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A GENERALISATION OF AN OSTROWSKI INEQUALITY IN INNER PRODUCT SPACES

SEVER S. DRAGOMIR AND ANCA C. GOŞA

ABSTRACT. A generalisation of inner product spaces of an inequality due to Ostrowski and applications for sequences and integrals are given.

1. INTRODUCTION

In 1951, A.M. Ostrowski [2, p. 289] obtained the following result (see also [1, p. 92]).

Theorem 1. *Suppose that $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ are real n -tuples such that $\mathbf{a} \neq 0$ and*

$$(1.1) \quad \sum_{i=1}^n a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^n b_i x_i = 1.$$

Then

$$(1.2) \quad \sum_{i=1}^n x_i^2 \geq \frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2},$$

with equality if and only if

$$(1.3) \quad x_k = \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2}, \quad k = 1, \dots, n.$$

Another similar result due to Ostrowski which is far less known and obtained in the same work [2, p. 130] (see also [1, p. 94]), is the following one.

Theorem 2. *Let \mathbf{a} , \mathbf{b} and \mathbf{x} be n -tuples of real numbers with $\mathbf{a} \neq 0$ and*

$$(1.4) \quad \sum_{i=1}^n a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^n x_i^2 = 1.$$

Then

$$(1.5) \quad \frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2}{\sum_{i=1}^n a_i^2} \geq \left(\sum_{i=1}^n b_i x_i \right)^2.$$

If \mathbf{a} and \mathbf{b} are not proportional, then the equality holds in (1.5) iff

$$(1.6) \quad x_k = q \cdot \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{(\sum_{k=1}^n a_k^2)^{\frac{1}{2}} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2 \right]^{\frac{1}{2}}}, \quad k \in \{1, \dots, n\},$$

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with $q \in \{-1, 1, \}$.

The case of equality which was neither mentioned in [1] nor in [2] is considered in Remark 1.

In the present paper, by the use of an elementary argument based on Schwarz's inequality, a natural generalisation in inner-product spaces of (1.5) is given. The case of equality is analyzed. Applications for sequences and integrals are also provided.

2. THE RESULTS

The following theorem holds.

Theorem 3. *Let $(H, \langle \cdot, \cdot \rangle)$ be a real or complex inner product space and $a, b \in H$ two linearly independent vectors. If $x \in H$ is such that*

$$(i) \quad \langle x, a \rangle = 0 \text{ and } \|x\| = 1,$$

then

$$(2.1) \quad \frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \geq |\langle x, b \rangle|^2.$$

The equality holds in (2.1) iff

$$(2.2) \quad x = \nu \left(b - \frac{\langle a, b \rangle}{\|a\|^2} \cdot a \right),$$

where $\nu \in \mathbb{K}$ (\mathbb{C}, \mathbb{R}) is such that

$$(2.3) \quad |\nu| = \frac{\|a\|}{\left[\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right]^{\frac{1}{2}}}.$$

Proof. We use Schwarz's inequality in the inner product space H , i.e.,

$$(2.4) \quad \|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2, \quad u, v \in H$$

with equality iff there is a scalar $\alpha \in \mathbb{K}$ such that

$$(2.5) \quad u = \alpha v.$$

If we apply (2.4) for $u = z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c$, $v = d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c$, where $c \neq 0$ and $c, d, z \in H$, and taking into account that

$$\begin{aligned} \left\| z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2}{\|c\|^2}, \\ \left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2}{\|c\|^2} \end{aligned}$$

and

$$\left\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\rangle = \frac{\langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle}{\|c\|^2},$$

we deduce the inequality

$$(2.6) \quad \left[\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \geq \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^2$$

with equality iff there is a $\beta \in \mathbb{K}$ such that

$$(2.7) \quad z = \frac{\langle z, c \rangle}{\|c\|^2} \cdot c + \beta \left(d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right).$$

If in (2.6) we choose $z = x$, $c = a$ and $d = b$, where a and x satisfy (i), then we deduce

$$\|a\|^2 \left[\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right] \geq \left[\langle x, b \rangle \|a\|^2 \right]^2$$

which is clearly equivalent to (2.1).

The equality holds in (2.1) iff

$$x = \nu \left(b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right),$$

where $\nu \in \mathbb{K}$ satisfies the condition

$$(2.8) \quad 1 = \|x\| = |\nu| \left\| b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right\| = |\nu| \left[\frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \right]^{\frac{1}{2}},$$

and the theorem is thus proved. \square

The following particular cases hold.

1. If $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \ell^2(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, where

$$\ell^2(\mathbb{K}) := \left\{ x = (x_i)_{i \in \mathbb{N}}, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with \mathbf{a}, \mathbf{b} linearly independent and

$$(a) \quad \sum_{i=1}^{\infty} x_i \overline{a_i} = 0, \quad \sum_{i=1}^{\infty} |x_i|^2 = 1,$$

then

$$(2.9) \quad \frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2}{\sum_{i=1}^{\infty} |a_i|^2} \geq \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right|^2.$$

The equality holds in (2.9) iff

$$(2.10) \quad x_i = \nu \left[b_i - \frac{\sum_{k=1}^{\infty} a_k \overline{b_k}}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \{1, 2, \dots\}$$

with $\nu \in \mathbb{K}$ is such that

$$(2.11) \quad |\nu| = \frac{\left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}}{\left[\sum_{k=1}^{\infty} |a_k|^2 \sum_{k=1}^{\infty} |b_k|^2 - \left| \sum_{k=1}^{\infty} a_k \overline{b_k} \right|^2 \right]^{\frac{1}{2}}}.$$

Remark 1. The case of equality in (1.5) is obviously a particular case of the above. We omit the details.

2. If $f, g, h \in L^2(\Omega, m)$, where Ω is an m -measurable space and

$$L^2(\Omega, m) := \left\{ f : \Omega \rightarrow \mathbb{K}, \int_{\Omega} |f(x)|^2 dm(x) < \infty \right\},$$

with f, g being linearly independent and

$$(2.12) \quad \int_{\Omega} h(x) \overline{f(x)} dm(x) = 0, \quad \int_{\Omega} |h(x)|^2 dm(x) = 1,$$

then

$$(2.13) \quad \frac{\int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2}{\int_{\Omega} |f(x)|^2 dm(x)} \geq \left| \int_{\Omega} h(x) \overline{g(x)} dm(x) \right|^2.$$

The equality holds in (2.13) iff

$$h(x) = \nu \left[g(x) - \frac{\int_{\Omega} g(x) \overline{f(x)} dm(x)}{\int_{\Omega} |f(x)|^2 dm(x)} f(x) \right] \quad \text{for a.e. } x \in \Omega$$

and $\nu \in \mathbb{K}$ with

$$|\nu| = \frac{\left(\int_{\Omega} |f(x)|^2 dm(x) \right)^{\frac{1}{2}}}{\left[\int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2 \right]^{\frac{1}{2}}}.$$

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SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY,
PO Box 14428, MCMC 8001, VICTORIA, AUSTRALIA.

E-mail address: sever@matilda.vu.edu.au

URL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

COLLEGE NO. 12, REŞIŢA, JUD. CARAŞ-SEVERIN, RO-1700, REŞIŢA, ROMANIA.

E-mail address: ancagosa@hotmail.com