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# KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

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ABSTRACT. We prove an equivalent relation between Ky Fan-typed inequalities and certain bounds for the differences of means. We also generalize a result of H. Alzer, S. Ruscheweyh and L. Salinas.

## 1. INTRODUCTION

Let  $P_{n,r}(\mathbf{x})$  be the generalized weighted power means:  $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^n \omega_i x_i^r)^{\frac{1}{r}}$ , where  $\omega_i > 0, 1 \leq i \leq n$  with  $\sum_{i=1}^n \omega_i = 1$  and  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . Here  $P_{n,0}(\mathbf{x})$  denotes the limit of  $P_{n,r}(\mathbf{x})$  as  $r \rightarrow 0^+$ . We shall write  $P_{n,r}$  for  $P_{n,r}(\mathbf{x})$  when there is no risk of confusion.

In this paper, we always assume  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . To any given  $\mathbf{x}$  we associate  $\mathbf{x}' = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$  and write  $A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1}$ . When  $1 - x_i \geq 0$  for all  $i$ , we define  $A'_n = P_{n,1}(\mathbf{x}')$  and similarly for  $G'_n, H'_n$ . We also let  $\sigma_n = \sum_{i=1}^n \omega_i [x_i - A_n]^2$ .

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published in the monograph *Inequalities* by Beckenbach and Bellman [7]:

**Theorem 1.1.** For  $x_i \in (0, 1/2]$ ,

$$(1.1) \quad \frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}$$

with equality holding if and only if  $x_1 = \dots = x_n$ .

In this paper we consider the validity of the following additive Ky Fan-typed inequalities (with  $x_1 < x_n < 1$ ):

$$(1.2) \quad \frac{x_1}{1 - x_1} < \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} < \frac{x_n}{1 - x_n}$$

Note by a change of variables,  $x_i \rightarrow 1 - x_i$ , the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). One can deduce (see [9]) theorem 1.1 from the case  $r = 1, s = 0, x_n \leq 1/2$  in (1.2), which is a result of H. Alzer [5]. P. Gao [9] later proved the validity of (1.2) for  $r = 1, -1 \leq s < 1, x_n \leq 1/2$ .

What's worth mentioning is a nice result of P. Mercer [12], who showed the validity of  $r = 1, s = 0$  in (1.2) is a consequence of a result of D.I. Cartwright and M.J. Field [8], who established the validity of  $r = 1, s = 0$  for the following bounds for the differences between power means ( $r > s$ ):

$$(1.3) \quad \frac{r - s}{2x_1} \sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r - s}{2x_n} \sigma_n$$

where the constant  $(r - s)/2$  is best possible (see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all  $r > s$ . We refer the reader to the survey article [2] and the references therein for an account of Ky Fan's inequality and to the articles [4], [5], [10], [11] for other interesting refinements and extensions of (1.3).

Mercer's result reveals a close relation between (1.3) and (1.2) and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed  $r$  and  $s$ . As a consequence of

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this result, we will give a characterization of the validity of (1.3) for  $r = 1$  or  $s = 1$ . A solution of an open problem from [11] is also given.

Among numerous sharpenings of Ky Fan's inequality in the literature, we have the following inequalities connecting the three classical means (with  $\omega_i = 1/n$  here):

$$(1.4) \quad \left(\frac{H_n}{H'_n}\right)^{n-1} \frac{A_n}{A'_n} \leq \left(\frac{G_n}{G'_n}\right)^n \leq \left(\frac{A_n}{A'_n}\right)^{n-1} \frac{H_n}{H'_n}$$

The right-hand side inequality of (1.4) is due to P.F.Wang and W.L.Wang[14] and the left-hand side inequality was proved recently by H. Alzer, S. Ruscheweyh and L. Salinas[6].

It is natural to ask whether one can extend the above inequality to the weighted case and by using the same idea as in [6], we will show this indeed is true in section 5.

