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for Mappings of Bounded Variation and their
Applications*

This is the Published version of the following publication

Tseng, Kuei-Lin, Yang, Gou-Sheng and Dragomir, Sever S (2003)
Generalizations of Weighted Trapezoidal Inequality for Mappings of Bounded
Variation and their Applications. RGMIA research report collection, 6 (1).

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Generalizations of Weighted Trapezoidal Inequality for Mappings of Bounded Variation and Their Applications

Kuei-Lin Tseng, Gou-Sheng Yang, and Sever S. Dragomir

ABSTRACT. In this paper, we establish some generalizations of weighted trapezoid inequality for mappings of bounded variation, and give several applications for r -moment, the expectation of a continuous random variable and the Beta mapping.

1. Introduction

The *trapezoid inequality* states that if f'' exists and is bounded on (a, b) , then

$$(1.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty,$$

where

$$\|f''\|_\infty := \sup_{x \in (a,b)} |f''| < \infty.$$

Now if we assume that $I_n : a = x_0 < x_1 < \dots < x_n = b$ is a partition of the interval $[a, b]$ and f is as above, then we can approximate the integral $\int_a^b f(x) dx$ by the *trapezoidal quadrature formula* $A_T(f, I_n)$, having an error given by $R_T(f, I_n)$, where

$$A_T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] l_i,$$

and the remainder satisfies the estimation

$$|R_T(f, I_n)| \leq \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} l_i^3,$$

with $l_i := x_{i+1} - x_i$ for $i = 0, 1, \dots, n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2]-[9].

Recently, Cerone-Dragomir-Pearce [4] proved the following two trapezoid type inequalities:

2000 *Mathematics Subject Classification*. Primary 26D15; Secondary 41A55.

Key words and phrases. Trapezoid Inequality, Mappings of bounded variation, Random Variable, r-moments, expectation, Beta function.

THEOREM 1. *Let $f : [a, b] \rightarrow R$ be a mapping of bounded variation. Then*

$$(1.2) \quad \left| \int_a^b f(t) dt - [(x-a)f(a) + (b-x)f(b)] \right| \\ \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] V_a^b(f),$$

for all $x \in [a, b]$, where $V_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ is the best possible.

Let I_n, l_i ($i = 0, 1, \dots, n-1$) be as above and let $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) be intermediate points. Define the sum

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})].$$

We have the following result concerning the approximation of the integral $\int_a^b f(x) dx$ in terms of T_P .

THEOREM 2. *Let f be defined as in Theorem 1, then we have*

$$\int_a^b f(x) dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi).$$

The remainder term $R_P(f, I_n, \xi)$ satisfies the estimate

$$(1.3) \quad |R_P(f, I_n, \xi)| \\ \leq \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] V_a^b(f) \leq \nu(l) V_a^b(f),$$

where $\nu(l) := \max \{l_i \mid i = 0, 1, \dots, n-1\}$. The constant $\frac{1}{2}$ is the best possible.

In this paper, we establish weighted generalizations of Theorems 1-2, and give several applications for r -moment, the expectation of a continuous random variable and the Beta mapping.

2. Some Integral Inequalities

We may state the following result.

THEOREM 3. *Let $g : [a, b] \rightarrow R$ be non-negative and continuous and let $h : [a, b] \rightarrow R$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Suppose f is defined as in Theorem 1. Then*

$$(2.1) \quad \left| \int_a^b f(t)g(t) dt - [(x-h(a))f(a) + (h(b)-x)f(b)] \right| \\ \leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a)+h(b)}{2} \right| \right] V_a^b(f),$$

for all $x \in [h(a), h(b)]$. The constant $\frac{1}{2}$ is the best possible.

