On a New Generalization of Martins' Inequality

ON A NEW GENERALIZATION OF MARTINS' INEQUALITY

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ABSTRACT. Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m-1}$ is increasing. Then the following inequality between ratios of the power means and of the geometic means holds:

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}},$$

where r is a positive number, $a_i!$ denotes the sequence factorial defined by $\prod_{i=1}^{n} a_i$. The upper bound is the best possible.

1. Introduction

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [1], and the right-hand side Martins's inequality [7].

Alzer's inequality has invoked the interest of several mathematicians, we refer the reader to [5, 9, 16, 18] and the references therein.

Recently, F. Qi and L. Debnath in [15] proved that: Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - k a_k^r} \ge \left(\frac{a_{k+2}}{a_{k+1}}\right)^r \tag{2}$$

for a given positive real number r and $k \in \mathbb{N}$. Then

$$\frac{a_n}{a_{n+m}} \le \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{3}$$

The lower bound of (3) is the best possible.

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In [12, 13, 14, 19, 20, 21], F. Qi and others proved the following inequalities and other more general results:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}, \tag{4}$$

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{5}$$

where $n, m \in \mathbb{N}$, k is a nonnegative integer, a a positive constant, and b a nonegative constant. The equalities in (4) and (5) is valid for $n = 1$ and $m = 1$.

In [17], the following monotonicity results for the gamma function were obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{6}$$

is decreasing in $x \ge 1$ for fixed $y \ge 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \le \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{7}$$

In [11, 15], F. Qi proved that: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{8}$$

where r is a given positive real number. The lower bound is the best possible.

In [4, 18], some more general results for the lower bound of ratio of power means $\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r}$ for positive sequence $\{a_i\}_{i\in\mathbb{N}}$ were obtained.

An open problem in [10, 11] asked for the validity of the following inequality:

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}, \tag{9}$$

where $r > 0$, $n, m \in \mathbb{N}$, $k \in \mathbb{Z}^+$.

Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \ge a_i^2$ for $i \ge 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \le a_i^2$ for $i \ge 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence. See [8, p. 284].

In [3], the open problem mentioned above was solved and generalized affirmatively: Let $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence satisfying $(a_{\ell+1}/a_\ell)^\ell \ge (a_\ell/a_{\ell-1})^{\ell-1}$ for any positive integer $\ell > 1$, then $\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \sqrt[n]{a_n!}\big/\sqrt[n+m]{a_{n+m}!}$, where r is a positive number, $n, m \in \mathbb{N}$, and $a_i!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$. The upper bound is best possible.

The purpose of this paper is to give a new generalization of inequality (9) as follows.

Theorem 1. *Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i} - 1\right]\right\}_{i=1}^{n+m-1}$ is* *increasing. Then the following inequality between ratios of the power means and of the geometic means holds:*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Bigg/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{10}$$

where r is a positive number and $a_i!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$. The upper bound is the best possible.

As an easy consequence of Theorem 1 by taking $\{a_i\}_{i=1}^{n+m} = \{(i+k+b)^\alpha\}_{i=1}^{n+m}$ for a positive constant α, we have

Corollary 1. *Let α be a positive real number, k a nonnegative integer and b a real number such that $k+b>0$, and $m,n \in \mathbb{N}$. If the sequence*

$$\left\{ i\left[\left(1+\frac{1}{i+k+b}\right)^\alpha - 1\right]\right\}_{i=1}^{n+m-1} \tag{11}$$

is increasing, then for any real number $r>0$, we have

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k}[(i+b)^\alpha]^r}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}[(i+b)^\alpha]^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+b)^\alpha}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i+b)^\alpha}}. \tag{12}$$

The upper bound is the best possible.

Remark 1. By letting $\alpha = 1$ and $b = 0$ in (12), we recover inequality (9).

Taking $\alpha = 2$ in Corollary 1 leads to the following

Corollary 2. *Let k be a nonnegative integer, and b a real number such that $k+b \geq \frac{1}{2}$, and $m,n \in \mathbb{N}$. Then, for any real number $r>0$, we have*

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k}[(i+b)^2]^r}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}[(i+b)^2]^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+b)^2}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i+b)^2}}. \tag{13}$$

The upper bound is the best possible.

