Integral Expression and Inequalities of Mathieu Type Series

INTEGRAL EXPRESSION AND INEQUALITIES OF MATHIEU TYPE SERIES

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Abstract. Let $r > 0$ be a positive real number and $a = (a_1, a_2, \dots, a_k, \dots)$ a positive sequence such that the series $g(x) = \sum_{k=1}^{\infty} e^{-a_k x}$ converges for $x > 0$, then $\sum_{k=1}^{\infty} a_k/(a_k^2 + r^2)^2 = \frac{1}{2r}\int_0^{\infty} x g(x)\sin(rx)\,\mathrm{d}x$.

If $a = (a_1, a_2, \dots, a_k, \dots)$ is a positive arithmetic sequence with difference $d > 0$, then several inequalities of Mathieu type series $\sum_{k=1}^{\infty} a_k/(a_k^2 + r^2)^2$ are obtained for $r > 0$ under some conditions on a.

1. Introduction

In 1890, Mathieu defined $S(r)$ in [12] as

$$S(r) = \sum_{k=1}^{\infty} \frac{2k}{(k^2 + r^2)^2}, \quad r > 0, \tag{1}$$

and conjectured that $S(r) < \frac{1}{r^2}$. We call formula (1) Mathieu's series.

In [2, 11], Berg and Makai proved

$$\frac{1}{r^2 + \frac{1}{2}} < S(r) < \frac{1}{r^2}. \tag{2}$$

H. Alzer, J. L. Brenner and O. G. Ruehr in [1] obtained

$$\frac{1}{r^2 + \frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2 + \frac{1}{6}}, \tag{3}$$

where ζ denotes the zeta function and the number $\zeta(3)$ is the best possible.

The integral form of Mathieu's series (1) was given in [6, 7] by

$$S(r) = \frac{1}{r}\int_0^{\infty} \frac{x}{e^x - 1}\sin(rx)\,\mathrm{d}x. \tag{4}$$

Recently, the following results were obtained in [14, 15]:

(1) Let Φ_1 and Φ_2 be two integrable functions sush that $\frac{x}{e^x-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{x}{e^x-1}$ are increasing. Then, for any positive number r, we have

$$\frac{1}{r}\int_0^{\infty} \Phi_2(x)\sin(rx)\,\mathrm{d}x \leqslant S(r) \leqslant \frac{1}{r}\int_0^{\infty} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{5}$$

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(2) For any positive number r, we have

$$S(r) \geqslant \frac{1}{8r(1+r^2)^3}\left[16r(r^2-3)+\pi^3(r^2+1)^3\operatorname{sech}^2\left(\frac{\pi r}{2}\right)\tanh\left(\frac{\pi r}{2}\right)\right]. \tag{6}$$

(3) For positive number $r > 0$, we have

$$\frac{4\left(1+r^2\right)\left(e^{-\pi/r}+e^{-\pi/(2r)}\right)-4r^2-1}{\left(e^{-\pi/r}-1\right)\left(1+r^2\right)\left(1+4r^2\right)} \leqslant S(r) \leqslant \frac{\left(1+4r^2\right)\left(e^{-\pi/r}-e^{-\pi/(2r)}\right)-4\left(1+r^2\right)}{\left(e^{-\pi/r}-1\right)\left(1+r^2\right)\left(1+4r^2\right)}. \tag{7}$$

(4) For any positive number $r > 0$, we have

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2} < \frac{1}{r}\int_0^{\pi/r}\frac{x}{e^x-1}\sin(rx)\,\mathrm{d}x < \frac{1+\exp(-\frac{\pi}{2r})}{r^2+\frac{1}{4}}. \tag{8}$$

(5) Suppose r is a positive number, then, for any positive real number α, we have

$$\frac{1}{r^2+\frac{1}{2}} < \sum_{k=1}^{\infty}\frac{2k^{\alpha}}{(k^{2\alpha}+r^2)^2} < \frac{1}{r^2}. \tag{9}$$

In [9, 14, 15], the following open problem was proposed by B.-N. Guo and F. Qi respectively: Let

$$S(r,t,\alpha) = \sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^{\alpha}+r^2)^{t+1}} \tag{10}$$

for $t > 0$, $r > 0$ and $\alpha > 0$. Can one obtain an integral expression of $S(r,t,\alpha)$? Give some sharp inequalities for the series $S(r,t,\alpha)$.

