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A LOWER BOUND FOR RATIO OF POWER MEANS

FENG QI, BAI-NI GUO, AND CHAO-PING CHEN

ABSTRACT. Let n and m be natural numbers. Suppose $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by $P_n(r) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{1/r}$. Then $P_n(r)/P_{n+m}(r) \geq a_n/a_{n+m}$. The lower bound is the best possible.

1. Introduction

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [1], and the right-hand side Martins' inequality [8].

Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence.

In [2], Martins' inequality was generalized as follows: Let $\{a_i\}_{i\in\mathbb{N}}$ be an increasing, logarithmically concave, positive, and nonconstant sequence satisfying $(a_{\ell+1}/a_\ell)^\ell \geq (a_\ell/a_{\ell-1})^{\ell-1}$ for any positive integer $\ell > 1$, then

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{2}$$

where r is a positive number, $n, m \in \mathbb{N}$, and $a_i!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$. The upper bound is best possible.

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Recently, in [14], another generalization of Martins' inequality was obtained: Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\bigl\{i\bigl[\frac{a_{i+1}}{a_i}-1\bigr]\bigr\}_{i=1}^{n+m-1}$ is increasing. Then the inequality (2) between ratios of the power means and of the geometic means holds. The upper bound is the best possible.

Alzer's inequality has invoked the interest of several mathematicians including, for examples, P. Cerone [3], Ch.-P. Chen [3], S. S. Dragomir [3], N. Elezović [4], B.-N. Guo [5, 16, 17], J.-Ch. Kuang [6], L. Debnath [15], Zh. Liu [7], Q.-M. Luo [18], N. Ozeki [9], J. Pečarić [4], J. Sándor [19, 20], J. S. Ume [21], the first author [10]–[13] of this paper, and so on.

In [22], a general form of Alzer's inequality was obtained: Let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing positive sequence, and let m be a natural number. If $\{a_i\}_{i=1}^{\infty}$ is logarithmically concave and the sequence $\bigl\{\bigl(\frac{a_{n+1}}{a_n}\bigr)^n\bigr\}_{i=1}^{\infty}$ is increasing, then

$$\frac{a_n}{a_{n+m}} < \left(\frac{\frac{1}{n}\sum_{i=1}^{n} a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{3}$$

In this short note, utilizing the mathematical induction, we obtain the following

Theorem 1. *Let n and m be natural numbers. Suppose $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by*

$$P_n(r) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}}. \tag{4}$$

Then the sequence $\left\{\frac{P_i(r)}{a_i}\right\}_{i=1}^{n+m}$ is decreasing for any given positive real number r, that is,

$$\frac{P_n(r)}{P_{n+m}(r)} \geq \frac{a_n}{a_{n+m}}. \tag{5}$$

The lower bound in (5) *is the best possible.*

Considering that the exponential functions a^{x^α} and a^{α^x} for given constants $\alpha \geq 1$ and $a > 1$ is logarithmically convex on $[0, \infty)$, as a corollary of Theorem 1, we have

Corollary 1. *Let $\alpha \geq 1$ and $a > 1$ be two constants. For any given real number r, the following inequalities hold:*

$$\frac{a^{(n+k)^\alpha}}{a^{(n+m+k)^\alpha}} \leq \left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} a^{i^\alpha r}}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} a^{i^\alpha r}}\right)^{1/r}, \tag{6}$$

$$\frac{a^{\alpha^{n+k}}}{a^{\alpha^{n+m+k}}} \leq \left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} a^{\alpha^i r}}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} a^{\alpha^i r}} \right)^{1/r}, \tag{7}$$

where n and m are natural numbers, and k is a nonnegative integer. The lower bounds above are the best possible.

2. Proof of Theorem 1

The inequality (5) is equivalent to

$$\frac{\frac{1}{n}\sum_{i=1}^{n} a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r} \geq \frac{a_n^r}{a_{n+m}^r}, \tag{8}$$

that is,

$$\frac{1}{(n+m)a_{n+m}^r}\sum_{i=1}^{n+m} a_i^r \leq \frac{1}{na_n^r}\sum_{i=1}^{n} a_i^r. \tag{9}$$

This is also equivalent to

$$\frac{1}{(n+1)a_{n+1}^r}\sum_{i=1}^{n+1} a_i^r \leq \frac{1}{na_n^r}\sum_{i=1}^{n} a_i^r. \tag{10}$$

Since

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r, \tag{11}$$

inequality (10) reduces to

$$\sum_{i=1}^{n} a_i^r \geq \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}. \tag{12}$$

It is easy to see that inequality (12) holds for $n = 1$.

Assume that inequality (12) holds for some $n > 1$. Using the principle of mathematical induction, considering equality (11) and the inductive hypothesis, it is easy to show that the induction for inequality (12) on $n+1$ can be written as

$$\frac{(n+2)a_{n+2}^r - (n+1)a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r} \geq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r, \tag{13}$$

which can be rearranged as

$$k\left[\left(\frac{a_{n+1}}{a_{n+2}}\right)^r - \left(\frac{a_n}{a_{n+1}}\right)^r\right] + \left(\frac{a_{n+1}}{a_{n+2}}\right)^r \leq 1. \tag{14}$$

Since the sequence $\{a_i\}_{i=1}^{n+m}$ is increasing, we have $\frac{a_{n+1}}{a_{n+2}} \leq 1$ and $\left(\frac{a_{n+1}}{a_{n+2}}\right)^r \leq 1$. From the logarithmical convexity of the sequence $\{a_i\}_{i=1}^{n+m}$, it follows that $\frac{a_{n+1}}{a_{n+2}} \leq \frac{a_n}{a_{n+1}}$ and $\left(\frac{a_{n+1}}{a_{n+2}}\right)^r - \left(\frac{a_n}{a_{n+1}}\right)^r \leq 0$. Therefore, inequality (14) is valid. Thus, the inequality (5) holds.

It can easily be shown by L'Hospital rule that

$$\lim_{r\to\infty} \frac{P_n(r)}{P_{n+m}(r)} = \frac{a_n}{a_{n+m}}. \tag{15}$$

Hence, the lower bound in (5) is the best possible. The proof is complete.

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