Some Inequalities Relating to Upper and Lower Bounds for the Riemann-Stieltjes Integral

SOME INEQUALITIES RELATING TO UPPER AND LOWER BOUNDS FOR THE RIEMANN–STIELTJES INTEGRAL

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Abstract. Some new inequalities are obtained relating to the generalized trapezoid and midpoint rules for the Riemann–Stieltjes integral with a convex integrand and monotone nondecreasing integrator. Results are deduced for the special case of weighted Riemann integrals.

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1 Introduction

There is a long and substantial history on approximation of the Riemann–Stieltjes integral, which precludes other than a limited overview within the page limits of the present article. We note, for example, that the seminal paper of Darst and Pollard [9], although not the first in the area, goes back nearly 40 years.

In considering the approximation of the Riemann–Stieltjes integral $\int_a^b f(t)\ du(t)$ ($a < b$ finite) by the generalized trapezoid formula

$$[u(b) - u(x)] f(b) + [u(x) - u(a)] f(a), \qquad x \in [a, b], \tag{1}$$

it is convenient to define the error functional

$$T(f, u; a, b; x) := \int_a^b f(t)\, du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a).$$

Suppose that
(a) $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, and
(b) $u : [a, b] \to \mathbb{R}$ is of $r - H$–Hölder type
(that is, $|u(t) - u(s)| \le H |t - s|^r$ for any $t, s \in [a, b]$, where $r \in (0, 1]$ and $H > 0$ are given).

In [15], the authors showed that if (a) and (b) hold, then

$$|T(f, u; a, b; x)| \le H \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \qquad x \in [a, b],$$

where as usual $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a, b]$.

The dual case, in which f is of $q - K$–Hölder type and u of bounded variation, was treated in [5]. The authors obtained the bound

$$\begin{aligned} |T(f, u; a, b; x)| &\le K \left[(x-a)^q \bigvee_a^x (u) + (b-x)^q \bigvee_x^b (u)\right] \\ &\le \begin{cases} K\left[(x-a)^q + (b-x)^q\right]\left[\frac{1}{2}\bigvee_a^b (u) + \frac{1}{2}\left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right] \\ K\left[(x-a)^{q\alpha} + (b-x)^{q\alpha}\right]^{\frac{1}{\alpha}} \left[\left[\bigvee_a^x (u)\right]^\beta - \left[\bigvee_x^b (u)\right]^\beta\right]^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ K\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b (u) \end{cases} \end{aligned}$$

for any $x \in [a,b]$.

The coresponding situations where bounded variation is replaced by monotonicity were considered by Cheung and Dragomir in [8], while the cases in which one function was of Hölder type and the other Lipschitzian were considered in [3]. For other recent results estimating the error $T(f,u;a,b,x)$ for absolutely continuous integrands f and integrators u of bounded variation, see [6] and [4].

In seeking an Ostrowski type inequality for the Riemann–Stieltjes integral, Dragomir established the following result in [10].

Theorem 1. *Suppose (a) and (b) hold. Then for any $x \in [a,b]$,*

$$\begin{aligned}
\left|[u(b)-u(x)]f(x)-\int_a^b f(t)du(t)\right| &\le H\left[(x-a)^r \bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \\
&\le \begin{cases} H[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right] \\ H\left[(x-a)^{qr}+(b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f). \end{cases}
\end{aligned}$$

The dual case was considered in [13] and can be stated as follows.

Theorem 2. *Let $u:[a,b]\to\mathbb{R}$ be of bounded variation and $f:[a,b]\to\mathbb{R}$ of $r-H$–Hölder type. Then*

$$\left|[u(b)-u(x)]f(x)-\int_a^b f(t)\,du(t)\right| \le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(u)$$

for any $x\in[a,b]$.

Recently Mercer [19] has addressed the question of how x should be chosen in the general trapezoidal formula (1) to provide an analogue and generalization of Hadamard's inequality. He found that the choice $x=\int_a^b f(t)du(t)/(b-a)$ is appropriate for the second Hadamard inequality. Our Theorem 4 gives a shorter derivation of this result. Mercer's generalization of mid–point quadrature, corresponding to the first Hadamard inequality, appears as our Remark 6.

