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SOME LANDAU TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATES ARE HÖLDER CONTINUOUS

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ABSTRACT. Some inequalities of Landau type for functions whose derivatives satisfy Hölder's condition are pointed out.

1. INTRODUCTION

Let $I = \mathbb{R}_+$ or $I = \mathbb{R}$. If $f : I \rightarrow \mathbb{R}$ is twice differentiable and $f, f'' \in L_p(I)$, $p \in [1, \infty]$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of the function f , so that

$$(1.1) \quad \|f'\|_{p,I} \leq C_p(I) \|f\|_{\frac{1}{2},I}^{\frac{1}{2}} \cdot \|f''\|_{\frac{1}{2},I}^{\frac{1}{2}},$$

where $\|\cdot\|_{p,I}$ is the p -norm on the interval I , i.e, we recall

$$\|h\|_{\infty,I} := \operatorname{ess\,sup}_{t \in I} |h(t)|$$

and

$$\|h\|_{p,I} := \left(\int_I |h(t)|^p dt \right)^{\frac{1}{p}},$$

if $p \in [1, \infty)$.

The investigation of such inequalities was initiated by E. Landau [1] in 1913. He considered the case $p = \infty$ and showed that

$$(1.2) \quad C_{\infty}(\mathbb{R}_+) = 2 \quad \text{and} \quad C_{\infty}(\mathbb{R}) = \sqrt{2},$$

are the best constants for which (1.1) holds.

In 1932, G.H. Hardy and J.E. Littlewood [2] proved (1.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1.$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [3] showed that the best constant $C_p(\mathbb{R}_+)$ in (1.1) satisfies the estimate

$$(1.3) \quad C_p(\mathbb{R}_+) \leq 2 \quad \text{for} \quad p \in [1, \infty),$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Actually $C_p(\mathbb{R}) \leq \sqrt{2}$ (see [4] by R.R. Kallman and G.-C. Rota and [5] by Z. Ditzian).

For other results concerning this problem, see Chapter I of [7].

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2. SOME RESULTS FOR f BOUNDED AND f' HÖLDER CONTINUOUS

The following lemma is useful in what follows.

Lemma 1. *Let $C, D > 0$ and $r, u \in (0, 1]$. Consider the function $g_{r,u} : (0, \infty) \rightarrow \mathbb{R}$ given by*

$$(2.1) \quad g_{r,u}(\lambda) = \frac{C}{\lambda^u} + D\lambda^r.$$

Define $\lambda_0 := \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0, \infty)$. Then, for $\lambda_1 \in (0, \infty)$ we have the bound

$$(2.2) \quad \inf_{\lambda \in (0, \lambda_1]} g_{r,u}(\lambda) = \begin{cases} \frac{r+u}{u^{\frac{r+u}{r+u}} \cdot r^{\frac{r}{r+u}}} \cdot C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}} & \text{if } \lambda_1 \geq \lambda_0, \\ \frac{C}{\lambda_1^u} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \lambda_0. \end{cases}$$

Proof. We observe that

$$g'_{r,u}(\lambda) = \frac{rD\lambda^{r+u} - Cu}{\lambda^{u+1}}, \quad \lambda \in (0, \infty).$$

The unique solution of the equation $g'_{r,u}(\lambda) = 0$, $\lambda \in (0, \infty)$ is $\lambda_0 = \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0, \infty)$. The function $g_{r,u}$ is decreasing on $(0, \lambda_0)$ and increasing on (λ_0, ∞) . The global minimum for $g_{r,u}$ on $(0, \infty)$ is

$$\begin{aligned} g_{r,u}(\lambda_0) &= \frac{C}{\left(\frac{uC}{rD}\right)^{\frac{u}{r+u}}} + D \left(\frac{uC}{rD}\right)^{\frac{r}{r+u}} = \frac{C(rD)^{\frac{u}{r+u}}}{(uC)^{\frac{u}{r+u}}} + \frac{D(uC)^{\frac{r}{r+u}}}{(rD)^{\frac{r}{r+u}}} \\ &= \frac{CrD + DuC}{(uC)^{\frac{u}{r+u}} (rD)^{\frac{r}{r+u}}} = \frac{CD(r+u)}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}} \cdot C^{\frac{u}{r+u}} \cdot D^{\frac{r}{r+u}}} \\ &= \frac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}}} C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}}, \end{aligned}$$

which proves that equality (2.2) ■

The following particular cases are useful:

