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# FEJÉR-TYPE INEQUALITIES (I)

K.-L. TSENG, S.-R. HWANG, AND S.S. DRAGOMIR

ABSTRACT. In this paper, we establish some new Fejér-type inequalities for convex functions.

## 1. INTRODUCTION

Throughout this paper, let  $f : [a, b] \rightarrow \mathbb{R}$  be convex, and  $g : [a, b] \rightarrow [0, \infty)$  be integrable and symmetric to  $\frac{a+b}{2}$ . We define the following functions on  $[0, 1]$  that are associated with the well known Hermite-Hadamard inequality [1]

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},$$

namely

$$I(t) = \int_a^b \frac{1}{2} \left[ f\left(t \frac{x+a}{2} + (1-t) \frac{a+b}{2}\right) + f\left(t \frac{x+b}{2} + (1-t) \frac{a+b}{2}\right) \right] g(x) dx;$$

$$J(t) = \int_a^b \frac{1}{2} \left[ f\left(t \frac{x+a}{2} + (1-t) \frac{3a+b}{4}\right) + f\left(t \frac{x+b}{2} + (1-t) \frac{a+3b}{4}\right) \right] g(x) dx;$$

$$M(t) = \int_a^{\frac{a+b}{2}} \frac{1}{2} \left[ f\left(ta + (1-t) \frac{x+a}{2}\right) + f\left(t \frac{a+b}{2} + (1-t) \frac{x+b}{2}\right) \right] g(x) dx \\ + \int_{\frac{a+b}{2}}^b \frac{1}{2} \left[ f\left(t \frac{a+b}{2} + (1-t) \frac{x+a}{2}\right) + f\left(tb + (1-t) \frac{x+b}{2}\right) \right] g(x) dx;$$

and

$$N(t) = \int_a^b \frac{1}{2} \left[ f\left(ta + (1-t) \frac{x+a}{2}\right) + f\left(tb + (1-t) \frac{x+b}{2}\right) \right] g(x) dx.$$

For some results which generalize, improve, and extend the famous integral inequality (1.1), see [2] – [6].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1):

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**Theorem A.** Let  $f$  be defined as above, and let  $H$  be defined on  $[0, 1]$  by

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

Then  $H$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have

$$(1.2) \quad f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In [6], Yang and Hong established the following theorem which is a refinement of the second inequality in (1.1):

**Theorem B.** Let  $f$  be defined as above, and let  $P$  be defined on  $[0, 1]$  by

$$P(t) = \frac{1}{2(b-a)} \int_a^b \left[ f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] dx.$$

Then  $P$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have

$$(1.3) \quad \frac{1}{b-a} \int_a^b f(x) dx = P(0) \leq P(t) \leq P(1) = \frac{f(a) + f(b)}{2}.$$

In [3], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

**Theorem C.** Let  $f, g$  be defined as above. Then

$$(1.4) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx$$

is known as Fejér inequality.

In this paper, we establish some Fejér-type inequalities related to the functions  $I, J, M, N$  introduced above.

## 2. MAIN RESULTS

In order to prove our main results, we need the following lemma:

**Lemma 1** (see [4]). Let  $f$  be defined as above and let  $a \leq A \leq C \leq D \leq B \leq b$  with  $A + B = C + D$ . Then

$$f(C) + f(D) \leq f(A) + f(B).$$

Now, we are ready to state and prove our results.

**Theorem 2.** Let  $f, g, I$  be defined as above. Then  $I$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have the following Fejér-type inequality

$$(2.1) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx = I(0) \leq I(t) \leq I(1) \\ = \int_a^b \frac{1}{2} \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] g(x) dx.$$

*Proof.* It is easily observed from the convexity of  $f$  that  $I$  is convex on  $[0, 1]$ . Using simple integration techniques and under the hypothesis of  $g$ , the following identity holds on  $[0, 1]$ ,

$$(2.2) \quad I(t) = \int_a^{\frac{a+b}{2}} \left[ f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right] g(2x-a) dx.$$

