq I_n}D(f,I,p,x;\Phi)=D_n(f,p,x;\Phi)\geq 0, \tag{3.5}$$

$$\begin{aligned} & D_n(f,p,x;\Phi) \\ & \geq \max_{1\leq i<j\leq n}\left\{(p_i+p_j)\Phi\left[\frac{p_i f(x_i)+p_j f(x_j)}{p_i+p_j}-f\left(\frac{p_i x_i+p_j x_j}{p_i+p_j}\right)\right]\right\}\geq 0 \end{aligned} \tag{3.6}$$

and

$$D_n(f,p,x;\Phi)\geq D_{n-1}(f,p,x;\Phi)\geq ...\geq D_2(f,p,x;\Phi)\geq 0. \tag{3.7}$$

The proof is obvious by the monotonicity property of the functional $D(f,\cdot,p,x;\Phi)$ as an index set function.

If we use the superadditivity property, then we can state the following result as well:

Corollary 5. *Let $f \in Conv(C,\mathbb{R}), p \in J^{+}(\mathbb{R})$ and $x \in J_{*}(C)$. Assume that $\Phi : [0,\infty) \to \mathbb{R}$ is monotonic nondecreasing and concave where is defined. Then*

$$
\begin{aligned}
(3.8)\quad & P_{2n}\Phi\left[\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_i f(x_i) - f\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_i x_i\right)\right] \\
&\geq \sum_{i=1}^{n}p_{2i}\Phi\left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right] \\
&+\sum_{i=1}^{n}p_{2i-1}\Phi\left[\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}x_{2i-1}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\quad & P_{2n+1}\Phi\left[\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_i f(x_i) - f\left(\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_i x_i\right)\right] \\
&\geq \sum_{i=1}^{n}p_{2i}\Phi\left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right] \\
&+\sum_{i=1}^{n}p_{2i+1}\Phi\left[\frac{1}{\sum_{i=1}^{n}p_{2i+1}}\sum_{i=1}^{n}p_{2i+1}f(x_{2i+1}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i+1}}\sum_{i=1}^{n}p_{2i+1}x_{2i+1}\right)\right].
\end{aligned}
$$

Remark 2. *If we consider the functional defined in (2.7), namely*

$$K(f,I,p,x) := \left[\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right]^{P_I}$$

then by Theorem 4 we have that

$$(3.10)\qquad K(f,I\cup H,p,x) \geq K(f,I,p,x)\cdot K(f,H,p,x)$$

for any $I,H \in P_f(\mathbb{N})$ with $I\cap H = \varnothing$ meaning that the functional $K(f,\cdot,p,x)$ is supermultiplicative as an index set mapping.

This fact obviously imply the following multiplicative inequalities of interest:

$$
\begin{aligned}
(3.11)\quad & \left[\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_i f(x_i) - f\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_i x_i\right)\right]^{P_{2n}} \\
&\geq \left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right]^{\sum_{i=1}^{n}p_{2i}} \\
&\times\left[\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}x_{2i-1}\right)\right]^{\sum_{i=1}^{n}p_{2i-1}}
\end{aligned}
$$

and where $f \in Conv(C,\mathbb{R}), p \in J^{+}(\mathbb{R})$ and $x \in J_{*}(C)$.

Moreover, if we consider the functional defined in (2.10) by

$$D_s(f,I,p,x) := \left[P_I^{s-1}\sum_{i\in I}p_i f(x_i) - P_I^{s} f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right]^{s} \geq 0$$

where $s \in (0,1]$ *and introduce the associated functional*

$$F_s(f,I,p,x) := P_I^{s-1} \sum_{i\in I} p_i f(x_i) - P_I^s f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right), \tag{3.12}$$

then by denoting

$$F_{s,n}(f,p,x) := F_s(f,I_n,p,x) = P_n^{s-1} \sum_{i=1}^{n} p_i f(x_i) - P_n^s f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \tag{3.13}$$

where $I_n = \{1,...,n\}$ *, we have that the sequence* $\{F_{s,n}(f,p,x)\}_{n\geq 2}$ *is nondecreasing and the following bounds are valid*

$$\max_{I\sqsubseteq I_n} F_s(f,I,p,x) = F_{s,n}(f,p,x) \tag{3.14}$$

and

$$\begin{aligned} F_{s,n}(f,p,x) &\\ \geq \max_{1\leq i<j\leq n} &\left\{\frac{p_i f(x_i) + p_j f(x_j)}{(p_i+p_j)^{1-s}} - (p_i+p_j)^s f\left(\frac{p_i x_i + p_j x_j}{p_i+p_j}\right)\right\} \geq 0. \end{aligned} \tag{3.15}$$