Corollary 1. *Let $C, D > 0$ and $r \in (0, 1]$. Consider the function $g_r : (0, \infty) \rightarrow \mathbb{R}$ given by*

$$g_r(\lambda) = \frac{C}{\lambda} + D\lambda^r.$$

Define $\bar{\lambda}_0 = \left(\frac{C}{rD}\right)^{\frac{1}{r+1}} \in (0, \infty)$. Then for $\lambda_1 \in (0, \infty)$ one has

$$(2.3) \quad \inf_{\lambda \in (0, \lambda_1]} g_r(\lambda) = \begin{cases} \frac{r+1}{r^{\frac{r}{r+1}}} \cdot C^{\frac{r}{r+1}} \cdot D^{\frac{1}{r+1}} & \text{if } \lambda_1 \geq \bar{\lambda}_0, \\ \frac{C}{\lambda_1} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \bar{\lambda}_0. \end{cases}$$

Corollary 2. *Let $C, D > 0$ and $u \in (0, 1]$. Consider the function $g_u : (0, \infty) \rightarrow \mathbb{R}$ given by*

$$g_u(\lambda) = \frac{C}{\lambda^u} + D\lambda.$$

Define $\widetilde{\lambda}_0 = \left(\frac{uC}{D}\right)^{\frac{1}{1+u}} \in (0, \infty)$. Then for $\lambda_1 \in (0, \infty)$ one has

$$(2.4) \quad \inf_{\lambda \in (0, \lambda_1]} g_u(\lambda) = \begin{cases} \frac{1+u}{u^{1+u}} \cdot C^{\frac{1}{1+u}} \cdot D^{\frac{u}{1+u}} & \text{if } \lambda_1 \geq \widetilde{\lambda}_0, \\ \frac{C}{\lambda_1^u} + D\lambda_1 & \text{if } 0 < \lambda_1 < \widetilde{\lambda}_0. \end{cases}$$

Remark 1. If $r = u = 1$ then the following bound holds

$$(2.5) \quad \inf_{\lambda \in (0, \lambda_1]} \left(\frac{C}{\lambda} + D\lambda \right) = \begin{cases} 2\sqrt{CD} & \text{if } \lambda_1 \geq \sqrt{\frac{C}{D}}, \\ \frac{C}{\lambda_1} + D\lambda_1 & \text{if } 0 < \lambda_1 < \sqrt{\frac{C}{D}}. \end{cases}$$

The following theorem holds:

Theorem 1. Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ a locally absolutely continuous function on I . If $f \in L_\infty(I)$ and the derivative $f' : I \rightarrow \mathbb{R}$ satisfies the Hölder condition:

$$(2.6) \quad |f'(t) - f'(s)| \leq H|t - s|^r \text{ for any } t, s \in I,$$

where $H > 0$ and $r \in (0, 1]$ are given, then $f' \in L_\infty(I)$ and one has the inequalities

$$(2.7) \quad \|f'\|_{L_\infty} \leq \begin{cases} 2^{\frac{r}{r+1}} \left(1 + \frac{1}{r}\right)^{\frac{r}{r+1}} \|f\|_{L_\infty}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \quad \text{if } m(I) \geq 2^{\frac{r+2}{r+1}} \left(\frac{\|f\|_{L_\infty}}{H}\right)^{\frac{1}{r+1}} \left(1 + \frac{1}{r}\right)^{\frac{1}{r+1}}, \\ \frac{4\|f\|_{L_\infty}}{m(I)} + \frac{H}{2^r(r+1)} [m(I)]^r \\ \quad \text{if } 0 \leq m(I) \leq 2^{\frac{r+2}{r+1}} \left(\frac{\|f\|_{L_\infty}}{H}\right)^{\frac{1}{r+1}} \left(1 + \frac{1}{r}\right). \end{cases}$$

Proof. We start with the following identity

$$(2.8) \quad f(t) = f(a) + (t-a)f'(a) + \int_a^t [f'(s) - f'(a)] ds$$

to get

$$(2.9) \quad |f'(a)| \leq \left| \frac{f(t) - f(a)}{t-a} \right| + \frac{1}{|t-a|} \left| \int_a^t |f'(s) - f'(a)| ds \right|,$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Since f' is of $r-H$ -Hölder type, then

$$(2.10) \quad \left| \int_a^t |f'(s) - f'(a)| ds \right| \leq H \left| \int_a^t |s-a|^r ds \right| = \frac{H}{r+1} |t-a|^{r+1}.$$