Considering $\{a_i\}_{i\in\mathbb{N}} = \left\{e^{i^\alpha}\right\}_{i\in\mathbb{N}}$ in Theorem 1 and standard argument gives us the following

Corollary 3. *Let $m,n \in \mathbb{N}$. If the constant $0<\alpha<1$ such that inequality*

$$\frac{e^{(1+x)^\alpha} - e^{x^\alpha}}{x^{\alpha-1} - (1+x)^{\alpha-1}} \geq \alpha x e^{(1+x)^\alpha} \tag{14}$$

holds with x on $[1,\infty)$, then, for any real number $r>0$, we have

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} e^{i^\alpha r}}{\frac{1}{n+m}\sum_{i=1}^{n+m} e^{i^\alpha r}}\right)^{1/r} < \exp\left[\frac{1}{n}\sum_{i=1}^{n} i^\alpha - \frac{1}{n+m}\sum_{i=1}^{n+m} i^\alpha\right]. \tag{15}$$

The upper bound is the best possible.

2. Lemmas

To prove our main results, the following lemmas are neccessary.

Lemma 1. *Let* $n, m \in \mathbb{N}$, *and* $\{a_i\}_{i=1}^{n+m+1}$ *a nonconstant positive sequence such that the sequence* $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ *is increasing, then the sequence*

$$\left\{\frac{\sqrt[i]{a_i!}}{a_{i+1}}\right\}_{i=1}^{n+m} \tag{16}$$

is decreasing. As a simple consequence, we have the following

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} > \frac{a_{n+1}}{a_{n+m+1}}, \tag{17}$$

where $a_n!$ *denotes the sequence factorial defined by* $\prod_{i=1}^n a_i$.

Proof. For $1 \le i \le n+m-1$, the monotonicity of the sequence (16) is equivalent to the following

$$\frac{\sqrt[i]{a_i!}}{a_{i+1}} \ge \frac{\sqrt[i+1]{a_{i+1}!}}{a_{i+2}}, \tag{18}$$

$$\iff \qquad \left(\prod_{k=1}^{i}\frac{a_k}{a_{i+1}}\right)^{1/i} \ge \left(\prod_{k=1}^{i+1}\frac{a_k}{a_{i+2}}\right)^{1/(i+1)},$$

$$\iff \qquad \frac{1}{i}\sum_{k=1}^{i}\ln\frac{a_k}{a_{i+1}} \ge \frac{1}{i+1}\sum_{k=1}^{i+1}\ln\frac{a_k}{a_{i+2}},$$

$$\iff \qquad \frac{i}{i+1}\sum_{k=1}^{i+1}\ln\frac{a_k}{a_{i+2}} \le \sum_{k=1}^{i}\ln\frac{a_k}{a_{i+1}}. \tag{19}$$

Since $\ln x$ is concave on $(0,\infty)$, by definition of concaveness, it follows that, for $1 \le k \le i$,

$$\begin{aligned}
&\frac{k}{i+1}\ln\frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1}\ln\frac{a_k}{a_{i+2}}\\
&\le \ln\left(\frac{k}{i+1}\cdot\frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1}\cdot\frac{a_k}{a_{i+2}}\right)\\
&= \ln\left(\frac{ka_{k+1}+(i-k+1)a_k}{(i+1)a_{i+2}}\right).
\end{aligned} \tag{20}$$

Since the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ is increasing, we have, for $1 \le i \le n+m-1$ and $1 \le k \le i$, the following

$$\frac{(i+1)a_{i+2}}{a_{i+1}} - (i+1) \ge \frac{ia_{i+1}}{a_i} - i,$$

$$\iff \qquad \frac{(i+1)a_{i+2}}{a_{i+1}} - (i+1) \ge \frac{ka_{k+1}}{a_k} - k,$$

$$\iff \qquad \frac{ka_{k+1}+(i-k+1)a_k}{a_k} \le \frac{(i+1)a_{i+2}}{a_{i+1}},$$

$$\iff \qquad \frac{ka_{k+1}+(i-k+1)a_k}{(i+1)a_{i+2}} \le \frac{a_k}{a_{i+1}}.$$

Combining the last line above with (20) yields

$$\frac{k}{i+1}\ln\frac{a_{k+1}}{a_{i+2}}+\frac{i-k+1}{i+1}\ln\frac{a_k}{a_{i+2}}\le\ln\frac{a_k}{a_{i+1}}. \tag{21}$$

Summing up on both sides of (21) with k from 1 to i and simplifying reveals inequality (19). The monotonicity follows.