4. Applications for Norm Inequalities

Let $(X, \|\cdot\|)$ be a real or complex normed linear space. It is well known that the function $f_p : X \to \mathbb{R}$, $f_p(x) = \|x\|^p, p \geq 1$ is convex on X. Assume that $p = (p_1,...,p_n)$ and $q = (q_1,...,q_n)$ are probability distributions with all q_j nonzero. In [11] we obtained the following refinements of the generalised triangle inequality:

$$\begin{aligned} \max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j\|x_j\|^p - \left\|\sum_{j=1}^{n} q_j x_j\right\|^p\right] &\geq \sum_{j=1}^{n} p_j\|x_j\|^p - \left\|\sum_{j=1}^{n} p_j x_j\right\|^p \\ &\geq \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j\|x_j\|^p - \left\|\sum_{j=1}^{n} q_j x_j\right\|^p\right] \quad (\geq 0) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \max_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\left\|\sum_{j=1}^{n} x_j\right\|^p\right] &\geq \sum_{j=1}^{n} p_j\|x_j\|^p - \left\|\sum_{j=1}^{n} p_j x_j\right\|^p \\ &\geq \min_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\left\|\sum_{j=1}^{n} x_j\right\|^p\right] \quad (\geq 0) \end{aligned} \tag{4.2}$$

for all $p \geq 1$.

We remark that, for $p = 1$ one may get out of the previous results the following inequalities that are intimately related with the generalised triangle inequality in normed spaces:

$$\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n}q_j\left\|x_j\right\|-\left\|\sum_{j=1}^{n}q_jx_j\right\|\right]\geq\sum_{j=1}^{n}p_j\left\|x_j\right\|-\left\|\sum_{j=1}^{n}p_jx_j\right\|$$
$$\geq\min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n}q_j\left\|x_j\right\|-\left\|\sum_{j=1}^{n}q_jx_j\right\|\right]\quad(\geq 0), \tag{4.3}$$

$$\max_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\left\|x_j\right\|-\left\|\sum_{j=1}^{n}x_j\right\|\right]\geq\sum_{j=1}^{n}p_j\left\|x_j\right\|-\left\|\sum_{j=1}^{n}p_jx_j\right\|$$
$$\geq\min_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\left\|x_j\right\|-\left\|\sum_{j=1}^{n}x_j\right\|\right]\quad(\geq 0). \tag{4.4}$$

If in (4.4) we take

$$p_j:=\frac{1}{\left\|x_j\right\|}/\sum_{k=1}^{n}\frac{1}{\left\|x_k\right\|}\text{ with }x_j\neq 0\text{ for all }j\in\{1,...,n\},$$

then, by rearranging the inequality, we get the result:

$$\max_{1\leq j\leq n}\{\left\|x_j\right\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\left\|x_j\right\|}\right\|\right]\geq\sum_{j=1}^{n}\left\|x_j\right\|-\left\|\sum_{j=1}^{n}x_j\right\|$$
$$\geq\min_{1\leq j\leq n}\{\left\|x_j\right\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\left\|x_j\right\|}\right\|\right]. \tag{4.5}$$

We note that the inequality (4.5) has been obtained in a different way by M. Kato. K.-S. Saito & T. Tamura in [17] where an analysis of the equality case for strictly convex spaces has been performed as well.

We can state the following result that provides a generalization of the inequality (4.1) above:

Proposition 1. *Let $(X,\|\cdot\|)$ be a normed linear space, $x=(x_1,...,x_n)$ an n-tuple of vectors in X, $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ are probability distributions with all q_j nonzero. If $t\geq 1$ and $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\Phi\left[\sum_{i=1}^{n}q_i\left\|x_i\right\|^t-\left\|\sum_{i=1}^{n}q_ix_i\right\|^t\right]$$
$$\geq\Phi\left[\sum_{i=1}^{n}p_i\left\|x_i\right\|^t-\left\|\sum_{i=1}^{n}p_ix_i\right\|^t\right]$$
$$\geq\min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\Phi\left[\sum_{i=1}^{n}q_i\left\|x_i\right\|^t-\left\|\sum_{i=1}^{n}q_ix_i\right\|^t\right] \tag{4.6}$$

and, in particular,

$$n \max_{1\leq i\leq n}\{p_i\}\Phi\left[n^{-1}\sum_{i=1}^{n}\|x_i\|^t - n^{-t}\left\|\sum_{i=1}^{n}x_i\right\|^t\right] \geq \Phi\left[\sum_{i=1}^{n}p_i\|x_i\|^t - \left\|\sum_{i=1}^{n}q_i x_i\right\|^t\right] \geq n\min_{1\leq i\leq n}\{p_i\}\Phi\left[n^{-1}\sum_{i=1}^{n}\|x_i\|^t - n^{-t}\left\|\sum_{i=1}^{n}x_i\right\|^t\right]. \tag{4.7}$$

The proof follows from Corollary 1 and the details are omitted.