In [17], the open problem stated above was considered and an integral expression of $S(r,t,2)$ was obtained: Let $a > 0$ and $p \in \mathbb{N}$, then

$$S(a,p,2) = \sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^{p+1}} = \frac{2}{(2a)^p p!}\int_0^{\infty}\frac{t^p\cos(\frac{p\pi}{2}-at)}{e^t-1}\,\mathrm{d}t - 2\sum_{k=1}^{p}\frac{(k-1)(2a)^{k-2p-1}}{k!(p-k+1)}\binom{-(p+1)}{p-k}\int_0^{\infty}\frac{t^k\cos[\frac{\pi}{2}(2p-k+1)-at]}{e^t-1}\,\mathrm{d}t. \tag{11}$$

Using the quadrature formulas, some new inequalities of Mathieu series (1) were established in [8]. By the help of Laplace transform, the open problem mentioned above was partially solved, for example, among other things, an integral expression for $S\big(r,\frac{1}{2},2\big)$ was given as follows:

$$S\Big(r,\frac{1}{2},2\Big) = \frac{2}{r}\int_0^{\infty}\frac{tJ_0(rt)}{e^t-1}\,\mathrm{d}t, \tag{12}$$

where J_0 is Bessel function of order zero.

There has been a much rich literature on the study of Mathieu's series, for example, [4, 5, 11, 16, 18, 19, 20], also see [3, 10, 13]

In this paper, we are about to investigate the following Mathieu type series

$$S(r,a) = \sum_{k=1}^{\infty}\frac{a_k}{(a_k^2+r^2)^2}, \tag{13}$$

where $a = (a_1, a_2, \dots, a_k, \dots)$ is a positive sequence satisfying $\lim_{k\to\infty} a_k = \infty$, and obtain an integral expression and some inequalities of $S(r, a)$ under some suitable conditons. At last, two open problems are proposed.

2. Integral expression of Mathieu type series (13)

Let $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive sequence satisfying $\lim_{k\to\infty} a_k = \infty$, let

$$b_k(r, a) = \frac{a_k}{(a_k^2 + r^2)^2} \tag{14}$$

and

$$S(r, a) = \sum_{k=1}^{\infty} b_k(r, a). \tag{15}$$

Theorem 1. *Let $r > 0$ and $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive sequence such that the series*

$$g(x) \triangleq \sum_{k=1}^{\infty} e^{-a_k x} \tag{16}$$

converges for $x > 0$. Then we have

$$S(r, a) = \frac{1}{2r} \int_0^{\infty} x g(x) \sin(rx)\, \mathrm{d}x. \tag{17}$$

Proof. By direct computation, we have

$$b_k(r, a) = \frac{i}{4r}\left[\frac{1}{(a_k + ir)^2} - \frac{1}{(a_k - ir)^2}\right], \tag{18}$$

where $i^2 = -1$.

From the definition of gamma function, it is easy to see that

$$\frac{\Gamma(t)}{u^t} = \int_0^{\infty} x^{t-1} e^{-ux}\, \mathrm{d}x. \tag{19}$$

Set $u = a_k \pm ir$ in formula (19), then

$$\frac{\Gamma(t)}{(a_k \pm ir)^t} = \int_0^{\infty} x^{t-1} e^{-(a_k \pm ir)x}\, \mathrm{d}x, \tag{20}$$

$$\Gamma(t)\left[\frac{1}{(a_k + ir)^t} - \frac{1}{(a_k - ir)^t}\right] = -2i \int_0^{\infty} x^{t-1} e^{-x a_k} \sin(rx)\, \mathrm{d}x, \tag{21}$$

$$\Gamma(t)\sum_{k=1}^{\infty}\left[\frac{1}{(a_k + ir)^t} - \frac{1}{(a_k - ir)^t}\right] = -2i \int_0^{\infty} g(x) x^{t-1} \sin(rx)\, \mathrm{d}x. \tag{22}$$

Since $\Gamma(2) = 1$, we have

$$S(r; f) = \frac{1}{2r} \int_0^{\infty} g(x) x \sin(rx)\, \mathrm{d}x. \tag{23}$$

The proof is complete. □

Remark 1. If let $a_k = k$ in (17), then we can easily obtain the formula (4) in [7]. Note that the proof of Theorem 1 uses the technique which was used by O. E. Emersleben in [7].

Corollary 1. *If $a = (a_1, a_2, \dots, a_k, \dots)$ is a positive arithmetic sequence with difference $d > 0$, then for any positive real number $r > 0$, we have*

$$S(r, a) = \frac{1}{2r} \int_0^\infty \frac{x e^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, \mathrm{d}x. \tag{24}$$

Proof. Since $a = (a_1, a_2, \dots, a_k, \dots)$ is an arithmetic sequence with difference $d > 0$, then $\{e^{-xa_k}\}_{k=1}^\infty$ is a geometric sequence with constant ratio $e^{-dx} < 1$, thus

$$g(x) = \sum_{k=1}^\infty e^{-xa_k} = \frac{e^{(d-a_1)x}}{e^{dx} - 1}.$$

Then formula (24) follows from (17). □

3. Inequalities of Mathieu type series (24)

Now we give a general estimate of Mathieu type series (24) as follows.

