New Inequalities of Simpson's Type for s-Convex Functions with Applications

NEW INEQUALITIES OF SIMPSON'S TYPE FOR s–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In terms of the first derivative, some inequalities of Simpson's type based on s-convexity and concavity are introduced. Best Midpoint type inequalities are given. Error estimates for special means and some numerical quadrature rules are also obtained.

1. Introduction

Suppose $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty} := \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. The following inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4 \tag{1.1}$$

holds, and it is well known in the literature as Simpson's inequality.

It is well known that if the mapping f is neither four times differentiable nor its fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

In recent years many authors have established error estimations for the Simpson's inequality; for refinements, counterparts, generalizations and new Simpson-type inequalities, see [3] – [10], [12] and [19] – [24].

Dragomir in [8] pointed out some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth. Some of the important results are presented below.

Theorem 1. *Suppose $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{3}\|f'\|_1 \tag{1.2}$$

holds, where $\|f'\|_1 = \int_a^b |f'(x)|\,dx$.

The bound of (1.2) for L-Lipschitzian mappings was given in [8] by $\frac{5}{36}L(b-a)$. Also, the following inequality was obtained in [8].

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Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then we have the inequality:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p, \tag{1.3}$$

where, $(\frac{1}{p})+(\frac{1}{q})=1$, $p>1$.

In [15] some inequalities of Hermite-Hadamard type for differentiable convex mappings were presented as follows:

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.4}$$

A more general result related to (1.4) was established in [16] – [18].

In [8], Hudzik and Maligranda considered among others the class of functions which are *s-convex in the second sense.* This class is defined in the following way: a function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x+\beta y) \le \alpha^s f(x)+\beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions is usually denoted by K_s^2. It can be easily seen that for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$.

In [13], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense:

Theorem 4. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s–convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.5}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.5). The above inequalities are sharp.

For recent results and generalizations concerning Hadamard's inequality see [1, 2] and [14] – [18].

The aim of this paper is to establish Simpson type inequalities based on s-convexity and concavity. Using these results we can estimate the error(f) in the Simpson's formula without going through its higher derivatives which may not exist, not be bounded or may be hard to find.

2. Inequalities of Simpson type for s–Convex

In order to prove our main theorems, we need the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx = (b-a)\int_0^1 p(t)\,f'(tb + (1-t)a)\,dt, \tag{2.1}$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right), \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}.$$

Proof. We note that

$$\begin{aligned} I &= \int_0^1 p(t)\,f'(tb + (1-t)a)\,dt \\ &= \int_0^{1/2}\left(t - \frac{1}{6}\right) f'(tb + (1-t)a)\,dt + \int_{1/2}^1 \left(t - \frac{5}{6}\right) f'(tb + (1-t)a)\,dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I &= \left(t - \frac{1}{6}\right)\frac{f(tb + (1-t)a)}{b-a}\bigg|_0^{1/2} - \int_0^{1/2}\frac{f(tb + (1-t)a)}{b-a}dt \\ &\quad + \left(t - \frac{5}{6}\right)\frac{f(tb + (1-t)a)}{b-a}\bigg|_{1/2}^{1} - \int_{1/2}^1 \frac{f(tb + (1-t)a)}{b-a}dt \\ &= \frac{1}{6(b-a)}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_0^1 \frac{f(tb + (1-t)a)}{b-a}dt. \end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$, we obtain

$$(b-a)\cdot I = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dt$$

which gives the desired representation (2.1). ∎

The next theorem gives a new refinement of the Simpson inequality for s-convex functions.

Theorem 5. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad \le (b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2 + 3s + 2)}\left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{2.2}$$

Proof. From Lemma 1, and since f is s-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le (b-a)\left|\int_0^1 s(t)\,f'(tb+(1-t)a)\,dt\right| \\
&\le (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt \\
&\qquad + (b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt \\
&\le (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&\qquad + (b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&= (b-a)\int_0^{1/6}\left(\frac{1}{6}-t\right)\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&\qquad + (b-a)\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&\qquad + (b-a)\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&\qquad + (b-a)\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left(t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right)dt \\
&= (b-a)\,\frac{6^{-s}-9\,(2)^{-s}+(5)^{s+2}\,6^{-s}+3s-12}{18\,(s^2+3s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],
\end{aligned}$$

which completes the proof. ∎

Therefore, we can deduce the following result for convex functions.