For some related Čebyšev and Grüss type results, see Anastassiou [1], [2], Dragomir [11], [14], Dragomir and Fedotov [16], [17], Zheng Liu [18] (which sharpens [16]) and Rakhmail [20].

In this paper we derive several broadly related results connected with the generalized trapezoid and midpoint rules for the Riemann–Stieltjes integral. The situation of weighted Riemann integrals arises in the special case when the integrator u possesses a derivative. This case, often pertinent to applications, is treated in remarks. We begin in Section 2 with an inequality linking the integral means of two functions over a common interval. This derives from a Čebyšev–type inequality established for the Riemann–Stieltjes integral and is a prelude to Section 3, in which a related result is proved involving a Riemann–Stieltjes integral. Theorem 4 in Section 3 is concerned with an upper bound to the Riemann–Stieltjes integral. In Section 4 we derive lower bounds involving subgradients.

2 Intertwining Means

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}$, with f convex and g monotonic nondecreasing. If either*
(i) g is concave and $f(b)>f(a)$ or
(ii) g is convex and $f(b)<f(a)$,

then

$$\left[g(b)-\frac{1}{b-a}\int_a^b g(t)\,dt\right]f(a) + \left[\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)\right]f(b) \geq \frac{g(b)-g(a)}{b-a}\int_a^b f(t)\,dt. \tag{2}$$

Proof. For $h, u : [a,b] \to \mathbb{R}$ with h monotonic nondecreasing and u convex, we have the Čebyšev–type inequality

$$\int_a^b h(t)\,du(t) \geq \frac{u(b)-u(a)}{b-a}\int_a^b h(t)\,dt \tag{3}$$

established in [12] for the Riemann–Stieltjes integral. This provides

$$\begin{aligned}\int_a^b (t-x)\,df(t) &\geq \frac{f(b)-f(a)}{b-a}\int_a^b (t-x)\,dt\\ &= \left(\frac{a+b}{2}-x\right)[f(b)-f(a)]\end{aligned}$$

for any $x \in [a,b]$.

Coupled with the equality

$$(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt = \int_a^b (t-x)\,df(t)$$

established for any $x \in [a,b]$ in [7], this gives

$$(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt \geq \left(\frac{a+b}{2}-x\right)[f(b)-f(a)] \tag{4}$$

for any $x \in [a,b]$.

Since g is monotonic nondecreasing, integrating (4) in the Riemann–Stieltjes sense over g leads to

$$f(b)\int_a^b (b-x)\,dg(x) + f(a)\int_a^b (x-a)\,dg(x)-[g(b)-g(a)]\int_a^b f(t)\,dt \geq [f(b)-f(a)]\int_a^b\left(\frac{a+b}{2}-x\right)dg(x). \tag{5}$$

If (i) holds, then by (3)

$$\begin{aligned}\int_a^b\left(\frac{a+b}{2}-x\right)dg(x) &= \int_a^b\left(x-\frac{a+b}{2}\right)d(-g(x))\\ &\geq \frac{g(a)-g(b)}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)dx\\ &= 0\end{aligned}$$

and thus

$$[f(b)-f(a)]\int_a^b\left(\frac{a+b}{2}-x\right)dg(x) \geq 0. \tag{6}$$

Similarly (6) holds under (ii).

By (5) and (6) we conclude that under (i) or (ii)

$$f(b)\int_a^b (b-x)\,dg(x)+f(a)\int_a^b (x-a)\,dg(x) \geq [g(b)-g(a)]\int_a^b f(t)\,dt$$

and since

$$\int_a^b (b-x)\,dg(x) = \int_a^b g(t)\,dt - (b-a)\,g(a)$$

and

$$\int_a^b (x-a)\,dg(x) = (b-a)\,g(b) - \int_a^b g(t)\,dt,$$

we derive the desired inequality (2). □

Remark 1. *For the function $g(t) = t$, $t \in [a,b]$, which is both convex and concave, we obtain from (2) the second part of the Hermite–Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}$$

that holds for the convex function $f : [a,b] \to \mathbb{R}$.