Now, if $p = (p_1, ..., p_n)$ are positive weights with $P_n = \sum_{i=1}^{n} p_i > 0$ and $x = (x_1, ..., x_n)$ is an n-tuple of vectors in X, then by defining the functional

$$D_n(t, \|\cdot\|, p, x; \Phi) = P_n\Phi\left[P_n^{-1}\sum_{i=1}^{n}p_i\|x_i\|^t - P_n^{-t}\left\|\sum_{i=1}^{n}p_i x_i\right\|^t\right] \tag{4.8}$$

we can state the following result as well:

Proposition 2. *If $t \geq 1$ and $\Phi : [0,\infty) \to [0,\infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$D_n(t, \|\cdot\|, p, x; \Phi) \geq \max_{1\leq i<j\leq n}\left\{(p_i+p_j)\Phi\left[\frac{p_i\|x_i\|^t + p_j\|x_j\|^t}{p_i+p_j} - \left\|\frac{p_i x_i + p_j x_j}{p_i+p_j}\right\|^t\right]\right\} \geq 0 \tag{4.9}$$

and

$$D_n(t, \|\cdot\|, p, x; \Phi) \geq D_{n-1}(t, \|\cdot\|, p, x; \Phi) \geq ... \geq D_2(t, \|\cdot\|, p, x; \Phi) \geq 0. \tag{4.10}$$

The proof follows from Corollary 4 and the details are omitted.

5. Applications for f-Divergences

Given a convex function $f : [0,\infty) \to \mathbb{R}$, the $f-$*divergence functional*

$$I_f(p,q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{5.1}$$

was introduced by Csiszár [3]-[4] as a generalized measure of information, a *"distance function"* on the set of probability distribution $\mathbb{P}^n$. The restriction here to discrete distributions is only for convenience, similar results hold for general distributions. As in Csiszár [3]-[4] , we interpret undefined expressions by

$$f(0) = \lim_{t\to 0+} f(t), \quad 0f\left(\tfrac{0}{0}\right) = 0,$$
$$0f\left(\tfrac{a}{0}\right) = \lim_{\varepsilon\to 0+}\varepsilon f\left(\tfrac{a}{\varepsilon}\right) = a\lim_{t\to\infty}\tfrac{f(t)}{t}, \ a > 0.$$

The following results were essentially given by Csiszár and Körner [5].

Proposition 3. *(Joint Convexity) If $f : [0,\infty) \to \mathbb{R}$ is convex, then $I_f(p,q)$ is jointly convex in p and q.*

Proposition 4. *(Jensen's inequality) Let $f:[0,\infty)\to\mathbb{R}$ be convex. Then for any $p,q\in[0,\infty)^n$ with $P_n:=\sum_{i=1}^{n}p_i>0$, $Q_n:=\sum_{i=1}^{n}q_i>0$, we have the inequality*

$$I_f(p,q)\geq Q_n f\left(\frac{P_n}{Q_n}\right). \tag{5.2}$$

If f is strictly convex, equality holds in (5.2) iff

$$\frac{p_1}{q_1}=\frac{p_2}{q_2}=...=\frac{p_n}{q_n}.$$

It is natural to consider the following corollary.

Corollary 6. *(Nonnegativity) Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalized, i.e.,*

$$f(1)=0. \tag{5.3}$$

Then for any $p,q\in[0,\infty)^n$ with $P_n=Q_n$, we have the inequality

$$I_f(p,q)\geq 0. \tag{5.4}$$

If f is strictly convex, equality holds in (5.4) iff

$$p_i=q_i \ \text{ for all } \ i\in\{1,...,n\}.$$

In particular, if p,q are probability vectors, then Corollary 6 shows that, for strictly convex and normalized $f:[0,\infty)\to\mathbb{R}$ that

$$I_f(p,q)\geq 0 \ \text{ and } I_f(p,q)=0 \ \text{ iff } p=q. \tag{5.5}$$

We now give some examples of divergence measures in Information Theory which are particular cases of $f-$divergences.

Kullback-Leibler distance ([20]). The *Kullback-Leibler distance* $D(\cdot,\cdot)$ is defined by

$$D(p,q):=\sum_{i=1}^{n}p_i\log\left(\frac{p_i}{q_i}\right).$$

If we choose $f(t)=t\ln t$, $t>0$, then obviously

$$I_f(p,q)=D(p,q).$$

Variational distance (l_1-distance). The *variational distance* $V(\cdot,\cdot)$ is defined by

$$V(p,q):=\sum_{i=1}^{n}|p_i-q_i|.$$

If we choose $f(t)=|t-1|$, $t\in[0,\infty)$, then we have

$$I_f(p,q)=V(p,q).$$

Hellinger discrimination ([2]). The *Hellinger discrimination* is defined by $\sqrt{2h^2(\cdot,\cdot)}$, where $h^2(\cdot,\cdot)$ is given by

$$h^2(p,q):=\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i}-\sqrt{q_i}\right)^2.$$