## 2. THE MAIN THEOREM

**Theorem 2.1.** *For fixed  $r > s$ , the following inequalities are equivalent:*

(i). inequality (1.2) for  $x_n \leq 1/2$ ; (ii). inequality (1.2); (iii). inequality (1.3).

*Proof.* (iii)  $\Rightarrow$  (ii) follows from a similar argument as given in [12], (ii)  $\Rightarrow$  (i) is trivial, so it suffices to show (i)  $\Rightarrow$  (iii):

Fix  $r > s$ , assuming (1.2) holds for  $x_n \leq 1/2$ . Without loss of generality, we can assume  $x_1 < x_n$ . For a given  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ , let  $\mathbf{y} = (\epsilon x_1, \epsilon x_2, \dots, \epsilon x_n)$ . We can choose  $\epsilon$  small so that  $\epsilon x_n \leq 1/2$  and now apply the right-hand side inequality (1.2) for  $\mathbf{y}$ , we get

$$(2.1) \quad x_n(P_{n,r}(\mathbf{x}) - P_{n,s}(\mathbf{x})) > \frac{1 - \epsilon x_n}{\epsilon^2} (P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}'))$$

By letting  $\epsilon$  tend to 0, it is easy to verify the limit of the expression on the right-hand side of (2.1) is  $(r - s)\sigma_n/2$ . We can consider the left-hand side of (2.1) by a similar argument and this completes the proof.  $\square$

## 3. AN APPLICATION OF THEOREM 2.1

**Lemma 3.1.** *If inequality (1.3) holds for  $r > s$  then  $0 \leq r + s \leq 3$ .*

*Proof.* Let  $n = 2$ , write  $\omega_1 = 1 - q$ ,  $\omega_2 = q$ ,  $x_1 = 1$  and  $x_2 = 1 + t$  with  $t \geq -1$ . Let

$$D(t; r, s, q) = \frac{r - s}{2} \sum_{i=1}^2 w_i [x_i - A_2]^2 - P_{2,r} + P_{2,s}$$

For  $t \geq 0$  then  $D(t; r, s, q) \geq 0$  implies the validity of the left-hand side inequality of (1.3) while for  $-1 \leq t \leq 0$ ,  $D(t; r, s, q) \leq 0$  implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of  $D(t; r, s, q)$  around  $t = 0$ , it is readily seen that  $D(0; r, s, q) = D^{(1)}(0; r, s, q) = D^{(2)}(0; r, s, q) = 0$ . Thus by the Lagrangian remainder term of the Taylor expansion:

$$D(t; r, s, q) = \frac{D^{(3)}(\theta t; r, s, q)}{3!} t^3$$

with  $0 < \theta < 1$ .

Since

$$\lim_{t \rightarrow 0^+} D^{(3)}(\theta t; r, s, q) = D^{(3)}(0; r, s, q)$$

a necessary condition for (1.3) to hold is  $D^{(3)}(0; r, s, q) > 0$  for  $0 \leq q \leq 1$ . Calculation yields

$$D^{(3)}(0; r, s, q) = (r - s)q(q - 1)((3 - 2r - 2s)q - (3 - r - s))$$

It is easy to check that this is equivalent to  $0 \leq r + s \leq 3$ .  $\square$

**Theorem 3.1.** *If  $r = 1$ , inequality (1.3) holds if and only if  $-1 \leq s < 1$ . If  $s = 1$ , inequality (1.3) holds if and only if  $1 < r \leq 2$ .*

*Proof.* A result of P.Gao[9] shows the validity of (1.2) for  $r = 1, -1 \leq s < 1, x_n \leq 1/2$  and a similar result of him[10] shows the validity of (1.2) for  $s = 1, 1 < r \leq 2, x_n \leq 1/2$ . Thus it follows from theorem 2.1 that (1.3) holds for  $r = 1, -1 \leq s < 1$  and  $s = 1, 1 < r \leq 2$ . This proves the “if” part of the statement and the “only if” part follows from the previous lemma.  $\square$