Remark 2. *Suppose $g(t) = \int_a^t w(s)\,ds$ with $w(s) \ge 0$, $s \in [a,b]$. Then if either*
(i) $f(b) > f(a)$ and w is decreasing, or
(ii) $f(b) < f(a)$ and w in increasing,
we have from Theorem 3 that

$$f(b)\int_a^b (t-a)\,w(s)\,ds + f(a)\int_a^b (b-t)\,w(s)\,ds \ge \int_a^b w(s)\,ds \cdot \int_a^b f(t)\,dt.$$

3 An Upper Bound for the R–S Integral

The following result is complementary to Theorem 3 and does not involve the supplementary conditions (i), (ii). Theorem 4 is due to Mercer [19, Theorem 1]. Our proof is slightly shorter.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be convex and $g : [a,b] \to \mathbb{R}$ monotonic nondecreasing. Then*

$$f(a)\left[\frac{1}{b-a}\int_a^b g(t)\,dt - g(a)\right] + f(b)\left[g(b) - \frac{1}{b-a}\int_a^b g(t)\,dt\right] \ge \int_a^b f(t)\,dg(t). \tag{7}$$

Proof. Since f is convex, we have for any $t \in [a,b]$ that

$$\frac{(b-t)\,f(a) + (t-a)\,f(b)}{b-a} \ge f\left[\frac{(b-t)\,a + (t-a)\,b}{b-a}\right] = f(t).$$

Integrating in the Riemann–Stieltjes sense with the integrator g, we have

$$\frac{1}{b-a}\left[f(a)\int_a^b (b-t)\,dg(t) + f(b)\int_a^b (t-a)\,dg(t)\right] \ge \int_a^b f(t)\,dg(t). \tag{8}$$

However

$$\int_a^b (b-t)\,dg(t) = \int_a^b g(t)\,dt - (b-a)\,g(a)$$

and

$$\int_a^b (t-a)\,dg(t) = (b-a)\,g(b) - \int_a^b g(t)\,dt$$

and by (8) we deduce the desired result (7). □

Remark 3. *As noted by Mercer [19], if $g(t) = t$, then we get from (7) the second Hermite–Hadamard inequality*

$$\frac{f(a)+f(b)}{2} \ge \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Remark 4. *If* $g(t) = \int_a^t w(s)\,ds$, *with* $w(s) \geq 0$ *for* $s \in [a, b]$, *then*

$$\begin{aligned} \int_a^b g(t)\,dt &= \int_a^b \left(\int_a^t w(s)\,ds \right) dt \\ &= t \int_a^t w(s)\,ds \Big|_a^b - \int_a^b t w(t)\,dt \\ &= b \int_a^b w(s)\,ds - \int_a^b t w(t)\,dt. \end{aligned}$$

Therefore

$$\frac{1}{b-a} \int_a^b g(t)\,dt - g(a) = \frac{1}{b-a} \int_a^b (b-t)\,w(t)\,dt$$

and

$$\begin{aligned} g(b) - \frac{1}{b-a} \int_a^b g(t)\,dt &= \int_a^b w(s)\,ds - \frac{1}{b-a} \left[b \int_a^b w(s)\,ds - \int_a^b t w(t)\,dt \right] \\ &= \frac{1}{b-a} \int_a^b (t-a)\,w(s)\,ds. \end{aligned}$$

In conclusion, we obtain from (7) that

$$\frac{f(a)}{b-a} \int_a^b (b-t) w(t) dt + \frac{f(b)}{b-a} \int_a^b (t-a) w(t) dt \geq \int_a^b f(t) w(t) dt, \tag{9}$$

where w *is a positive integrable function on* $[a, b]$.

4 Lower Bounds for the R–S Integral

The results of this section involve subgradients.