PROOF. Let $x \in [h(a), h(b)]$. Using integration by parts, we have the following identity

$$\begin{aligned}
 (2.2) \quad & \int_a^b (x - h(t)) df(t) \\
 &= (x - h(t)) f(t) \Big|_a^b + \int_a^b f(t)g(t) dt \\
 &= \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)].
 \end{aligned}$$

It is well known [1, p.159] that if $\mu, \nu : [a, b] \rightarrow R$ are such that μ is continuous on $[a, b]$ and ν is of bounded variation on $[a, b]$, then $\int_a^b \mu(t) d\nu(t)$ exists and [1, p.177]

$$(2.3) \quad \left| \int_a^b \mu(t) d\nu(t) \right| \leq \sup_{x \in [a, b]} |\mu(t)| V_a^b(\nu).$$

Now, using identity (2.2) and inequality (2.3), we have

$$\begin{aligned}
 (2.4) \quad & \left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
 & \leq \sup_{t \in [a, b]} |x - h(t)| V_a^b(f).
 \end{aligned}$$

Since $x - h(t)$ is decreasing on $[a, b]$, $h(a) \leq x \leq h(b)$ and $h'(t) = g(t)$ on $[a, b]$, we have

$$\begin{aligned}
 (2.5) \quad & \sup_{t \in [a, b]} |x - h(t)| \\
 &= \max \{x - h(a), h(b) - x\} \\
 &= \frac{h(b) - h(a)}{2} + \left| x - \frac{h(a) + h(b)}{2} \right| \\
 &= \frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right|.
 \end{aligned}$$

Thus, by (2.4) and (2.5), we obtain (2.1).

Let

$$\begin{aligned}
 g(t) &\equiv 1, \quad t \in [a, b], \\
 h(t) &= t, \quad t \in [a, b], \\
 f(t) &= \begin{cases} 0 & \text{as } t = a \\ 1 & \text{as } t \in (a, b) \\ 0 & \text{as } t = b, \end{cases}
 \end{aligned}$$

and $x = \frac{a+b}{2}$. Then, we can see that the constant $\frac{1}{2}$ is best possible. This completes the proof. ■

REMARK 1. (1) If we choose $g(t) \equiv 1, h(t) = t$ on $[a, b]$, then the inequality (2.1) reduces to (1.2).

(2) If we choose $x = \frac{h(a)+h(b)}{2}$, then we get

$$(2.6) \quad \left| \int_a^b f(t)g(t) dt - \frac{f(a)+f(b)}{2} \int_a^b g(t) dt \right| \leq \frac{1}{2} \int_a^b g(t) dt \cdot V_a^b(f),$$

which is the "weighted trapezoid" inequality.

Under the conditions of Theorem 3, we have the following corollaries.

COROLLARY 1. Let $f \in C^{(1)}[a, b]$. Then we have the inequality

$$(2.7) \quad \left| \int_a^b f(t)g(t) dt - [(x-h(a))f(a) + (h(b)-x)f(b)] \right| \\ \leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a)+h(b)}{2} \right| \right] \|f'\|_1,$$

for all $x \in [h(a), h(b)]$, where $\|\cdot\|_1$ is the L_1 -norm, namely

$$\|f'\|_1 := \int_a^b |f'(t)| dt.$$

COROLLARY 2. Let $f : [a, b] \rightarrow R$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality

$$(2.8) \quad \left| \int_a^b f(t)g(t) dt - [(x-h(a))f(a) + (h(b)-x)f(b)] \right| \\ \leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a)+h(b)}{2} \right| \right] (b-a)L,$$

for all $x \in [h(a), h(b)]$.

COROLLARY 3. Let $f : [a, b] \rightarrow R$ be a monotonic mapping. Then we have the inequality

$$(2.9) \quad \left| \int_a^b f(t)g(t) dt - [(x-h(a))f(a) + (h(b)-x)f(b)] \right| \\ \leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a)+h(b)}{2} \right| \right] \cdot |f(b) - f(a)|,$$

for all $x \in [h(a), h(b)]$.

REMARK 2. The following inequality is well-known in the literature as the Fejér inequality (see for example [10]):

$$(2.10) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt \leq \int_a^b f(t)g(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b g(t) dt,$$

where $f : [a, b] \rightarrow R$ is convex and $g : [a, b] \rightarrow R$ is non-negative integrable and symmetric to $\frac{a+b}{2}$. Using the above results and (2.6), we obtain the following error bound of the second inequality in (2.10),

$$(2.11) \quad 0 \leq \frac{f(a)+f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \leq \frac{1}{2} \int_a^b g(t) dt \cdot V_a^b(f),$$

provided that f is of bounded variation on $[a, b]$.