We note here a special case of the above corollary answers an open problem of A. M. Mercer[11], namely, we have shown:

$$(3.1) \quad \frac{1}{x_1} \sigma_n \geq A_n - H_n \geq \frac{1}{x_n} \sigma_n$$

#### 4. TWO LEMMAS

**Lemma 4.1.** *Let  $x, b, u, v$  be real numbers with  $0 < x \leq b, u \geq 1, v \geq 1$ , then  $f(u, v, x, b) \leq 0$  where*

$$(4.1) \quad f(u, v, x, b) = \frac{u + v - 1}{ux + vb} + \frac{1}{x^2(u/x + v/b)} - \frac{1}{x} - \frac{u + v - 2}{b^2(u + v)^2} v(x - b)$$

*with equality holding if and only if  $x = b$  or  $u = v = 1$ .*

*Proof.* Let  $x < b$  and  $u > 1, v > 1$ . We have

$$\begin{aligned} f(u, v, x, b) &= v(b - x) \left( -\frac{(u - 1)b + (v - 1)x}{x(bv + ux)(bu + vx)} + \frac{(u - 1) + (v - 1)}{b^2(u + v)^2} \right) \\ &< \frac{v(b - x)}{xb^2(u + v)^2} [((u - 1) + (v - 1))x - (u - 1)b - (v - 1)x] \\ &= -\frac{v(u - 1)(b - x)^2}{xb^2(u + v)^2} < 0 \end{aligned}$$

Since  $b^2(u + v)^2 > (bv + ux)(bu + vx)$ . Thus we conclude that  $f(u, v, x, b) < 0$  for  $0 < x \leq b, u \geq 1, v \geq 1$ .  $\square$

**Lemma 4.2.** *Let  $x, a, b, u, v, s$  be real numbers with  $0 < x \leq a \leq b, u \geq 1, v \geq 1, u + v \geq 2$  and  $0 \leq s \leq v$ , then*

$$(4.2) \quad \begin{aligned} &\frac{u + v - 1}{ux + sa + (v - s)b} + \frac{1}{x^2(u/x + s/a + (v - s)/b)} - \frac{1}{x} \\ &- \frac{u + v - 2}{b^2(u + v)^2} (s(x - a) + (v - s)(x - b)) \leq 0 \end{aligned}$$

*with equality holding if and only if one of the following cases is true: 1.  $x = a = b$ ; 2.  $a = b, u = v = 1$ ; 3.  $s = 0, u = v = 1$ ; 4.  $s = 0, x = b$ ; 5.  $s = 0, x = a$ ; 6.  $s = u = v = 1$ .*

*Proof.* Let  $M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b\}$ . Furthermore, we define  $H(s, a)$  as the expression on the left-hand side of (4.2), where  $(s, a) \in M$ . It suffices to show  $H(s, a) < 0$ . We denote the absolute minimum of  $H$  by  $m = (s_0, a_0)$ . If  $m$  is an interior point of  $M$ , then we obtain

$$0 = \frac{1}{s} \frac{\partial H}{\partial a} - \frac{1}{a - b} \frac{\partial H}{\partial s} \Big|_{(s,a)=(s_0,a_0)} = \frac{a - b}{x^2 a^2 b (u/x + s/a + (v - s)/b)^2} < 0$$

Hence,  $m$  is a boundary point of  $M$ , so that we get

$$m \in \{(s_0, x), (s_0, b), (0, a_0), (v, a_0)\}$$

Using lemma 4.1 we obtain

$$H(s_0, x) = f(u + s_0, v - s_0, x, b) < 0$$

$$H(s_0, b) = H(0, a_0) = f(u, v, x, b) < 0$$

$$H(v, a_0) = f(u, v, x, a_0) < 0$$

Thus, we get: if  $(s, a) \in M$ , then  $H(s, a) \leq 0$ . The conditions for equality can be easily checked by using lemma 4.1. □

## 5. A SHARPENING OF KY FAN'S INEQUALITY

In this section we prove the following theorem:

**Theorem 5.1.** For  $0 < x_1 \leq \cdots \leq x_n$ ,  $q = \min\{\omega_i\}$

$$(5.1) \quad \frac{1-2q}{2x_1^2}\sigma_n \geq (1-q)\ln A_n + q\ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2}\sigma_n$$

$$(5.2) \quad \frac{1-2q}{2x_1^2}\sigma_n \geq \ln G_n - q\ln A_n - (1-q)\ln H_n \geq \frac{1-2q}{2x_n^2}\sigma_n$$

with equality holding if and only if  $q = 1/2$  or  $x_1 = \cdots = x_n$ .