Theorem 5. *Let* $f : [a, b] \to \mathbb{R}$ *be convex and* $g : [a, b] \to \mathbb{R}$ *monotonic nondecreasing. Then for any* $\lambda \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right) \right]$, *we have*

$$\begin{aligned} &\frac{1}{g(b) - g(a)} \int_a^b f(t)\,dg(t) - f\left(\frac{a+b}{2}\right) \\ &\geq \frac{\lambda}{g(b) - g(a)} \left[\frac{g(a) + g(b)}{2} (b-a) - \int_a^b g(t) dt \right]. \end{aligned} \tag{10}$$

Proof. Since f is convex, then for any $t \in [a, b]$

$$f(t) - f\left(\frac{a+b}{2}\right) \geq \lambda \left(t - \frac{a+b}{2} \right).$$

On integrating in the Riemann–Stieltjes sense with the monotone nondecreasing integrator g, we have

$$\begin{aligned} \int_a^b f(t)\,dg(t) - f\left(\frac{a+b}{2}\right) [g(b) - g(a)] &\geq \lambda \int_a^b \left(t - \frac{a+b}{2} \right) dg(t) \\ &= \lambda \left[\frac{g(a) + g(b)}{2} (b-a) - \int_a^b g(t)\,dt \right], \end{aligned}$$

which proves the desired inequality (10). □

Corollary 1. *Let* f *and* g *be as in Theorem 5. If*
(a) $0 \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right) \right]$, *or*

(b) $0 < f'_-\left(\frac{a+b}{2}\right)$ *and* g *is convex, or*
(c) $f'_+\left(\frac{a+b}{2}\right) < 0$ *and* g *is concave,*
then

$$\frac{1}{g(b)-g(a)}\int_a^b f(t)\,dg(t) \geq f\left(\frac{a+b}{2}\right).$$

Remark 5. *If* $g(t) = \int_a^t w(s)\,ds$, $t \in [a,b]$ *with* $w(s) \geq 0$, *then for any* $\lambda \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right)\right]$ *we have*

$$\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt - f\left(\frac{a+b}{2}\right) \geq \frac{\lambda}{\int_a^b w(s)\,ds}\left[\int_a^b \left(t-\frac{a+b}{2}\right)w(s)\,ds\right].$$

Theorem 6. *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous convex function and* $u : [a,b] \to \mathbb{R}$ *a monotonic nondecreasing function on* $[a,b]$. *Then for any* $x \in (a,b)$ *and* $\lambda(x) \in [f'_-(x), f'_+(x)]$, *we have*

$$\int_a^b f(t)\,du(t) \geq [u(b)-u(a)]\,f(x) + \lambda(x)\left[(b-x)u(b)+(x-a)\,u(a) - \int_a^b u(t)\,dt\right] \tag{11}$$

or, equivalently,

$$u(b)[f(b)-f(x)] + u(a)[f(x)-f(a)] + \lambda(x)\left[\int_a^b u(t)\,dt - (b-x)\,u(b) - (x-a)\,u(a)\right] \geq \int_a^b u(t)\,df(t). \tag{12}$$

Proof. The function f, being convex, satisfies the gradient inequality

$$f(t) - f(x) \geq \lambda(x)(t-x) \quad \text{for any } t \in [a,b], \tag{13}$$

where $\lambda(x) \in \partial(f)(x) = [f'_-(x), f'_+(x)]$.

Since the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we get on integrating (13) that

$$\int_a^b [f(t)-f(x)]\,du(t) \geq \lambda(x)\int_a^b (t-x)\,du(t),$$

which, on observing that

$$\int_a^b (t-x)\,du(t) = (b-x)\,u(b) + (x-a)\,u(a) - \int_a^b u(t)\,dt$$

and

$$\int_a^b [f(t)-f(x)]\,du(t) = \int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)],$$

leads to the desired inequality (11).

The integration by parts formula for the Stieltjes integral provides

$$\int_a^b f(t)\,du(t) - [u(b)-u(a)]\,f(x) = u(b)[f(b)-f(x)] + u(a)[f(x)-f(a)] - \int_a^b u(t)\,df(t),$$

which leads to the equivalence of (11) and (12). □

Corollary 2. *With the assumptions of Theorem 6 for f and u and $\lambda\left(\frac{a+b}{2}\right) \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right)\right]$, we have the inequalities*

$$\begin{aligned}\int_a^b f(t)\,du(t) &\geq [u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)\\ &\quad+\lambda\left(\frac{a+b}{2}\right)\left[\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(t)\,dt\right]\end{aligned}$$

and

$$\begin{aligned}&u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]+u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\\ &+\lambda\left(\frac{a+b}{2}\right)\left[\int_a^b u(t)\,dt-\frac{u(a)+u(b)}{2}(b-a)\right]\\ &\geq\int_a^b u(t)\,df(t).\end{aligned}$$