It is obvious that if $f(t)=\frac{1}{2}\left(\sqrt{t}-1\right)^2$, then

$$I_f(p,q)=h^2(p,q).$$

Triangular discrimination ([24]). We define *triangular discrimination* between p and q by

$$\Delta(p,q)=\sum_{i=1}^{n}\frac{|p_i-q_i|^2}{p_i+q_i}.$$

It is obvious that if $f(t)=\frac{(t-1)^2}{t+1}$, $t\in(0,\infty)$, then

$$I_f(p,q)=\Delta(p,q).$$

Note that $\sqrt{\Delta(p,q)}$ is known in the literature as the Le Cam distance.

χ^2-**distance.** We define the χ^2-*distance* (chi-square distance) by

$$D_{\chi^2}(p,q):=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}.$$

It is clear that if $f(t)=(t-1)^2$, $t\in[0,\infty)$, then

$$I_f(p,q)=D_{\chi^2}(p,q).$$

Rényi's divergences ([23]). For $\alpha\in\mathbb{R}\backslash\{0,1\}$, consider

$$\rho_\alpha(p,q):=\sum_{i=1}^{n}p_i^\alpha q_i^{1-\alpha}.$$

It is obvious that if $f(t)=t^\alpha$ $(t\in(0,\infty))$, then

$$I_f(p,q)=\rho_\alpha(p,q).$$

Rényi's divergences $R_\alpha(p,q):=\frac{1}{\alpha(\alpha-1)}\ln[\rho_\alpha(p,q)]$ have been introduced for all real orders $\alpha\neq0$, $\alpha\neq1$ (and continuously extended for $\alpha=0$ and $\alpha=1$) in [21], where the reader may find many inequalities valid for these divergences, without, as well as with, some restrictions for p and q.

For other examples of divergence measures, see the paper [18] and the books [21] and [25], where further references are given.

For a function $f:(0,\infty)\to\mathbb{R}$ we denote by $f^{\#}$ the function defined on $(0,\infty)$ by the equation $f^{\#}(x):=f\left(\frac{1}{x}\right)$. With this notation we have

$$I_{f^{\#}}(p,q)=\sum_{i=1}^{n}q_if^{\#}\left(\frac{p_i}{q_i}\right)=\sum_{i=1}^{n}q_if\left(\frac{q_i}{p_i}\right).\tag{5.6}$$

By the use of Corollary 1 we can state the following result for the f-divergences.

Proposition 5. *Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalized and p,q two probability distributions such that $R:=\max_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}<\infty$ and $r:=\min_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}>0$. If $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$\begin{aligned}R\Phi\left[I_{f^{\#}}(p,q)-f\left(D_{\chi^2}(q,p)+1\right)\right]&\geq\Phi\left[I_f(q,p)\right]\\&\geq r\Phi\left[I_{f^{\#}}(p,q)-f\left(D_{\chi^2}(q,p)+1\right)\right].\end{aligned}\tag{5.7}$$

Proof. Utilising the inequality (2.5) we have

$$R\Phi\left[\sum_{i=1}^{n} q_i f\left(\frac{q_i}{p_i}\right) - f\left(\sum_{i=1}^{n}\frac{q_i^2}{p_i}\right)\right] \geq \Phi\left[\sum_{i=1}^{n} p_i f\left(\frac{q_i}{p_i}\right) - f(1)\right] \geq r\Phi\left[\sum_{i=1}^{n} q_i f\left(\frac{q_i}{p_i}\right) - f\left(\sum_{i=1}^{n}\frac{q_i^2}{p_i}\right)\right]. \tag{5.8}$$

Since

$$\sum_{i=1}^{n}\frac{q_i^2}{p_i} = D_{\chi^2}(q,p) + 1,$$

then by (5.8) we deduce the desired result (5.7). □

Finally, by the use of Corollary 4 we also have the following lower bound for the f-divergence:

Proposition 6. *Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalized and p,q two probability distributions. If $\Phi:[0,\infty)\to[0,\infty)$ is monotonic nondecreasing and concave where is defined, then we have:*

$$\Phi\left[I_f(q,p)\right] \geq \max_{1\leq i<j\leq n}\left\{(p_i+p_j)\Phi\left[\frac{p_i f\left(\frac{q_i}{p_i}\right)+p_j f\left(\frac{q_j}{p_j}\right)}{p_i+p_j} - f\left(\frac{q_i+q_j}{p_i+p_j}\right)\right]\right\} \geq 0. \tag{5.9}$$

Remark 3. *If one chooses different examples of convex functions generating the particular divergences mentioned at the beginning of the section, that one can obtain various inequalities of interest. However the details are not presented here.*

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