Theorem 2. *Let $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive arithmetic sequence with difference $d > 0$. Let Φ_1 and Φ_2 be two integrable functions such that $\frac{xe^{(d-a_1)x}}{e^{dx}-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{xe^{(d-a_1)x}}{e^{dx}-1}$ are increasing. Then for any positive number r,*

$$\frac{1}{2r} \int_0^\infty \Phi_2(x) \sin(rx) \, \mathrm{d}x \leqslant S(r, a) \leqslant \frac{1}{2r} \int_0^\infty \Phi_1(x) \sin(rx) \, \mathrm{d}x. \tag{25}$$

Proof. The function $\psi(x) = \sin(rx)$ has a period $\frac{2\pi}{r}$, and $\psi(x) = -\psi\left(x + \frac{\pi}{r}\right)$.

Since $f(x) = \frac{xe^{(d-a_1)x}}{e^{dx}-1} - \Phi_1(x)$ is increasing, for any $\alpha > 0$, we have $f(x + \alpha) \geqslant f(x)$. Therefore, from Lemma 1 or Corollary 1 in [14, 15], we obtain

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \left[\frac{xe^{(d-a_1)x}}{e^{dx} - 1} - \Phi_1(x)\right] \sin(rx) \, \mathrm{d}x \leqslant 0, \tag{26}$$

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, \mathrm{d}x \leqslant \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) \, \mathrm{d}x. \tag{27}$$

Then, from formula (13), we have

$$\begin{aligned} S(r, a) &= \frac{1}{2r} \sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, \mathrm{d}x \\ &\leqslant \frac{1}{2r} \sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) \, \mathrm{d}x \\ &= \frac{1}{2r} \int_0^\infty \Phi_1(x) \sin(rx) \, \mathrm{d}x. \end{aligned} \tag{28}$$

The right hand side of inequality (25) follows.

Similar arguments yield the left hand side of inequality (25). □

For $x > 0$, we have

$$\frac{1}{e^x} < \frac{x}{e^x - 1} < \frac{1}{e^{x/2}}. \tag{29}$$

Theorem 3. *Let $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive arithmetic sequence with difference $d > 0$ and $a_1 > \frac{d}{2}$. For positive number $r > 0$, we have*

$$\frac{1}{d}\left\{\frac{\left(1+e^{-\pi a_1/r}\right)}{2\left(a_1^2+r^2\right)\left(1-e^{-2\pi a_1/r}\right)}-\frac{2\left[e^{-\pi(2a_1-d)/r}+e^{-\pi(2a_1-d)/(2r)}\right]}{\left[(2a_1-d)^2+4r^2\right]\left[1-e^{-\pi(2a_1-d)/r}\right]}\right\}\leqslant S(r,a)$$
$$\leqslant\frac{1}{d}\left\{\frac{2\left[1+e^{-\pi(2a_1-d)/(2r)}\right]}{\left[(2a_1-d)^2+4r^2\right]\left[1-e^{-\pi(2a_1-d)/r}\right]}-\frac{\left(e^{-2\pi a_1/r}+e^{-\pi a_1/r}\right)}{2\left(a_1^2+r^2\right)\left(1-e^{-2\pi a_1/r}\right)}\right\}. \tag{30}$$

Proof. For $r > 0$, using (24), by direct calculation, we have

$$S(r,a)=\frac{1}{2r}\sum_{k=0}^{\infty}\left[\int_{2k\pi/r}^{(2k+1)\pi/r}+\int_{(2k+1)\pi/r}^{(2k+2)\pi/r}\right]\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x. \tag{31}$$

The inequality (29) gives us

$$\begin{aligned}\frac{r\left(1+e^{-\pi a_1/r}\right)}{d\left(a_1^2+r^2\right)\left(1-e^{-2\pi a_1/r}\right)}&=\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{a_1x}}\,\mathrm{d}x\\&\leqslant\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\\leqslant\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{(a_1-\frac{d}{2})x}}\,\mathrm{d}x&=\frac{4r\left[1+e^{-\pi(2a_1-d)/(2r)}\right]}{d\left[(2a_1-d)^2+4r^2\right]\left[1-e^{-\pi(2a_1-d)/r}\right]}\end{aligned} \tag{32}$$

and

$$\begin{aligned}-\frac{4r\left[e^{-\pi(2a_1-d)/r}+e^{-\pi(2a_1-d)/(2r)}\right]}{d\left[(2a_1-d)^2+4r^2\right]\left[1-e^{-\pi(2a_1-d)/r}\right]}&=\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{(a_1-\frac{d}{2})x}}\,\mathrm{d}x\\&\leqslant\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\&\leqslant\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{a_1x}}\,\mathrm{d}x=-\frac{r\left(e^{-2\pi a_1/r}+e^{-\pi a_1/r}\right)}{d\left(a_1^2+r^2\right)\left(1-e^{-2\pi a_1/r}\right)}.\end{aligned} \tag{33}$$