Remark 6. *Since u is monotone nondecreasing, we have by the second mean–value theorem for integrals that there exists $x=c\in[a,b]$ for which*

$$(b-x)\,u(b)+(x-a)\,u(a)=\int_a^b u(t)\,dt.$$

If $u(b)>u(a)$, we have

$$c=\frac{bu(b)-au(a)-\int_a^b u(t)\,dt}{u(b)-u(a)}$$

and (11) provides

$$\int_a^b f(t)\,du(t)\geq[u(b)-u(a)]\,f\left[\frac{bu(b)-au(a)-\int_a^b u(t)\,dt}{u(b)-u(a)}\right]. \tag{14}$$

This has been shown by Mercer [19], who uses it to give a lower bound for $\int_a^b f(t)du(t)$. For $u(t)=t$, (14) reduces to the lower Hadamard inequality

$$\int_a^b f(t)dt\geq(b-a)f\left(\frac{a+b}{2}\right)$$

relating to mid–point quadrature.

References

[1] G.A. Anastassiou, Grüss type inequalities for the Stieltjes integral, *Nonlin. Funct. Anal. Appl.* **12** (2007), 583–593.

[2] G.A. Anastassiou, Chebyshev–Grüss type and comparison of integral means inequalities for the Stieltjes integral, *Panam. Math. J.* **17** (2007), 91–109.

[3] N.S. Barnett, W.-S. Cheung, S.S. Dragomir and A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, Preprint, *RGMIA Res. Rep. Coll.* (2006) Article 9 http://rgmia.vu.edu.au/v9n4.html

[4] P. Cerone, W.-S. Cheung and S.S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Comput. Math. Appl.* **54** (2007), 183–191.

[5] P. Cerone and S.S. Dragomir, New bounds for the three-point rule involving the Riemann–Stieltjes integral, *Advances in Statistics, Combinatorics and Related Areas*, 53–62, World Sci. Publ., River Edge, NJ, 2002.

[6] P. Cerone and S.S. Dragomir, Approximation of the Stieltjes integral and applications in numerical integration, *Appl. Math.* **51** (2006), 37–47.

[7] P. Cerone, S.S. Dragomir and C.E.M. Pearce, A generalized trapezoid inequality for functions of bounded variation, *Turkish J. Math.*, **24** (2000), 147–163.

[8] W.-S. Cheung and S.S. Dragomir, Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions, *Bull. Austral. Math. Soc.* **75** (2007), 299–311.

[9] R. Darst and H. Pollard, An inequality for the Riemann–Stieltjes integral, *Proc. Amer. Math. Soc.* **25** (1970), 912–913.

[10] S.S. Dragomir, On the Ostrowski's inequality for Riemann–Stieltjes integral, *Korean J. Appl. Math.* **7** (2000), 477–485.

[11] S.S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense* **16** (2003), 373–382.

[12] S.S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.* **26** (2004), 89–122.

[13] S.S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM* **5** (2001), 35–45.

[14] S.S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.* **20** (2007), 123–130.

[15] S.S. Dragomir, C. Buse, M.V. Boldea and L. Braescu, A generalization of the trapezoidal rule for the Riemann–Stieltjes integral and applications, *Nonlinear Anal. Forum* **6** (2001), 337–351.

[16] S.S. Dragomir and I.A. Fedotov, An inequality of Grüss' type for Riemann–Stieltjes integral and applications for special means, *Tamkang J. Math.* **29** (1998), 287–292.

[17] S.S. Dragomir and I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, *Nonlinear Funct. Anal. Appl.* **6** (2001), 425–438.

[18] Zheng Liu, Refinement of an inequality of Grüss type for Riemann–Stieltjes integral, *Soochow J. Math.* **30** (2004), 483–489.

[19] P.R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann–Stieltjes integral, *J. Math. Anal. Applic.* **344** (2008), 921–926.

[20] R.T. Rakhmail, The inverse Chebyshev inequality for Stieltjes integrals, *Math. Anal.* (Russian), 9–15 Leningrad Gos. Ped. Inst. (1990).