REMARK 3. If f is convex and Lipschitzian with the constant L on $[a, b]$, g is defined as in Remark 2 and $x = \frac{h(a)+h(b)}{2}$, then we get from (2.8) and (2.10) that

$$(2.12) \quad 0 \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \leq \frac{(b-a)L}{2} \int_a^b g(t) dt.$$

REMARK 4. If f is convex and monotonic on $[a, b]$, g is defined as in Remark 2 and $x = \frac{h(a)+h(b)}{2}$, then we get from (2.9) and (2.10) that

$$(2.13) \quad \begin{aligned} 0 &\leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \\ &\leq \frac{|f(b) - f(a)|}{2} \int_a^b g(t) dt. \end{aligned}$$

REMARK 5. If f is continuous, differentiable, convex on $[a, b]$ and $f' \in L_1(a, b)$, g is defined as in Remark 2 and $x = \frac{h(a)+h(b)}{2}$, then we get from (2.7) and (2.10) that

$$(2.14) \quad 0 \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \leq \frac{\|f'\|_1}{2} \int_a^b g(t) dt.$$

3. Applications for Quadrature Rules

Throughout this section, let g and h be defined as in Theorem 3.

Let $f : [a, b] \rightarrow \mathbb{R}$, and let $I_n : a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$ and $\xi_i \in [h(x_i), h(x_{i+1})]$ ($i = 0, 1, \dots, n-1$) be intermediate points. Put $l_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t) dt$ and define the sum

$$T_P(f, g, h, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})].$$

We have the following concerning the approximation of the integral $\int_a^b f(t)g(t) dt$ in terms of T_P .

THEOREM 4. Let f be defined as in Theorem 3 and let

$$(3.1) \quad \int_a^b f(t)g(t) dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi).$$

Then, the remainder term $R_P(f, g, h, I_n, \xi)$ satisfies the estimate

$$(3.2) \quad \begin{aligned} &|R_P(f, g, h, I_n, \xi)| \\ &\leq \left[\frac{1}{2} \nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] V_a^b(f) \\ &\leq \nu(l) V_a^b(f), \end{aligned}$$

where $\nu(l) := \max \{l_i \mid i = 0, 1, \dots, n-1\}$. The constant $\frac{1}{2}$ is the best possible.

PROOF. Apply Theorem 3 on the intervals $[x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) to get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right| \\ &\leq \left[\frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] V_{x_i}^{x_{i+1}}(f), \end{aligned}$$

for all $i \in \{0, 1, \dots, n-1\}$.

Using this and the generalized triangle inequality, we have

$$\begin{aligned}
& |R_P(f, g, h, I_n, \xi)| \\
& \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i))f(x_i) + (h(x_{i+1}) - \xi_i)f(x_{i+1})] \right| \\
& \leq \sum_{i=0}^{n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] V_{x_i}^{x_{i+1}}(f) \\
& \leq \max_{i=0,1,\dots,n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \sum_{i=0}^{n-1} V_{x_i}^{x_{i+1}}(f) \\
& \leq \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] V_a^b(f)
\end{aligned}$$

and the first inequality in (3.2) is proved.

For the second inequality in (3.2), we observe that

$$\left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2}l_i \quad (i = 0, 1, \dots, n-1);$$

and then

$$\max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2}\nu(l).$$

Thus the theorem is proved. ■

REMARK 6. *If we choose $g(t) \equiv 1$, $h(t) = t$ on $[a, b]$, then the inequality (3.2) reduces to (1.3).*

The following corollaries can be useful in practice.

COROLLARY 4. *Let $f : [a, b] \rightarrow R$ be a Lipschitzian mapping with the constant $L > 0$, I_n be defined as above and choose $\xi_i = \frac{h(x_i) + h(x_{i+1})}{2}$ ($i = 0, 1, \dots, n-1$). Then we have the formula*

$$\begin{aligned}
(3.3) \quad & \int_a^b f(t)g(t) dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi) \\
& = \sum_{i=0}^{n-1} \left[\frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} g(t) dt \right] + R_P(f, g, h, I_n, \xi),
\end{aligned}$$

and the remainder satisfies the estimate

$$(3.4) \quad |R_P(f, g, h, I_n, \xi)| \leq \frac{\nu(l) \cdot L \cdot (b-a)}{2}.$$