Corollary 1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{5\,(b-a)}{72}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.3}$$

Remark 1. *We note that the obtained midpoint inequality (2.3) is better than the inequality (1.2).*

A best upper bound for the midpoint inequality in terms of first derivative may be stated as follows:

Corollary 2. *In Theorem 5, if* $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$*, then we have,*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \tag{2.4}$$
$$\leq (b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2+3s+2)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Corollary 3. *In Corollary 2, setting* $s = 1$*, we have,*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{5(b-a)}{72}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{2.5}$$

Remark 2. *We note that the obtained midpoint inequality (2.5) is better than the inequality (1.4).*

The corresponding version of the Simpson's inequality for powers in terms of the first derivative is incorporated in the following result:

Theorem 6. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is* s*-convex on* $[a,b]$*, for some fixed* $s \in (0,1]$ *and* $p > 1$*, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.6}$$
$$\leq (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{1}{(s+1)^{\frac{1}{q}}}\left[\left(\left|f'(a)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1, using the well known Hölder integral inequality, we have

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left|\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt\right|$$
$$\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt$$
$$+(b-a)\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt$$

$$\leq (b-a)\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+(b-a)\left(\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$=(b-a)\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+(b-a)\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_{1/2}^1|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}.$$

Since f is s–convex by (1.5), we have

$$\int_0^{1/2}|f'(tb+(1-t)a)|^q dt\leq\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1} \tag{2.7}$$

and

$$\int_{1/2}^1|f'(tb+(1-t)a)|^q dt\leq\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}, \tag{2.8}$$

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{1}{(s+1)^{\frac{1}{q}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],$$

which completes the proof. ∎

Corollary 4. *Let $f : I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, for some fixed $p>1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.9}$$
$$\leq 2^{-\frac{1}{q}}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 5. *In Theorem 6, if in addition* $|f'(a)| = |f'(b)| = 0$*, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{(s+1)^{\frac{1}{q}}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|, \tag{2.10}$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

The corresponding version of the midpoint inequality for powers in terms of the first derivative is observed in the following result:

Corollary 6. *In Theorem 6, if* $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$*, then we have,*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leq (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{1}{(s+1)^{\frac{1}{q}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned} \tag{2.11}$$

Another version of the Simpson inequality for powers in terms of the first derivative is obtained as follows:

Theorem 7. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is s-convex on* $[a,b]$*, for some fixed* $s \in (0,1]$ *and* $q \geq 1$*, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)}{[216(s^2+3s+2)]^{\frac{1}{q}}}\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left(\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]|f'(b)|^q\right.\right. \\ &\qquad\left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]|f'(a)|^q\right)^{\frac{1}{q}} \\ &\qquad+\left(\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]|f'(a)|^q\right. \\ &\qquad\left.\left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]|f'(b)|^q\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{2.12}$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq (b-a)\left|\int_0^1 s(t)\,f'(tb+(1-t)a)\,dt\right|\end{aligned}$$

$$\le (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|dt$$
$$+(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|dt$$
$$\le (b-a)\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+(b-a)\left(\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is s-convex, therefore we have

$$\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|^q dt$$
$$\le \int_0^{1/6}\left(\frac{1}{6}-t\right)\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt$$
$$+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt$$
$$=\frac{(3^{-s})\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)}{36\left(s^2+3s+2\right)}\left|f'(b)\right|^q$$
$$+\frac{5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24}{36\left(s^2+3s+2\right)}\left|f'(a)\right|^q$$

and

$$\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'\left(tb+(1-t)a\right)\right|^q dt$$
$$\le \int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt$$
$$+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt$$
$$=\frac{(3^{-s})\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)}{36\left(s^2+3s+2\right)}\left|f'(a)\right|^q$$
$$+\frac{5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24}{36\left(s^2+3s+2\right)}\left|f'(b)\right|^q.$$

Also, we note that

$$\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|dt=\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|dt=\frac{5}{72}.$$

Combining all the above inequalities gives the required result, which completes the proof. ∎

Theorem 8. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed*

$q \geq 1$, *then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29b+61a}{90}\right)\right|+\left|f'\left(\frac{61b+29a}{90}\right)\right|\right]. \tag{2.13}$$

Proof. First, we note that by the concavity of $|f'|^q$ and the power-mean inequality, we have

$$|f'(\alpha x+(1-\alpha)y)|^q \geq \alpha|f'(x)|^q+(1-\alpha)|f'(y)|^q.$$

Hence,

$$|f'(\alpha x+(1-\alpha)y)| \geq \alpha|f'(x)|+(1-\alpha)|f'(y)|,$$

so $|f'|$ is also concave.