Since $\{a_i\}_{i=1}^{n+m+1}$ is a nonconstant positive sequence, there exists at least one number $1\le i_0\le n+m-1$ such that $a_{i_0}\ne a_{i_0+1}$. The function $\ln x$ is strictly concave on $(0,\infty)$. Then, for any i such that $i_0\le i\le n+m-1$, we have

$$\begin{aligned}
&\frac{i_0}{i+1}\ln\frac{a_{i_0+1}}{a_{i+2}}+\frac{i-i_0+1}{i+1}\ln\frac{a_{i_0}}{a_{i+2}}\\
&<\ln\left(\frac{i_0}{i+1}\cdot\frac{a_{i_0+1}}{a_{i+2}}+\frac{i-i_0+1}{i+1}\cdot\frac{a_{i_0}}{a_{i+2}}\right)\\
&=\ln\left(\frac{i_0a_{i_0+1}+(i-i_0+1)a_{i_0}}{(i+1)a_{i+2}}\right)\\
&\le\ln\frac{a_{i_0}}{a_{i+1}},
\end{aligned} \tag{22}$$

notice that the last line follows from the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ being increasing. Therefore, for any i such that $i_0\le i\le n+m-1$, inequality (18) is strict. Inequality (17) is proved.

The proof is complete. □

Lemma 2. *Let $n>1$ be a positive integer and $\{a_i\}_{i=1}^{n}$ an increasing nonconstant positive sequence such that $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n-1}$ is increasing. Then the sequence*

$$\left\{\frac{a_i}{(a_i!)^{1/i}}\right\}_{i=1}^{n} \tag{23}$$

is increasing, and, for any positive integer ℓ satisfying $1\le\ell<n$,

$$\frac{a_\ell}{a_n}<\frac{(a_\ell!)^{1/\ell}}{(a_n!)^{1/n}}, \tag{24}$$

where $a_n!$ denotes the sequence factorial $\prod_{i=1}^{n}a_i$.

Proof. For $1\le\ell\le n-1$, the monotonicity of the sequence (23) is equivalent to

$$\begin{aligned}
&\frac{a_\ell}{(a_\ell!)^{1/\ell}}\le\frac{a_{\ell+1}}{(a_{\ell+1}!)^{1/(\ell+1)}},\\
\iff\quad&\left(\prod_{j=1}^{\ell}\frac{a_j}{a_\ell}\right)^{\frac{1}{\ell}}\ge\left(\prod_{j=1}^{\ell+1}\frac{a_j}{a_{\ell+1}}\right)^{\frac{1}{\ell+1}},\\
\iff\quad&\frac{1}{\ell}\sum_{j=1}^{\ell-1}\ln\frac{a_j}{a_\ell}\ge\frac{1}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}},\\
\iff\quad&\sum_{j=1}^{\ell-1}\ln\frac{a_j}{a_\ell}\ge\frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}}.
\end{aligned} \tag{25}$$

Since $\ln x$ is concave on $(0,\infty)$, by definition of concaveness, it follows that, for $1\le j\le \ell$,

$$\begin{aligned}
&\frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}}+\frac{\ell-j+1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\\
&\le \ln\biggl(\frac{j}{\ell+1}\cdot\frac{a_{j+1}}{a_{\ell+1}}+\frac{\ell-j+1}{\ell+1}\cdot\frac{a_j}{a_{\ell+1}}\biggr)\\
&=\ln\biggl(\frac{ja_{j+1}+(\ell-j+1)a_j}{(\ell+1)a_{\ell+1}}\biggr).
\end{aligned}\tag{26}$$

Straightforward computation gives us

$$\begin{aligned}
&\sum_{j=1}^{\ell}\biggl[\frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}}+\frac{\ell-j+1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\biggr]\\
&=\frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}}+\sum_{j=1}^{\ell}\biggl[\frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}}\biggr]-\sum_{j=1}^{\ell}\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\\
&=\frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}}+\sum_{j=2}^{\ell+1}\biggl[\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\biggr]-\sum_{j=1}^{\ell}\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\\
&=\frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}}.
\end{aligned}\tag{27}$$