So then by (2.9) and (2.10) we deduce

$$(2.11) \quad |f'(a)| \leq \frac{|f(t) - f(a)|}{|t-a|} + \frac{H}{r+1} |t-a|^r,$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Since $f \in L_\infty(I)$, then by (2.11) we obviously get that

$$(2.12) \quad |f'(a)| \leq \frac{2\|f\|_{L_\infty}}{|t-a|} + \frac{H}{r+1} |t-a|^r$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Now observe that for any $a \in I$ and any $s \in \left(0, \frac{m(I)}{2}\right)$ there exists a $t \in I$ so that $s = |t - a|$ and then, by (2.12), we deduce

$$(2.13) \quad |f'(a)| \leq \frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1}s^r$$

for a.e. $a \in I$ and every $s \in \left(0, \frac{m(I)}{2}\right)$. By taking the inequality (2.13) to the infimum over s on $\left(0, \frac{m(I)}{2}\right)$, we get that

$$(2.14) \quad |f'(a)| \leq \inf_{s \in (0, \frac{m(I)}{2})} \left[\frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1}s^r \right] = K$$

for a.e. $a \in I$.

If we take the essential supremum over $a \in I$ in (2.14), we conclude that

$$(2.15) \quad \|f'\|_{I,\infty} \leq K.$$

Making use of Corollary 1, we get

$$K = \begin{cases} \frac{r+1}{r} (2\|f\|_{I,\infty})^{\frac{r}{r+1}} \left(\frac{H}{r+1}\right)^{\frac{1}{r+1}} & \text{if } \frac{m(I)}{2} \geq \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}, \\ \frac{2\|f\|_{I,\infty}}{\frac{m(I)}{2}} + \frac{H}{r+1} \cdot \left(\frac{m(I)}{2}\right)^r & \text{if } 0 < \frac{m(I)}{2} < \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}. \end{cases}$$

giving the desired result (2.7). ■

The following result also holds

Corollary 3. *With the assumption in Theorem 1 and if f' is L -Lipschitz then*

$$(2.16) \quad \|f'\|_{I,\infty} \leq \begin{cases} 2\sqrt{\|f\|_{I,\infty} \cdot L} & \text{if } m(I) \geq \sqrt{\frac{\|f\|_{I,\infty}}{L}}; \\ \frac{4\|f\|_{I,\infty}}{m(I)} + \frac{H}{4}m(I) & \text{if } 0 < m(I) \leq \sqrt{\frac{\|f\|_{I,\infty}}{L}}. \end{cases}$$

Remark 2. *This result was obtained by Niculescu and Buşe in [6], see Theorem 3.*

3. SOME BOUNDS FOR f AND f' HÖLDER CONTINUOUS

The following result also holds:

Theorem 2. *Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ a locally absolutely continuous function on I . If f is $l - K$ -Hölder type, i.e. it satisfies the condition*

$$(3.1) \quad |f(t) - f(s)| \leq K|t - s|^l \quad \text{for any } t, s, \in \overset{\circ}{I},$$

where $K > 0$ and $l \in (0, 1)$ are given, and the derivative $f' : I \rightarrow \mathbb{R}$ satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality

$$(3.2) \quad \|f'\|_{I,\infty} \leq \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l}{1-l+r}} \cdot r^{\frac{1-l}{1-l+r}} \cdot (r+1)^{\frac{1-l}{r+1-l}}} K^{\frac{r}{r+1-l}} \cdot H^{\frac{1-l}{r+1-l}} \\ \quad \text{if } m(I) \geq 2 \left[\frac{(1-l)K}{H} \right]^{\frac{1}{1-l+r}} \left(1 + \frac{1}{r}\right)^{\frac{1}{1-l+r}}; \\ \frac{2(1-l)K}{[m(I)]^{1-l}} + \frac{H}{2^r(r+1)} [m(I)]^r \\ \quad \text{if } 0 < m(I) < 2 \left[\frac{(1-l)K}{H} \right]^{\frac{1}{1-l+r}} \left(1 + \frac{1}{r}\right)^{\frac{1}{1-l+r}}. \end{cases}$$

Proof. We know (see the proof of Theorem 1) that

$$(3.3) \quad |f'(a)| \leq \frac{|f(t) - f(a)|}{|t - a|} + \frac{H}{r+1} |t - a|^r$$

for any $t \in I$ and a.e. $a \in I$ with $a \neq t$.