Accordingly, by Lemma 1 and the Jensen integral inequality, we have

$$\begin{aligned}&\int_0^{1/2}\left|t-\frac{1}{6}\right|f'(tb+(1-t)a)\,dt \tag{2.14}\\ &\leq\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)\left|f'\left(\frac{\int_0^{1/2}\left|t-\frac{1}{6}\right|[tb+(1-t)a]\,dt}{\int_0^{1/2}\left|t-\frac{1}{6}\right|dt}\right)\right|\\ &=\frac{5}{72}\left|f'\left(\frac{29b+61a}{90}\right)\right|\end{aligned}$$

and

$$\begin{aligned}&\int_{1/2}^{1}\left|t-\frac{5}{6}\right|f'(tb+(1-t)a)\,dt \tag{2.15}\\ &\leq\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt\right)\left|f'\left(\frac{\int_{1/2}^{1}\left|t-\frac{5}{6}\right|[tb+(1-t)a]\,dt}{\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt}\right)\right|\\ &=\frac{5}{72}\left|f'\left(\frac{61b+29a}{90}\right)\right|.\end{aligned}$$

Therefore,

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29b+61a}{90}\right)\right|+\left|f'\left(\frac{61b+29a}{90}\right)\right|\right],$$

which completes the proof. ∎

Theorem 9. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right]. \tag{2.16}$$

Proof. From Lemma 1, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt$$
$$+(b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt.$$

Using the Hölder inequality, for $q>1$ and $p=\frac{q}{q-1}$, we obtain

$$(b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt$$
$$\leq (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_0^{1/2}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}},$$

and

$$(b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt$$
$$\leq (b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_{1/2}^{1}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

It is easy to check that

$$\int_0^{1/2}\left|t-\frac{1}{6}\right|^{\frac{q}{q-1}}dt=\int_{1/2}^{1}\left|t-\frac{5}{6}\right|^{\frac{q}{q-1}}dt=\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right).$$

Since $|f'|^q$ is concave on $[a,b]$ we can use Jensen's integral inequality to obtain

$$\int_0^{1/2}\left|f'(tb+(1-t)a)\right|^q dt=\int_0^{1/2}t^0\left|f'(tb+(1-t)a)\right|^q dt$$
$$\leq\left(\int_0^{1/2}t^0dt\right)\left|f'\left(\frac{\int_0^{1/2}(tb+(1-t)a)\,dt}{\int_0^{1/2}t^0dt}\right)\right|^q$$
$$=\frac{1}{2}\left|f'\left(2\int_0^{1/2}(tb+(1-t)a)\,dt\right)\right|^q$$
$$=\frac{1}{2}\left|f'\left(\frac{b+3a}{4}\right)\right|^q.$$

Analogously,

$$\int_{1/2}^{1}\left|f'(tb+(1-t)a)\right|^q dt\leq\frac{1}{2}\left|f'\left(\frac{3b+a}{4}\right)\right|^q.$$

Combining all the obtained inequalities, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
&\leq \frac{\left(b-a\right)}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left(\frac{1}{2}\right)^q\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right] \\
&\leq \left(b-a\right)\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right],
\end{aligned}$$

which completes the proof. ∎

Theorem 10. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is s–concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
&\qquad\leq \left(b-a\right)2^{(s-1)/q}\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right) \\
&\qquad\qquad\times\left[\left|f'\left(\frac{3a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{2}\right)\right|\right].
\end{aligned} \tag{2.17}$$