Substituting (32) and (33) into (31) yields (30). The proof is complete. □

Theorem 4. *Let $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive arithmetic sequence with difference $d > 0$ and $a_1 > \frac{d}{2}$. For any positive number $r > 0$, we have*

$$S(r,a)<\frac{1}{2r}\int_0^{\pi/r}\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x<\frac{2\left[1+e^{\pi(d-2a_1)/(2r)}\right]}{d[(d-2a_1)^2+4r^2]}. \tag{34}$$

Proof. Straightforward computation yields

$$\begin{aligned}
&\int_0^\infty \frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x-\int_0^{\pi/r}\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\
&\quad=\sum_{k=1}^\infty\int_{k\pi/r}^{(k+1)\pi/r}\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\
&\quad=\sum_{i=1}^\infty\left(\int_{2i\pi/r}^{(2i+1)\pi/r}+\int_{(2i-1)\pi/r}^{2i\pi/r}\right)\frac{xe^{(d-a_1)x}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\
&\quad=\frac{1}{r^2}\sum_{i=1}^\infty\left(\int_0^\pi+\int_{-\pi}^0\right)\frac{(2i\pi+x)\exp\frac{(d-a_1)(2i\pi+x)}{r}}{\exp\frac{d(2i\pi+x)}{r}-1}\sin(2i\pi+x)\,\mathrm{d}x\\
&\quad=\frac{1}{r^2}\sum_{i=1}^\infty\int_0^\pi\left[\frac{(2i\pi+x)\exp\frac{(d-a_1)(2i\pi+x)}{r}}{\exp\frac{d(2i\pi+x)}{r}-1}\right.\\
&\qquad\qquad\left.-\frac{[(2i-1)\pi+x]\exp\frac{(d-a_1)[(2i-1)\pi+x]}{r}}{\exp\frac{d[(2i-1)\pi+x]}{r}-1}\right]\sin x\,\mathrm{d}x. \qquad (35)
\end{aligned}$$

For $x>0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xe^{(d-a_1)x}}{e^{dx}-1}\right)=-\frac{x\bigl[a_1+\frac{1}{x}\bigl(\frac{dx}{e^{dx}-1}-1\bigr)\bigr]e^{(d-a_1)x}}{e^{dx}-1}, \qquad (36)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\left(\frac{dx}{e^{dx}-1}-1\right)\right]=\frac{1-2e^{dx}+e^{2dx}-d^2x^2e^{dx}}{x^2(e^{dx}-1)^2}, \qquad (37)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[1-2e^{dx}+e^{2dx}-d^2x^2e^{dx}\right]=2d^2xe^{dx}\left[\frac{e^{dx}-1}{dx}-1-\frac{dx}{2}\right]>0, \qquad (38)$$

$$\lim_{x\to0}\left[\frac{1}{x}\left(\frac{dx}{e^{dx}-1}-1\right)\right]=-\frac{d}{2}, \qquad (39)$$

then $\frac{\mathrm{d}}{\mathrm{d}x}\bigl[\frac{1}{x}\bigl(\frac{dx}{e^{dx}-1}-1\bigr)\bigr]>0$ and $\frac{1}{x}\bigl(\frac{dx}{e^{dx}-1}-1\bigr)>-\frac{d}{2}$. From $a_1>\frac{d}{2}$, it follows that $\frac{\mathrm{d}}{\mathrm{d}x}\bigl(\frac{xe^{(d-a_1)x}}{e^{dx}-1}\bigr)<0$, the function $\frac{xe^{(d-a_1)x}}{e^{dx}-1}$ decreases, and then for $x>0$ and $i\in\mathbb{N}$

$$\frac{(2i\pi+x)\exp\frac{(d-a_1)(2i\pi+x)}{r}}{\exp\frac{d(2i\pi+x)}{r}-1}<\frac{[(2i-1)\pi+x]\exp\frac{(d-a_1)[(2i-1)\pi+x]}{r}}{\exp\frac{d[(2i-1)\pi+x]}{r}-1}, \qquad (40)$$