Let  $t_1 < t_2$  in  $[0, 1]$ . By Lemma 1, the following inequality holds for all  $x \in [a, \frac{a+b}{2}]$ :

$$(2.3) \quad f\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + f\left(t_1(a+b-x) + (1-t_1)\frac{a+b}{2}\right) \leq f\left(t_2x + (1-t_2)\frac{a+b}{2}\right) + f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}\right).$$

Indeed, it holds when we make the choice:

$$\begin{aligned} A &= t_2x + (1-t_2)\frac{a+b}{2}, \\ C &= t_1x + (1-t_1)\frac{a+b}{2}, \\ D &= t_1(a+b-x) + (1-t_1)\frac{a+b}{2} \end{aligned}$$

and

$$B = t_2(a+b-x) + (1-t_2)\frac{a+b}{2}$$

in Lemma 1.

Multiplying the inequality (2.3) by  $g(2x-a)$ , integrating both sides over  $x$  on  $[a, \frac{a+b}{2}]$  and using identity (2.2), we derive  $I(t_1) \leq I(t_2)$ . Thus  $I$  is increasing on  $[0, 1]$  and then the inequality (2.1) holds. This completes the proof. ■

**Remark 3.** Let  $g(x) = \frac{1}{b-a}$  ( $x \in [a, b]$ ) in Theorem 2. Then  $I(t) = H(t)$  ( $t \in [0, 1]$ ) and the inequality (2.1) reduces to the inequality (1.2), where  $H$  is defined as in Theorem A.

**Theorem 4.** Let  $f, g, J$  be defined as above. Then  $J$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have the following Fejér-type inequality

$$(2.4) \quad \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \int_a^b g(x) dx = J(0) \leq J(t) \leq J(1) \\ = \frac{1}{2} \int_a^b \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] g(x) dx.$$

*Proof.* By using a similar method to that from Theorem 2, we can show that  $J$  is convex on  $[0, 1]$ , the identity

$$(2.5) \quad J(t) = \int_a^{\frac{3a+b}{4}} \left[ f\left(tx + (1-t)\frac{3a+b}{4}\right) + f\left(t\left(\frac{3a+b}{2} - x\right) + (1-t)\frac{3a+b}{4}\right) + f\left(t\left(x + \frac{b-a}{2}\right) + (1-t)\frac{a+3b}{4}\right) + f\left(t(a+b-x) + (1-t)\frac{a+3b}{4}\right) \right] g(2x-a) dx$$

holds on  $[0, 1]$  and the inequalities

$$(2.6) \quad f\left(t_1x + (1-t_1)\frac{3a+b}{4}\right) + f\left(t_1\left(\frac{3a+b}{2} - x\right) + (1-t_1)\frac{3a+b}{4}\right) \leq f\left(t_2x + (1-t_2)\frac{3a+b}{4}\right) + f\left(t_2\left(\frac{3a+b}{2} - x\right) + (1-t_2)\frac{3a+b}{4}\right),$$

$$(2.7) \quad f\left(t_1\left(x + \frac{b-a}{2}\right) + (1-t_1)\frac{a+3b}{4}\right) + f\left(t_1(a+b-x) + (1-t_1)\frac{a+3b}{4}\right) \leq f\left(t_2\left(x + \frac{b-a}{2}\right) + (1-t_2)\frac{a+3b}{4}\right) + f\left(t_2(a+b-x) + (1-t_2)\frac{a+3b}{4}\right)$$

hold for all  $t_1 < t_2$  in  $[0, 1]$  and  $x \in [a, \frac{3a+b}{4}]$ .

By (2.5) – (2.7) and using a similar method to that from Theorem 2, we can show that  $J$  is increasing on  $[0, 1]$  and (2.4) holds. This completes the proof. ■

The following result provides a comparison between the functions  $I$  and  $J$ .