From combining of (25), (26) and (27), it suffices to prove for $1\le j\le\ell$

$$\begin{aligned}
&&\frac{ja_{j+1}+(\ell-j+1)a_j}{(\ell+1)a_{\ell+1}}&\le\frac{a_j}{a_\ell},\\
\iff&&\frac{ja_{j+1}+(\ell-j+1)a_j}{a_j}&\le\frac{(\ell+1)a_{\ell+1}}{a_\ell},\\
\iff&&\frac{ja_{j+1}}{a_j}+\ell-j+1&\le\frac{(\ell+1)a_{\ell+1}}{a_\ell},\\
\iff&&(\ell+1)\biggl[\frac{a_{\ell+1}}{a_\ell}-1\biggr]&\ge j\biggl[\frac{a_{j+1}}{a_j}-1\biggr].
\end{aligned}\tag{28}$$

Since the sequences $\{a_i\}_{i=1}^n$ and $\bigl\{i\bigl[\frac{a_{i+1}}{a_i}-1\bigr]\bigr\}_{i=1}^{n-1}$ are increasing, the inequality (28) holds.

Moreover, the sequence $\{a_i\}_{i=1}^n$ is nonconstant positive, then there exists at least one number $1\le i_1\le n-1$ such that $a_{i_1}\ne a_{i_1+1}$. The function $\ln x$ is strictly concave on $(0,\infty)$. Then, for any ℓ such that $i_1<\ell\le n-1$, we have

$$\begin{aligned}
&\frac{i_1}{\ell+1}\ln\frac{a_{i_1+1}}{a_{\ell+1}}+\frac{\ell-i_1+1}{\ell+1}\ln\frac{a_{i_1}}{a_{\ell+1}}\\
&<\ln\biggl(\frac{i_1}{\ell+1}\cdot\frac{a_{i_1+1}}{a_{\ell+1}}+\frac{\ell-i_1+1}{\ell+1}\cdot\frac{a_{i_1}}{a_{\ell+1}}\biggr)\\
&=\ln\biggl(\frac{i_1a_{i_1+1}+(\ell-i_1+1)a_{i_1}}{(\ell+1)a_{\ell+1}}\biggr)\\
&\le\ln\frac{a_{i_1}}{a_\ell}.
\end{aligned}\tag{29}$$

Therefore, for any ℓ such that $i_1+1\le\ell<n$, inequality (25) is strict, and

$$\frac{a_\ell}{a_{\ell+1}}<\frac{(a_\ell!)^{1/\ell}}{(a_{\ell+1}!)^{1/(\ell+1)}},\tag{30}$$

and then inequality (24) is strict. The proof is complete. □

Remark 2. Some problems similar to Lemma 1 and Lemma 2 were discussed in [19] by the author and B.-N. Guo.

The methods proving Lemma 1 and Lemma 2 had been used in [18] and others.

Lemma 3 (König's inequality [2, p. 149] and [7, 22])**.** *Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be decreasing nonnegative n-tuples such that*

$$\prod_{i=1}^k b_i\le\prod_{i=1}^k a_i,\quad 1\le k\le n,\tag{31}$$

then, for $r>0$, we have

$$\sum_{i=1}^k b_i^r\le\sum_{i=1}^k a_i^r,\quad 1\le k\le n.\tag{32}$$

Remark 3. Lemma 3 is a well-known result due to König used to give a proof of Weyl's inequality (cf. Corollary 1.b.8 of [6, p. 24]).

3. Proofs of Theorem 1

Inequality (10) holds for $n=1$ by the power mean inequality and its case of equality.

For $n\ge2$, inequality (10) is equivalent to

$$\left(\frac{1}{n}\sum_{i=1}^n a_i^r\bigg/\frac{1}{n+1}\sum_{i=1}^{n+1}a_i^r\right)^{1/r}<\frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}},\tag{33}$$

which is equivalent to

$$\frac{1}{n}\sum_{i=1}^n\left(\frac{a_i}{\sqrt[n]{a_n!}}\right)^r<\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}}\right)^r.\tag{34}$$

Set

$$b_{jn+1}=b_{jn+2}=\cdots=b_{jn+n}=\frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}},\quad 0\le j\le n;\tag{35}$$

$$c_{j(n+1)+1}=c_{j(n+1)+2}=\cdots=c_{j(n+1)+(n+1)}=\frac{a_{n-j}}{\sqrt[n]{a_n!}},\quad 0\le j\le n-1.\tag{36}$$