Proof. We proceed similarly as in the proof of Theorem 9, by using (1.5) instead of Jensen's integral inequality for concave functions. For $|f'|^q$ s–concave, we have

$$\int_0^{1/2}\left|f'\left(tb+\left(1-t\right)a\right)\right|^q dt \leq 2^{s-1}\left|f'\left(\frac{3a+b}{2}\right)\right|^q,$$

and

$$\int_{1/2}^{1}\left|f'\left(tb+\left(1-t\right)a\right)\right|^q dt \leq 2^{s-1}\left|f'\left(\frac{a+3b}{2}\right)\right|^q,$$

so that,

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
&\qquad\leq \left(b-a\right)2^{(s-1)/q}\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right) \\
&\qquad\qquad\times\left[\left|f'\left(\frac{3a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{2}\right)\right|\right],
\end{aligned}$$

which completes the proof. ∎

Remark 3.

(1) *In Theorems 7 – 10, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, one can obtain new inequalities of midpoint type. However, the details are left to the interested reader.*
(2) *All of the above inequalities obviously hold for convex functions. Simply choose $s = 1$ in each of the results to obtain the desired ones.*

3. Applications to Special Means

Let $s \in (0,1]$ and $u, v, w \in \mathbb{R}$. We define a function $f : [0,\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} u, & t = 0; \\ vt^s + w, & t > 0. \end{cases}$$

If $v \geq 0$ and $0 \leq w \leq u$, then $f \in K_s^2$ (see [13]). Hence, for $u = w = 0$, $v = 1$, we have $f : [a,b] \to \mathbb{R}$, $f(t) = t^s$, $f \in K_s^2$.

In [13], the following result is given:

Let $f : I_1 \to \mathbb{R}_+$ be a non–decreasing and s–convex function on I_1 and $g : J \to I_2 \subseteq I_1$ be a non–negative convex function on J, then $f \circ g$ is s–convex on I_1.

A simple consequence of the previous result may be stated as follows:

Corollary 7. *Let $g : I \to I_1 \subseteq [0,\infty)$ be a non–negative convex function on I, then $g^s(x)$ is s–convex on $[0,\infty)$, $0 < s < 1$.*

For arbitrary real numbers α, β $(\alpha \neq \beta)$, we consider the following means:

(1) *The arithmetic mean*:

$$A = A(\alpha,\beta) := \frac{\alpha+\beta}{2}, \qquad \alpha,\beta \in \mathbb{R};$$

(2) *The logarithmic mean*:

$$L = L(\alpha,\beta) := \frac{b-a}{\ln b - \ln a}, \qquad \alpha,\beta \in \mathbb{R}, \alpha \neq \beta;$$

(3) *The generalized log-mean*:

$$L_p = L_p(\alpha,\beta) := \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \qquad p \in \mathbb{R}\backslash\{-1,0\}, \alpha,\beta \in \mathbb{R},\ \alpha \neq \beta.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $L \leq A$.

In the following, some new inequalities are derived for the above means.

(1) Consider $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^s$, $s \in (0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_s^s(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^s,b^s),$$

$$f\left(\frac{a+b}{2}\right) = A^s(a,b).$$

(a) Using the inequality (2.2), we obtain

$$\left|\frac{1}{3}A(a^s,b^s) + \frac{2}{3}A^s(a,b) - L_s^s(a,b)\right|$$
$$\leq s(b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2+3s+2)}\left[|a|^{s-1} + |b|^{s-1}\right].$$

For instance, if $s = 1$ then we get

$$|A(a,b) - L(a,b)| \leq \frac{5}{36}(b-a).$$

(b) Using the inequality (2.4), we have

$$\left|A^{s}(a,b)-L_{s}^{s}(a,b)\right| \le s(b-a)\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}6^{-s}+3s-12}{18(s^{2}+3s+2)}\left[|a|^{s-1}+|b|^{s-1}\right].$$

For instance, if $s=1$ then we obtain

$$|A(a,b)-L(a,b)| \le \frac{5}{72}(b-a).$$

(c) Using the inequality (2.6), we get

$$\left|\frac{1}{3}A(a^{s},b^{s})+\frac{2}{3}A^{s}(a,b)-L_{s}^{s}(a,b)\right| \le (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}}}\left[\left(\left|a^{s-1}\right|^{q}+\left|A^{s-1}(a,b)\right|^{q}\right)^{\frac{1}{q}} + \left(\left|A^{s-1}(a,b)\right|^{q}+\left|b^{s-1}\right|^{q}\right)^{\frac{1}{q}}\right],$$

where, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$|A(a,b)-L(a,b)| \le 2(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}, \qquad p>1.$$