thus, from inequality (29), we have

$$\begin{aligned}
\int_0^\infty\frac{xe^{(d-a_1)x}}{e^{dx}-1}\sin(rx)\,\mathrm{d}x&<\int_0^{\pi/r}\frac{xe^{(d-a_1)x}}{e^{dx}-1}\sin(rx)\,\mathrm{d}x\\
&<\int_0^{\pi/r}\frac{\sin(rx)}{de^{(a_1-\frac{d}{2})x}}\,\mathrm{d}x=\frac{4r\bigl[1+e^{\pi(d-2a_1)/(2r)}\bigr]}{d[(d-2a_1)^2+4r^2]}. \qquad (41)
\end{aligned}$$

Inequality (34) follows from combination of (41) with (24). □

Remark 2. The proof of Theorem 4 can be shortened by observing that

$$\frac{xe^{(d-a_1)x}}{e^{dx}-1}=\frac{x\exp\bigl[\bigl(\frac{d}{2}-a_1\bigr)x\bigr]}{2\sinh\bigl(\frac{dx}{2}\bigr)} \qquad (42)$$

and $\frac{\sinh x}{x}$ and $\exp\bigl[\bigl(a_1-\frac{d}{2}\bigr)x\bigr]$ are both increasing with $x>0$ for $a_1>\frac{d}{2}$.

This observation was given by Professor Lothar Berg at FB Mathematik der Universität, Universitätspl. 1, D-18055 Rostock, Germany, through an e-mail to the author on May 19, 2003.

By exploiting a technique presented by E. Makai in [11] and used by the author in [14], we obtain the following inequalities of Mathieu type series (13).

Theorem 5. *Suppose r is a positive number, then for a positive sequence $a = (a_1, a_2, \dots, a_k, \dots)$ and a positive real number $\alpha > 0$ satisfying $a_{k+1}^{\alpha/2} - a_k^{\alpha/2} = 1$, we have*

$$\frac{1}{2r^2+1} < \sum_{k=1}^{\infty} \frac{a_k^{\alpha/2}}{(a_k^{\alpha}+r^2)^2} < \frac{1}{2r^2}. \tag{43}$$

Proof. By standard argument, we obtain

$$\begin{aligned} &\frac{1}{\left(a_k^{\alpha/2}-\frac{1}{2}\right)^2+r^2-\frac{1}{4}} - \frac{1}{\left(a_k^{\alpha/2}+\frac{1}{2}\right)^2+r^2-\frac{1}{4}} \\ &= \frac{2a_k^{\alpha/2}}{\left(a_k^{\alpha}+r^2-a_k^{\alpha/2}\right)\left(a_k^{\alpha}+r^2+a_k^{\alpha/2}\right)} \\ &> \frac{2a_k^{\alpha/2}}{\left(a_k^{\alpha}+r^2\right)^2} \\ &> \frac{2a_k^{\alpha/2}}{\left(a_k^{\alpha}+r^2\right)^2+r^2+\frac{1}{4}} \\ &= \frac{1}{\left(a_k^{\alpha/2}-\frac{1}{2}\right)^2+r^2+\frac{1}{4}} - \frac{1}{\left(a_k^{\alpha/2}+\frac{1}{2}\right)^2+r^2+\frac{1}{4}}, \end{aligned}$$

summing for $k = 1, 2, \dots$ yields inequalities in (43). □

4. Two open problems

Now we will propose two open problems for interesting readers to discuss.

Open Problem 1. *Let $r > 0$, $t > 0$, $\alpha > 0$, $\beta > 0$ and $a = (a_1, a_2, \dots, a_k, \dots)$ be a positive sequence, define*

$$S(r,t,\alpha,\beta,a) = \sum_{k=1}^{\infty} \frac{a_k^{\beta}}{(a_k^{\alpha}+r^2)^t}. \tag{44}$$

1. *Under what conditions does the sequence $S(r,t,\alpha,\beta,a)$ converge?*
2. *Can one obtain an integral expression for the series $S(r,t,\alpha,\beta,a)$?*
3. *Can one establish a sharp double inequality for the series $S(r,t,\alpha,\beta,a)$?*

Open Problem 2. *For $r > 0$, we have*

$$\left[\int_0^{\infty} \frac{x\sin(rx)}{e^x-1}\,\mathrm{d}x\right]^2 > 2r^2 \int_0^{\infty} \frac{x^2 f(x)}{e^{r^2 x}}\,\mathrm{d}x, \tag{45}$$

where $f(x) = \sum_{k=1}^{\infty} ke^{-k^2 x}$.

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