**Theorem 5.** *Let  $f, g, I, J$  be defined as above. Then  $I(t) \leq J(t)$  on  $[0, 1]$ .*

*Proof.* By the identity

$$(2.8) \quad J(t) = \int_a^{\frac{a+b}{2}} \left[ f\left(tx + (1-t)\frac{3a+b}{4}\right) + f\left(t(a+b-x) + (1-t)\frac{a+3b}{4}\right) \right] g(2x-a) dx$$

on  $[0, 1]$ , (2.2) and using a similar method to that from Theorem 2, we can show that  $I(t) \leq J(t)$  on  $[0, 1]$ . The details are omitted. ■

Further, the following result incorporates the properties of the function  $M$  :

**Theorem 6.** *Let  $f, g, M$  be defined as above. Then  $M$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have the following Fejér-type inequality*

$$(2.9) \quad \int_a^b \frac{1}{2} \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] g(x) dx \\ = M(0) \leq M(t) \leq M(1) = \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \int_a^b g(x) dx.$$

*Proof.* Follows by the identity

$$(2.10) \quad M(t) = \int_a^{\frac{3a+b}{4}} \left[ f(ta + (1-t)x) + f\left(t\frac{a+b}{2} + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right. \\ \left. + f\left(t\frac{a+b}{2} + (1-t)\left(x + \frac{b-a}{2}\right)\right) + f(tb + (1-t)(a+b-x)) \right] \\ \times g(2x-a) dx$$

on  $[0, 1]$ . The details are left to the interested reader. ■

We now present a result concerning the properties of the function  $N$  :

**Theorem 7.** *Let  $f, g, N$  be defined as above. Then  $N$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have the following Fejér-type inequality*

$$(2.11) \quad \int_a^b \frac{1}{2} \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] g(x) dx \\ = N(0) \leq N(t) \leq N(1) = \frac{f(a)+f(b)}{2} \int_a^b g(x) dx.$$

*Proof.* By the identity

$$(2.12) \quad N(t) = \int_a^{\frac{a+b}{2}} [f(ta + (1-t)x) + f(tb + (1-t)(a+b-x))] g(2x-a) dx$$

on  $[0, 1]$  and using a similar method to that for Theorem 2, we can show that  $N$  is convex, increasing on  $[0, 1]$  and (2.11) holds. ■

**Remark 8.** *Let  $g(x) = \frac{1}{b-a}$  ( $x \in [a, b]$ ) in Theorem 7. Then  $N(t) = P(t)$  ( $t \in [0, 1]$ ) and the inequality (2.11) reduces to (1.3) where  $P$  is defined as in Theorem B.*

**Theorem 9.** *Let  $f, g, M, N$  be defined as above. Then  $M(t) \leq N(t)$  on  $[0, 1]$ .*

*Proof.* By the identity

$$(2.13) \quad N(t) = \int_a^{\frac{3a+b}{4}} \left[ f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right. \\ \left. + f(tb + (1-t)(a+b-x)) \right. \\ \left. + f\left(tb + (1-t)\left(x + \frac{b-a}{2}\right)\right) \right] g(2x-a) dx$$

on  $[0, 1]$ , (2.10) and using a similar method to that for Theorem 2, we can show that  $M(t) \leq N(t)$  on  $[0, 1]$ . This completes the proof. ■

The following Fejér-type inequality is a natural consequence of Theorems 2 – 9.

**Corollary 10.** *Let  $f, g$  be defined as above. Then we have*

$$\begin{aligned}
 (2.14) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \int_a^b g(x) dx \\
 &\leq \int_a^b \frac{1}{2} \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] g(x) dx \\
 &\leq \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] \int_a^b g(x) dx \\
 &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx.
 \end{aligned}$$

**Remark 11.** *Let  $g(x) = \frac{1}{b-a}$  ( $x \in [a, b]$ ) in Corollary 10. Then the inequality (2.14) reduces to*

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \\
 &\leq \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] \leq \frac{f(a) + f(b)}{2},
 \end{aligned}$$

which is a refinement of (1.1).

**Remark 12.** *In Corollary 10, the third inequality in (2.14) is the weighted generalization of Bullen's inequality [5]*

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right].$$

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