The Best Lower and Upper Bounds of Harmonic Sequence

THE BEST LOWER AND UPPER BOUNDS OF HARMONIC SEQUENCE

CHAO-PING CHEN AND FENG QI

Abstract. For any natural number $n \in \mathbb{N}$,

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \le \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \tag{1}$$

where $\gamma = 0.57721566490153286 \cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

1. Introduction

Let n be a natural number, then we have

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n}, \tag{2}$$

where $\gamma = 0.57721566 \cdots$ is Euler's constant.

The inequality (2) is called in literature Franel's inequality [4, Ex. 18]. Because of the well known importance of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$, there exists a very rich literature on inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$. For example, [1], [3, pp. 68–78] and references therein.

L. Tóth and S. Mare in [5, p. 264] proposed the following problems:

(1) Prove that for every positive integer n we have

$$\frac{1}{2n + \frac{2}{5}} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \tag{3}$$

where γ is Euler's constant.

(2) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In 1997 and 1999, K. Wu and B.-Ch. Yang in [8] and Sh.-R. Wei and B.-Ch. Yang in [7] verified inequality (3).

In this short note, we shall give the best lower and upper bounds of the sequence $\sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma$ and refine inequality (3), obtain the following

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Theorem 1. *For any natural number $n \in \mathbb{N}$, we have*

$$\frac{1}{2n+\frac{1}{1-\gamma}-2} \le \sum_{i=1}^{n}\frac{1}{i} - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}}, \tag{4}$$

where $\gamma = 0.57721566490153286\cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

2. Lemma

In order to prove inequality (3), the following lemma is necessary.

Lemma 1. *For $x > 0$, we have*

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x} \tag{5}$$

and

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}, \tag{6}$$

where $\psi = \frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t. \tag{7}$$

Proof. It is a well known fact ([1] and [6, p. 103]) that for $x > 0$ and a nonnegative integer m,

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{8}$$

and

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt}\,\mathrm{d}t. \tag{9}$$

The first Binet's formula ([1] and [6, p. 106]) states that for $x > 0$

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} - \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right)\frac{e^{-xt}}{t}\,\mathrm{d}t. \tag{10}$$

Differentiating (10), integrating by part and using formulas (9) and (8), it is deduced that

$$\psi(x+1) - \ln x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}t. \tag{11}$$

Using formulas (9) and (11) and the series expansion of e^x at $x = 0$ yields

$$\begin{aligned} &\psi(x+1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1} - \frac{1}{2} + \frac{1}{12}t\right)e^{-xt}\,\mathrm{d}t \\ &= \int_0^\infty \frac{12(e^t-1) - 12t - 6t(e^t-1) + t^2(e^t-1)}{12t(e^t-1)}e^{-xt}\,\mathrm{d}t \\ &= \int_0^\infty \left[\frac{1}{12t(e^t-1)}\sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!}t^n\right]e^{-xt}\,\mathrm{d}t \\ &> 0 \end{aligned} \tag{12}$$

and

$$\begin{aligned}\psi(x+1)-\ln x-\frac{1}{2x}&=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}\right)e^{-xt}\,\mathrm{d}t\\&=-\int_0^\infty\left[\frac{1}{2t(e^t-1)}\sum_{n=3}^\infty\frac{n-2}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\&<0.\end{aligned}\tag{13}$$

Hence, inequality (5) follows.

Differentiation of (11) immediately produces

$$\frac{1}{x}-\psi'(x+1)=\int_0^\infty\left(1-\frac{t}{e^t-1}\right)e^{-xt}\,\mathrm{d}t.\tag{14}$$

Exploiting formulas (9) and (14) and the series expansion of e^x at $x=0$ yields

$$\begin{aligned}&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}\\&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2\right)e^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\frac{1}{12(e^t-1)}\sum_{n=5}^\infty\frac{(n-3)(n-4)}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\&>0\end{aligned}\tag{15}$$

and

$$\begin{aligned}&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}\\&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2-\frac{1}{720}t^4\right)e^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\frac{1}{720(e^t-1)}\sum_{n=7}^\infty\left(\frac{720}{n!}-\frac{360}{(n-1)!}+\frac{60}{(n-2)!}-\frac{1}{(n-4)!}\right)t^n\right]e^{-xt}\,\mathrm{d}t.\end{aligned}\tag{16}$$

Noticing that for $n\ge 7$,

$$\begin{aligned}&\frac{720}{n!}-\frac{360}{(n-1)!}+\frac{60}{(n-2)!}-\frac{1}{(n-4)!}\\&=-\frac{120+218(n-7)+119(n-7)^2+22(n-7)^3+(n-7)^4}{n!}<0,\end{aligned}\tag{17}$$

we obtain

$$\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}<0.\tag{18}$$

Therefore, inequality (6) holds. The proof is complete. □

3. Proof of Theorem 1

In [1], [2, p. 593] and [6, p. 104] it is given that $\psi(n)=\sum_{k=1}^{n-1}\frac{1}{k}-\gamma$. Thus, inequality (4) can be rearranged as

$$\frac{1}{3}<\frac{1}{\psi(n+1)-\ln n}-2n\le\frac{1}{1-\gamma}-2.\tag{19}$$

Define for $x>0$

$$\phi(x)=\frac{1}{\psi(x+1)-\ln x}-2x.\tag{20}$$

Differentiating ϕ and utilizing (5) and (6) reveals that for $x > \frac{12}{5}$,

$$\begin{aligned}
&(\psi(x+1)-\ln x)^2\phi'(x)\\
&=\frac{1}{x}-\psi'(x+1)-2(\psi(x+1)-\ln x)^2\\
&<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-2\left(\frac{1}{2x}-\frac{1}{12x^2}\right)^2\\
&=\frac{12-5x}{360x^5}<0,
\end{aligned}\tag{21}$$

and $\phi(x)$ decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1)=\frac{1}{1-\gamma}-2=0.36527211862544155\cdots, \tag{22}$$

$$\phi(2)=\frac{1}{\frac{3}{2}-\gamma-\ln 2}-4=0.35469600731465752\cdots, \tag{23}$$

$$\phi(3)=\frac{1}{\frac{11}{6}-\gamma-\ln 3}-6=0.34898948531361115\cdots. \tag{24}$$

Therefore, the sequence

$$\phi(n)=\frac{1}{\psi(n+1)-\ln n}-2n, \quad n\in\mathbb{N} \tag{25}$$

is decreasing strictly, and for $n\in\mathbb{N}$

$$\lim_{n\to\infty}\phi(n)<\phi(n)\le\phi(1)=\frac{1}{1-\gamma}-2. \tag{26}$$

Making use of approximating expansion of ψ in [1], [2, p. 594], or [6, p. 108] gives

$$\psi(x)=\ln x-\frac{1}{2x}-\frac{1}{12x^2}+O(x^{-4}) \quad (x\to\infty), \tag{27}$$

and then

$$\lim_{n\to\infty}\phi(n)=\lim_{x\to\infty}\phi(x)=\lim_{x\to\infty}\frac{\frac{1}{3}+O(x^{-2})}{1+O(x^{-1})}=\frac{1}{3}. \tag{28}$$

The proof is complete.

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