A Refinement of Jensen's Discrete Inequality for Differentiable Convex Functions

A REFINEMENT OF JENSEN'S DISCRETE INEQUALITY FOR DIFFERENTIABLE CONVEX FUNCTIONS

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ABSTRACT. A refinement of Jensen's discrete inequality and applications for the celebrated Arithmetic Mean – Geometric Mean – Harmonc Mean inequality and Cauchy-Schwartz-Bunikowski inequality are pointed out.

1. Introduction

The following inequality is well known in literature as Jensen's inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i f\left(x_i\right), \tag{1.1}$$

provided $f:[a,b]\rightarrow\mathbb{R}$ is a convex function on $[a,b]$, $x_i\in[a,b]$, and $p_i\geq 0$ with $P_n:=\sum_{i=1}^{n}p_i>0$.

Its central role in Analytic Inequality Theory is determined by the fact that many other fundamental results such as: the Arithmetic Mean – Geometric Mean – Harmonic Mean inequality, or the Hölder and Minkowski inequalities, or even the Ky Fan inequality may be obtained from Jensen's inequality by appropriate choices of the function f.

There is an extensive literature devoted to Jensen's inequality concerning different generalizations, refinements, counterparts and converse results, see, for example [1] – [21].

The main aim of this paper is to point out a new refinement of this classical result. Two applications in connection with the celebrated $A-G-H-$means inequality and the Cauchy-Buniakowski-Schwartz inequality are mentioned as well.

2. A Refinement of Jensen's Inequality

The following refinement of Jensen's inequality holds.

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b) and $x_i\in(a,b)$, $p_i\geq 0$ with $P_n:=\sum_{i=1}^n p_i>0$. Then one has the inequality*

$$\begin{aligned}(2.1)\quad &\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\\ &\geq\left|\frac{1}{P_n}\sum_{i=1}^n p_i\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right.\\ &\left.-\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\cdot\frac{1}{P_n}\sum_{i=1}^n p_i\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\right|\geq 0.\end{aligned}$$

Proof. Since f is differentiable convex on (a,b), then for each $x,y\in(a,b)$, one has the inequality

$$(2.2)\qquad f(x)-f(y)\geq(x-y)f'(y).$$

Using the properties of the modulus, we have

$$\begin{aligned}(2.3)\qquad f(x)-f(y)-(x-y)f'(y)&=|f(x)-f(y)-(x-y)f'(y)|\\ &\geq||f(x)-f(y)|-|x-y||f'(y)||\end{aligned}$$

for each $x,y\in(a,b)$.

If we choose $y=\frac{1}{P_n}\sum_{j=1}^n p_j x_j$ and $x=x_i$, $i\in\{1,\dots,n\}$, then we have

$$\begin{aligned}(2.4)\quad &f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)-\left(x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\\ &\geq\left|\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|-\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right|\end{aligned}$$

for any $i\in\{1,\dots,n\}$.

If we multiply (2.4) by $p_i\geq 0$, sum over i from 1 to n, and divide by $P_n>0$, we deduce

$$\begin{aligned}&\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\\ &\qquad-\frac{1}{P_n}\sum_{i=1}^n p_i\left(x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\end{aligned}$$

$$\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| \left| f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| - \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \right| \left| f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \right|$$

$$\geq \left| \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \right.$$
$$\left. - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \right| \cdot \left| f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \right|.$$

Since

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) = 0,$$

the inequality (2.1) is proved. ∎

In particular, we have the following result for unweighted means.

Corollary 1. *With the above assumptions for f and x_i, one has the inequality*

$$\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \tag{2.5}$$
$$\geq \left| \frac{1}{n}\sum_{i=1}^{n} \left| x_i - f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right| \right.$$
$$\left. - \left| f'\left(\frac{x_1 + \cdots + x_n}{n}\right)\right| \cdot \frac{1}{n}\sum_{i=1}^{n} \left| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right| \right| \geq 0.$$

Remark 1. *Similar integral inequalities may be stated as well. We omit the details.*

3. A Refinement of $A. - G. - H.$ Inequality

For a positive n-tuple $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{p} = (p_1, \ldots, p_n)$ with $p_i \geq 0$ and $\sum_{i=1}^{n} p_i =: P_n > 0$, define

$$A_n(\bar{p}, \bar{x}) := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \quad \text{(the weighted arithmetic mean)},$$

$$G_n(\bar{p}, \bar{x}) := \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}} \quad \text{(the weighted geometric mean)},$$

$$H_n(\bar{p}, \bar{x}) := \frac{P_n}{\sum_{i=1}^{n} \frac{p_i}{x_i}} = \left[A_n\left(\bar{p}, \frac{1}{\bar{x}}\right)\right]^{-1} \quad \text{(the weighted harmonic mean)}.$$

The following inequality

$$A_n(\bar{p}, \bar{x}) \geq G_n(\bar{p}, \bar{x}) \geq H_n(\bar{p}, \bar{x}) \tag{3.1}$$

is well known in the literature as the Arithmetic Mean – Geometric Mean – Harmonic Mean ($A - G - H$)-means inequality.