Using the assumption that (3.1) holds, then, by (3.3) we may write that

$$(3.4) \quad |f'(a)| \leq \frac{K}{|t - a|^{1-l}} + \frac{H}{r+1} |t - a|^r$$

for any $t \in I$ and a.e. $a \in I$ with $t \neq a$.

Using a similar argument to the one in Theorem 1, we may conclude that $\|f'\|_{I,\infty} \leq S$, where

$$S = \inf_{\lambda \in (0, \frac{m(I)}{2})} \left[\frac{K}{\lambda^{1-l}} + \frac{H}{r+1} \lambda^r \right] \\ = \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l+r}{1-l+r}} \cdot r^{\frac{r}{1-l+r}}} K^{\frac{r}{r+1-l}} \cdot \left(\frac{H}{r+1} \right)^{\frac{1-l}{r+1-l}} & \text{if } \frac{m(I)}{2} \geq \left[\frac{(1-l)K}{r \frac{H}{r+1}} \right]^{\frac{1}{1-l+r}} \\ \frac{K}{(\frac{m(I)}{2})^{1-l}} + \frac{H}{r+1} \left(\frac{m(I)}{2} \right)^r & \text{if } 0 < \frac{m(I)}{2} \leq \left[\frac{(1-l)K}{r \frac{H}{r+1}} \right]^{\frac{1}{1-l+r}} \end{cases}$$

from where we deduce the desired inequality (3.2). ■

The following corollary is useful.

Corollary 4. *Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ a locally absolutely continuous function on I . If $f' \in L_p(I)$, $p > 1$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality:*

$$(3.5) \quad \|f'\|_{I,\infty} \leq \begin{cases} \frac{\frac{pr+1}{p} \cdot \frac{1}{r^{\frac{pr}{pr+1}} \cdot (r+1)^{\frac{1}{pr+1}}} \|f'\|_{I,p}^{\frac{pr}{pr+1}} H^{\frac{1}{pr+1}}}{\text{if } m(I) \geq 2 \left[\frac{\|f'\|_{I,p}}{pH} \right]^{\frac{p}{pr+1}} \cdot \left(1 + \frac{1}{r}\right)^{\frac{p}{pr+1}} ;} \\ \frac{\|f'\|_{I,p} \cdot 2^{\frac{1}{p}}}{[m(I)]^{\frac{1}{p}}} + \frac{H}{2^r(r+1)} [m(I)]^r \\ \text{if } 0 < m(I) < 2 \left[\frac{\|f'\|_{I,p}}{pH} \right]^{\frac{p}{pr+1}} \cdot \left(1 + \frac{1}{r}\right)^{\frac{p}{pr+1}} . \end{cases}$$

Proof. If $f' \in L_p(I)$, then we have

$$|f(b) - f(a)| = \left| \int_a^b f'(s) ds \right| \leq \left| \int_a^b |f'(s)| ds \right| \\ \leq |b - a|^{\frac{1}{q}} \left| \int_a^b |f'(s)|^p ds \right|^{\frac{1}{p}} \\ \leq |b - a|^{1 - \frac{1}{p}} \cdot \|f'\|_{I,p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, for a.e. $a, b \in I$.

Using Theorem 2 for $l = 1 - \frac{1}{p}$ and $K = \|f'\|_{I,p}$ we deduce the desired result (3.5). ■

Finally we may state the following corollary as well.

Corollary 5. *Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ a locally absolutely continuous function on I . If $f' \in L_1(I)$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality*

$$(3.6) \quad \|f'\|_{I,\infty} \leq \begin{cases} \left(1 + \frac{1}{r}\right)^{\frac{r}{r+1}} \cdot \|f'\|_{I,1}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \quad \text{if } m(I) \geq 2 \left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \cdot \left(1 + \frac{1}{r}\right)^{\frac{1}{r+1}} ; \\ \frac{2\|f'\|_{I,1}}{m(I)} + \frac{H}{2^{r(r+1)}} [m(I)]^r \\ \quad \text{if } 0 < m(I) < 2 \left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \left(1 + \frac{1}{r}\right)^{\frac{1}{r+1}} . \end{cases}$$

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