*Proof.* The proof uses the ideas in [6]. We will prove the right-hand side inequality of (5.1) and the proofs for other inequalities are similar. Fix  $0 < x = x_1, x_n = b$  with  $x_1 < x_n$ , we define

$$f_n(\mathbf{x}_n, q) = (1-q)\ln A_n + q\ln H_n - \ln G_n - \frac{1-2q}{2x_n^2}\sigma_n$$

where we regard  $A_n, G_n, H_n$  as functions of  $\mathbf{x}_n = (x_1, \dots, x_n)$ .

We then have

$$g_n(x_2, \dots, x_{n-1}) := \frac{1}{\omega_1} \frac{\partial f_n}{\partial x_1} = \frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1} - \frac{1-2q}{x_n^2}(x_1 - A_n)$$

We want to show  $g_n \leq 0$ . Let  $D = \{(x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-2} | 0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b\}$ . Let  $\mathbf{a} = (a_2, \dots, a_{n-1}) \in D$  be the absolute minimum of  $g_n$ . Next, we show that

$$(5.3) \quad \mathbf{a} = (x, \dots, x, a, \dots, a, b, \dots, b) \text{ with } x < a < b$$

where the numbers  $x, a$ , and  $b$  appear  $r, s$ , and  $t$  times, respectively, with  $r, s, t \geq 0, r+s+t = n-2$ .

Suppose not, this implies two components of  $\mathbf{a}$  have different values and are interior points of  $D$ . We denote these values by  $a_k$  and  $a_l$ . Partial differentiation leads to

$$(5.4) \quad \frac{B}{a_i^2} + C = 0$$

for  $i = k, l$ , where

$$B = q \frac{H_n^2}{x_1^2}, C = -\frac{1-q}{A_n^2} + \frac{1-2q}{x_n^2}$$

Since  $z \mapsto B/z^2 + C$  is strictly monotonic for  $z > 0$ , (5.4) yields  $a_k = a_l$ . This contradicts our assumption that  $a_k \neq a_l$ . Thus (5.3) is valid and it suffices to show  $g_n \leq 0$  for the case  $n = 2, 3$ .

When  $n = 2$ , by setting  $x_1 = x, x_2 = b, \omega_1/q = u, \omega_2/q = v$ , we can identify  $g_2/q$  as (4.1) and the result follows from lemma 4.1.

When  $n = 3$ , by setting  $x_1 = x, x_2 = a, x_3 = b, \omega_1/q = u, \omega_2/q = s, \omega_3/q = v - s$ , we can identify  $g_3/q$  as (4.2) and the result follows from lemma 4.2.

Thus we have shown that  $g_n = \frac{\partial f_n}{\partial x_n} \leq 0$  with equality holding if and only if  $n = 1$  or  $n = 2, q = 1/2$ . By letting  $x_1$  tend to  $x_2$ , we have

$$f_n(\mathbf{x}_n, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q')$$

where  $\mathbf{x}_{n-1} = (x_2, \dots, x_n)$  with weights  $\omega_1 + \omega_2, \dots, \omega_{n-1}, \omega_n$  and  $q' = \min\{\omega_1 + \omega_2, \dots, \omega_n\}$ . Here we have used the following inequality, which is a consequence of (3.1)(see [9]):

$$\ln A_n - \ln H_n \geq \frac{1}{x_n^2} \sigma_n$$

It then follows by induction that  $f_n > f_{n-1} > \dots > f_2 = 0$  when  $q = 1/2$  in  $f_2$  or else  $f_n > f_{n-1} > \dots > f_1 = 0$  and this completes the proof.  $\square$