COROLLARY 5. *Let $f : [a, b] \rightarrow R$ be a monotonic mapping and let ξ_i ($i = 0, 1, \dots, n-1$) be defined as in Corollary 4. Then we have the formula (3.3) and the remainder satisfies the estimate*

$$(3.5) \quad |R_P(f, g, h, I_n, \xi)| \leq \frac{\nu(l)}{2} \cdot |f(b) - f(a)|.$$

The case of equidistant division is embodied in the following corollary and remark:

COROLLARY 6. Suppose that $g(t) > 0$,

$$x_i = h^{-1} \left[h(a) + \frac{i(h(b) - h(a))}{n} \right] \quad (i = 0, 1, \dots, n)$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b) - h(a)}{n} = \frac{1}{n} \int_a^b g(t) dt \quad (i = 0, 1, \dots, n-1).$$

Let f be defined as in Theorem 4 and choose $\xi_i = \frac{h(x_i) + h(x_{i+1})}{2}$ ($i = 0, 1, \dots, n-1$).

Then we have the formula

$$(3.6) \quad \begin{aligned} \int_a^b f(t)g(t) dt &= T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi) \\ &= \frac{1}{2n} \int_a^b g(t) dt \cdot \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] + R_P(f, g, h, I_n, \xi), \end{aligned}$$

and the remainder satisfies the estimate

$$(3.7) \quad |R_P(f, g, h, I_n, \xi)| \leq \frac{1}{2n} \int_a^b g(t) dt \cdot V_a^b(f).$$

REMARK 7. If we want to approximate the integral $\int_a^b f(t)g(t) dt$ by $T_P(f, g, h, I_n, \xi)$ with an accuracy less than $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbb{N}$ points for the partition I_n , where

$$n_\varepsilon := \left\lceil \frac{1}{2\varepsilon} \int_a^b g(t) dt \cdot V_a^b(f) \right\rceil + 1,$$

and $[r]$ denotes the Gaussian integer of $r \in \mathbb{R}$.

4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g : [a, b] \rightarrow \mathbb{R}$ and the r -moment

$$E_r(X) := \int_a^b t^r g(t) dt,$$

which is assumed to be finite.

THEOREM 5. The inequality

$$(4.1) \quad \left| E_r(X) - \frac{a^r + b^r}{2} \right| \leq \frac{1}{2} |b^r - a^r|,$$

holds.

PROOF. If we put $f(t) = t^r$, $h(t) = \int_a^t g(x) dx$ ($t \in [a, b]$) and $x = \frac{h(a) + h(b)}{2}$ in Corollary 3, we obtain the inequality

$$(4.2) \quad \begin{aligned} &\left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \\ &\leq \frac{1}{2} \int_a^b g(t) dt \cdot |f(b) - f(a)|. \end{aligned}$$

Since

$$\int_a^b f(t)g(t) dt = E_r(X), \quad \int_a^b g(t) dt = 1,$$

$$\frac{f(a) + f(b)}{2} = \frac{a^r + b^r}{2}, \text{ and } |f(b) - f(a)| = |b^r - a^r|,$$

(4.1) follows from (4.2), immediately. This completes the proof. ■

If we choose $r = 1$ in Theorem 5, then we have the following remark:

REMARK 8. *The inequality*

$$(4.3) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{b-a}{2},$$

where $E(X)$ is the expectation of the random variable X .

5. Inequality for the Beta Mapping

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p > 0, q > 0.$$

THEOREM 6. *Let $p, q \geq 1$. Then the inequality*

$$(5.1) \quad \left| \beta(p, q) - \frac{1}{2np} \sum_{i=0}^{n-1} \left\{ \left[1 - \left(\frac{i}{n} \right)^{\frac{1}{p}} \right]^{q-1} + \left[1 - \left(\frac{i+1}{n} \right)^{\frac{1}{p}} \right]^{q-1} \right\} \right|$$

$$\leq \frac{1}{2np},$$

holds for any positive integer n .

PROOF. Let $p, q \geq 1$. If we put $a = 0, b = 1, f(t) = (1-t)^{q-1}, g(t) = t^{p-1}$ and $h(t) = \frac{t^p}{p}$ ($t \in [0, 1]$) in Corollary 6, we obtain the inequality (5.1). This completes the proof. ■

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