(2) Consider $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\frac{1}{x^{s}}\in K_{s}^{2}$ (by Corollary 7), $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_{a}^{b}f(x)\,dx=L_{-s}^{s}(a,b),$$

$$\frac{f(a)+f(b)}{2}=A\left(a^{-s},b^{-s}\right),$$

$$f\left(\frac{a+b}{2}\right)=A^{-s}(a,b).$$

(a) Using the inequality (2.2), we obtain

$$\left|\frac{1}{3}A\left(a^{-s},b^{-s}\right)+\frac{2}{3}A^{-s}(a,b)-L_{-s}^{s}(a,b)\right| \le s(b-a)\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}6^{-s}+3s-12}{18(s^{2}+3s+2)}\left[|a|^{-s-1}+|b|^{-s-1}\right].$$

For instance, if $s=1$ then we get

$$\left|\frac{1}{3}A\left(a^{-1},b^{-1}\right)+\frac{2}{3}A^{-1}(a,b)-L_{-1}(a,b)\right| \le \frac{5}{36}(b-a)\left[|a|^{-2}+|b|^{-2}\right].$$

(b) Using the inequality (2.4), we have

$$\left|A^{-s}(a,b)-L_{-s}^{s}(a,b)\right| \le s(b-a)\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}6^{-s}+3s-12}{18(s^{2}+3s+2)}\left[|a|^{-s-1}+|b|^{-s-1}\right].$$

For instance, if $s=1$ then we obtain

$$\left|A^{-1}(a,b)-L_{-1}(a,b)\right|\leq\frac{5}{72}(b-a)\left[|a|^{-2}+|b|^{-2}\right].$$

(c) Using the inequality (2.6), we get

$$\left|\frac{1}{3}A\left(a^{-s},b^{-s}\right)+\frac{2}{3}A^{-s}(a,b)-L_{-s}^{s}(a,b)\right|$$
$$\leq(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}}}\left[\left(\left|a^{-s-1}\right|^{q}+\left|A^{-s-1}(a,b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|A^{-s-1}(a,b)\right|^{q}+\left|b^{-s-1}\right|^{q}\right)^{\frac{1}{q}}\right],$$

where, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\left|\frac{1}{3}A\left(a^{-1},b^{-1}\right)+\frac{2}{3}A^{-1}(a,b)-L_{-1}(a,b)\right|$$
$$\leq(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}\left[\left(\left|a^{-2}\right|^{q}+\left|A^{-2}(a,b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|A^{-2}(a,b)\right|^{q}+\left|b^{-2}\right|^{q}\right)^{\frac{1}{q}}\right],\qquad p>1.$$

4. Applications to Some Numerical Quadrature Rules

Using the results of Section 2, we now provide some applications for numerical quadrature rules. Namely, we will consider the Simpson and Midpoint rules.

4.1. **Applications to Simpson's Formula.** Let d be a division of the interval $[a,b]$, i.e., $d: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$, $h_i=(x_{i+1}-x_i)/2$ and consider the Simpson's formula

$$S(f,d)=\sum_{i=0}^{n-1}\frac{f(x_i)+4f(x_i+h_i)+f(x_{i+1})}{6}(x_{i+1}-x_i).\tag{4.1}$$

It is well known that if the mapping $f:[a,b]\to\mathbb{R}$, is differentiable such that $f^{(4)}(x)$ exists on (a,b) and $M=\max_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$, then

$$I=\int_a^b f(x)\,dx=S(f,d)+E_S(f,d),\tag{4.2}$$

where the approximation error $E_S(f,d)$ of the integral I by Simpson's formula $S(f,d)$ satisfies

$$|E_S(f,d)|\leq\frac{K}{90}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^5.\tag{4.3}$$

It is clear that if the mapping f is not four times differentiable or the fourth derivative is not bounded on (a,b), then (4.2) cannot be applied. In the following we give many different estimations for the remainder term $E_S(f,d)$ in terms of the first derivative.