Using Theorem 1, we may improve this result as follows.

Proposition 1. *Suppose that $\bar{x}$, $\bar{p}$ are as above. Then we have the inequality*

$$\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})} \geq \exp\left[\left|A_n\left(\bar{p},\left|\ln\left(\frac{\bar{x}}{A_n(\bar{p},\bar{x})}\right)\right|\right)-A_n\left(\bar{p},\left|\frac{x-A_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right|\right)\right|\right] \geq 1, \tag{3.2}$$

where for a function h, we denote $h(\bar{x}) := (h(x_1),\dots,h(x_n))$.

Proof. Applying the inequality (2.1) for $f(x) = -\ln x$, we get

$$\ln\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right] \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\left|\ln\left(\frac{x_i}{A_n(p,x)}\right)\right| - A_n^{-1}(p,x)\cdot\frac{1}{P_n}\sum_{i=1}^{n} p_i\left|x_i - A_n(p,x)\right|\right| \geq 0,$$

from where we get the desired inequality (3.2). ∎

The following proposition also holds.

Proposition 2. *Suppose that $\bar{x}$, $\bar{p}$ are as above. Then we have the inequality:*

$$\frac{G_n(\bar{p},\bar{x})}{H_n(\bar{p},\bar{x})} \geq \exp\left[\left|A_n\left(\bar{p},\left|\ln\left(\frac{H_n(\bar{p},\bar{x})}{\bar{x}}\right)\right|\right)-A_n\left(\bar{p},\left|\frac{H_n(\bar{p},\bar{x})-\bar{x}}{\bar{x}}\right|\right)\right|\right] \geq 1, \tag{3.3}$$

Proof. Follows by Proposition 1 on choosing $\frac{1}{\bar{x}}$ instead of $\bar{x}$. ∎

4. A Refinement of Cauchy-Buniakowski-Schwartz's Inequality

The following inequality is well known in the literature as the Cauchy-Buniakowski-Schwartz inequality:

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{4.1}$$

for any $a_i, b_i \in \mathbb{R}$ $(i \in \{1,\dots,n\})$.

The following refinement of (4.1) holds.

Proposition 3. *If $a_i, b_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$, then one has the inequality;*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \frac{1}{\sum_{i=1}^{n} b_i^2}\left|\sum_{i=1}^{n}\left\|\begin{matrix}\frac{a_i^2}{\left(\sum_{j=1}^{n} a_j b_j\right)^2} & \frac{b_i^2}{\left(\sum_{j=1}^{n} b_j^2\right)^2}\end{matrix}\right\| - 2\left|\sum_{k=1}^{n} a_k b_k\right|\cdot\sum_{i=1}^{n}|b_i|\left|\sum_{j=1}^{n} b_j\left|\begin{matrix} a_i & b_i \\ a_j & b_j\end{matrix}\right|\right|\right| \geq 0. \tag{4.2}$$

Proof. If we apply Theorem 1 for $f(x) = x^2$, we get

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i^2 - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)^2\right| - 2\left|\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right|\right| \geq 0. \tag{4.3}$$

If in (4.3), we choose $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, $i \in \{1, \dots, n\}$, we get

$$\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} b_i^2} - \frac{\left(\sum_{i=1}^{n} a_i b_i\right)^2}{\left(\sum_{i=1}^{n} b_i^2\right)^2} \geq \left|\frac{1}{\sum_{i=1}^{n} b_i^2}\sum_{i=1}^{n} b_i^2 \cdot \left|\frac{a_i^2}{b_i^2} - \left(\frac{\sum_{j=1}^{n} a_j b_j}{\sum_{j=1}^{n} b_j^2}\right)^2\right| -2\left|\frac{\sum_{k=1}^{n} a_k b_k}{\sum_{i=1}^{n} b_i^2}\right| \cdot \frac{\sum_{i=1}^{n} b_i^2 \left|\frac{a_i}{b_i} - \sum_{j=1}^{n} a_j b_j \Big/ \sum_{j=1}^{n} b_j^2\right|}{\sum_{i=1}^{n} b_i^2}\right|, \tag{4.4}$$

which is clearly equivalent to (4.2). ∎

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