We note the above theorem gives a sharpening of Sierpiński's inequality[13], originally states for the unweighted case( $\omega_i = 1/n$ ) as:

$$H_n^{n-1} A_n \leq G_n \leq A_n^{n-1} H_n$$

The following corollary gives refinements of (1.4):

**Corollary 5.1.** For  $0 < x_1 \leq \dots \leq x_n < 1$ ,  $q = \min\{\omega_i\}$

$$(5.5) \quad \left( \frac{A_n'^{(1-q)} H_n'^q}{G_n'} \right)^{\frac{(1-x_1)^2}{x_1^2}} \geq \frac{A_n^{1-q} H_n^q}{G_n} \geq \left( \frac{A_n'^{(1-q)} H_n'^q}{G_n'} \right)^{\frac{(1-x_n)^2}{x_n^2}}$$

$$(5.6) \quad \left( \frac{G_n'}{A_n'^q H_n'^{(1-q)}} \right)^{\frac{(1-x_1)^2}{x_1^2}} \geq \frac{G_n}{A_n^q H_n^{1-q}} \geq \left( \frac{G_n'}{A_n'^q H_n'^{(1-q)}} \right)^{\frac{(1-x_n)^2}{x_n^2}}$$

with equality holding if and only if  $x_1 = x_2 = \dots = x_n$  or  $q = 1/2$ .

*Proof.* This is a direct consequence of theorem 5.1, following from a similar argument as in [12].  $\square$

## 6. CONCLUSION REMARKS

We note that if for  $x_n \leq 1/2$  one has

$$\left( \frac{x_1}{1-x_1} \right)^\beta < \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} < \left( \frac{x_n}{1-x_n} \right)^\alpha$$

then  $\beta \geq 1, \alpha \leq 1$ , otherwise by letting  $\epsilon$  tend to 0 in (2.1), we will get contradictions.

It was conjectured that an additive companion of (1.4) is true(see [1]):

$$(6.1) \quad n(G_n - G'_n) \leq (n-1)(A_n - A'_n) + H_n - H'_n$$

In [3], H. Alzer asked if the above conjecture is true, whether there exists a weighted version or not. Based on what we've got in this paper, it is natural to give the following conjecture of the weighed version of (6.1):

**Conjecture 6.1.** For  $0 < x_1 \leq \dots \leq x_n \leq 1/2$ ,  $q = \min\{\omega_i\}$

$$(6.2) \quad G_n - G'_n \leq (1-q)(A_n - A'_n) + q(H_n - H'_n)$$

Recently, H. Alzer, S. Ruscheweyh and L. Salinas[6] asked the following question: What is the largest number  $\alpha = \alpha(n)$  and what is the smallest number  $\beta = \beta(n)$  such that

$$\alpha(A_n - A'_n) + (1-\alpha)(H_n - H'_n) \leq G_n - G'_n \leq \beta(A_n - A'_n) + (1-\beta)(H_n - H'_n)$$

for all  $x_i \in (0, 1/2](j = 1, \dots, n)$ ?

We note here  $\alpha \leq 0$ , since the left-hand side inequality above can be written as:

$$(6.3) \quad \alpha A_n + (1-\alpha)H_n - G_n \leq \alpha A'_n + (1-\alpha)H'_n - G'_n$$

By a similar argument as in the proof of theorem 2.1, replacing  $(x_1, \dots, x_n)$  by  $(\epsilon x_1, \dots, \epsilon x_n)$  and letting  $\epsilon$  tend to 0 in (6.3), we find that (6.3) implies:

$$(6.4) \quad \alpha A_n + (1-\alpha)H_n - G_n \leq 0$$

for any  $\mathbf{x}$ . If we further let  $x_1$  tend to 0 in (6.4), we get

$$\alpha A_n \leq 0$$

which implies  $\alpha \leq 0$ .

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