Proposition 1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then in (4.2), for every division d of $[a, b]$, the following holds:*

$$|E_S(f, d)| \le \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|].$$

Proof. Applying Corollary 1 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, \dots, n-1)$ of the division d, we get

$$\left| \frac{(x_{i+1} - x_i)}{3} \left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1}) \right) - \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \\ \le \frac{5 (x_{i+1} - x_i)^2}{72} [|f'(x_i)| + |f'(x_{i+1})|].$$

Summing over i from 0 to $n - 1$ and taking into account that $|f'|$ is convex, we deduce, by the triangle inequality, that

$$\left| S(f, d) - \int_a^b f(x)\, dx \right| \le \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|].$$

which completes the proof. ∎

Proposition 2. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is convex on $[a, b]$, $p > 1$, then in (4.2), for every division d of $[a, b]$, the following holds:*

$$|E_S(f, d)| \le 2^{-\frac{1}{q}} \left(\frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\left(|f'(x_i)|^q + \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left(\left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right].$$

Proof. The proof is similar to that of Proposition 1, using the proof of Corollary 4. ∎

Proposition 3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $q \ge 1$, then in (4.2), for every division d of $[a, b]$, the following holds:*

$$|E_S(f, d)| \le \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\left| f'\left(\frac{29x_{i+1} + 61x_i}{90}\right) \right| + \left| f'\left(\frac{61x_{i+1} + 29x_i}{90}\right) \right| \right].$$

Proof. The proof is similar to that of Proposition 1, using the proof of Theorem 8. ∎

Proposition 4. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some* *fixed $q>1$, then in (4.2), for every division d of $[a,b]$, the following holds:*

$$\left|E_S(f,d)\right| \le \left(\frac{2q-1}{q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right) \times \sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'\left(\frac{3x_{i+1}+x_i}{4}\right)\right|+\left|f'\left(\frac{x_{i+1}+3x_i}{4}\right)\right|\right].$$

Proof. The proof is similar to that of Proposition 1, using the proof of Theorem 9. ∎

4.2. **Applications to the Midpoint Formula.** Let d be a division of the interval $[a,b]$, i.e., $d: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$, and consider the midpoint formula

$$M(f,d)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,f\left(\frac{x_i+x_{i+1}}{2}\right). \tag{4.4}$$

It is well known that if the mapping $f:[a,b]\to\mathbb{R}$, is differentiable such that $f''(x)$ exists on (a,b) and $K=\sup_{x\in(a,b)}|f''(x)|<\infty$, then

$$I=\int_a^b f(x)\,dx = M(f,d)+E_M(f,d), \tag{4.5}$$

where the approximation error $E_M(f,d)$ of the integral I by the midpoint formula $M(f,d)$ satisfies

$$\left|E_M(f,d)\right| \le \frac{\tilde{K}}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{4.6}$$

In the following, we propose some new estimates for the remainder term $E_M(f,d)$ in terms of the first derivative which are better than the estimations of [18].

Proposition 5. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then in (4.5), for every division d of $[a,b]$, the following holds:*

$$\left|E_M(f,d)\right| \le \frac{5}{72}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)|+|f'(x_{i+1})|\right]$$

Proof. Applying Corollary 3 on the subintervals $[x_i,x_{i+1}]$, $(i=0,1,\ldots,n-1)$ of the division d, we get

$$\left|(x_{i+1}-x_i)\,f\left(\frac{x_i+x_{i+1}}{2}\right)-\int_{x_i}^{x_{i+1}}f(x)\,dx\right| \le \frac{5(x_{i+1}-x_i)^2}{72}\left[|f'(x_i)|+|f'(x_{i+1})|\right].$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|$ is convex, we deduce that

$$\left|E_M(f,d)\right| \le \frac{5}{72}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)|+|f'(x_{i+1})|\right]$$

which completes the proof. ∎

Proposition 6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^{p/(p-1)}$ is convex on $[a, b]$, $p > 1$, then in (4.5), for every division d of $[a, b]$, the following holds:*

$$
\begin{aligned}
|E_M(f,d)| &\le 2^{-\frac{1}{q}} \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\
&\quad \times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left[\left(|f'(x_i)|^q + \left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Proof. The proof is similar to that of Proposition 5, using Corollary 6. ∎

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