Direct calculation yields

$$\begin{aligned}\sum_{i=1}^{n(n+1)}b_i^r&=\sum_{j=0}^n\sum_{k=1}^n b_{jn+k}^r\\&=n\sum_{j=0}^n\left(\frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}}\right)^r\\&=n\sum_{i=1}^{n+1}\left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}}\right)^r\end{aligned}\tag{37}$$

and

$$\sum_{i=1}^{n(n+1)} c_i^r = (n+1)\sum_{i=1}^{n}\left(\frac{a_i}{\sqrt[n]{a_n!}}\right)^r. \tag{38}$$

Since $\{a_i\}_{i=1}^{n+1}$ is increasing, the sequence $\{b_i\}_{i=1}^{n(n+1)}$ and $\{c_i\}_{i=1}^{n(n+1)}$ are decreasing. Therefore, by Lemma 3, to obtain inequality (34), it is sufficient to prove inequality

$$b_m! \ge c_m! \tag{39}$$

for $1 \le m \le n(n+1)$.

It is easy to see that $b_{n(n+1)}! = c_{n(n+1)}! = 1$. Thus, inequality (39) is equivalent to

$$\prod_{i=m}^{n(n+1)} b_i \le \prod_{i=m}^{n(n+1)} c_i \tag{40}$$

for $2 \le m \le n(n+1)$.

For $0 \le \ell \le n$ and $0 \le j \le n-2$, we have $2 \le (n-\ell)n + (n-j) = (n-\ell)(n+1) + (\ell - j) \le n(n+1)$. Then

$$\prod_{i=(n-\ell)n+(n-j)}^{n(n+1)} b_i = \frac{(a_{\ell+1})^{j+1}(a_\ell!)^n}{(a_{n+1}!)^{\frac{\ell n+j+1}{n+1}}}; \tag{41}$$

$$\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i = \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell > j; \tag{42}$$

$$\begin{aligned}\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i &= \prod_{i=(n-\ell-1)(n+1)+(n+1+\ell-j)}^{n(n+1)} c_i \\ &= \frac{(a_{\ell+1})^{j-\ell+1}(a_\ell!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell \le j;\end{aligned} \tag{43}$$

where $a_0 = 1$.

The last term in (43) is bigger than the right term in (42), so, without loss of generality, we can assume $j < \ell$. Therefore, from formulae (41) and (42), inequality (40) is reduced to

$$\frac{(a_{\ell+1})^{j+1}(a_\ell!)^n(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_{n+1}!)^\ell} \le \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^\ell (a_n!)^{\frac{j+1}{n}}}, \tag{44}$$

that is

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_\ell!)(a_\ell)^{j-\ell+1}} \le \frac{(a_{n+1})^\ell (a_n!)^{\frac{-\ell}{n}}}{(a_n!)^{\frac{j-\ell+1}{n}}}, \tag{45}$$

this is further equivalent to

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{a_\ell!(a_\ell)^{j-\ell+1}(a_n!)^{\frac{\ell-j-1}{n}}} \le \frac{(a_{n+1})^\ell}{(a_n!)^{\frac{\ell}{n}}}, \tag{46}$$

which can be rearranged as

$$\left(\frac{a_{\ell+1}}{a_\ell}\cdot\frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}}\right)^{\frac{j+1}{\ell}} \le \frac{\sqrt[\ell]{a_\ell!}}{a_\ell}\cdot\frac{a_{n+1}}{\sqrt[n+1]{a_{n+1}!}}, \quad j+1 \le \ell \le n. \tag{47}$$

Utilizing Lemma 2 and the logarithmical concaveness of the sequence $\{a_i\}_{i=1}^{n+1}$ yields

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > \frac{a_n}{a_{n+1}} \geq \frac{a_\ell}{a_{\ell+1}}. \tag{48}$$

Since $\frac{j+1}{\ell} \leq 1$ and $\frac{a_{\ell+1}}{a_\ell} \cdot \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > 1$ by (48), thus, to obtain (47), it suffices to prove

$$a_{\ell+1} \sqrt[n]{a_n!} < a_{n+1} \sqrt[\ell]{a_\ell!}, \tag{49}$$

this follows from Lemma 1.

Since the sequence $\{a_i\}_{i=1}^{n+m}$ is nonconstant, the inequality (10) is strict.

By the L'Hospital rule, easy calculation produces

$$\lim_{r\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} = \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{50}$$

thus, the upper bound is the best possible. The proof is complete.

Remark 4. Recently, some new inequalities for the ratios of the mean values of functions were established in [23].

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