Generalizations of Weighted Trapezoidal Inequality for Monotonic Mappings and Its Applications

GENERALIZATIONS OF WEIGHTED TRAPEZOIDAL INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATIONS

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Abstract. In this paper, we establish some generalizations of weighted trapezoid inequality for monotonic mappings, and give several applications for $r-moment$, the expectation of a continuous random variable and the Beta mapping.

1. Introduction

The *trapezoid inequality*, states that if f'' exists and is bounded on (a,b), then

$$\left|\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^3}{12}\left\|f''\right\|_\infty, \tag{1.1}$$

where

$$\left\|f''\right\|_\infty:=\sup_{x\in(a,b)}\left|f''\right|<\infty.$$

Now if we assume that $I_n:a=x_0<x_1<\cdots<x_n=b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *trapezoidal quadrature formula* $A_T(f,I_n)$, having an error given by $R_T(f,I_n)$, where

$$A_T(f,I_n):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]l_i, \tag{1.2}$$

and the remainder satisfies the estimation

$$\left|R_T(f,I_n)\right|\leq\frac{1}{12}\left\|f''\right\|_\infty\sum_{i=0}^{n-1}l_i^3, \tag{1.3}$$

with $l_i:=x_{i+1}-x_i$ for $i=0,1,\ldots,n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2] – [8].

Recently, Cerone-Dragomir [3] proved the following two trapezoid type inequalities:

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Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic non-decreasing mapping. Then*

$$\begin{aligned}(1.4)\qquad & \left|\int_a^b f(t)dt - \left[(x-a)\,f\,(a) + (b-x)\,f\,(b)\right]\right| \\ &\leq (b-x)\,f\,(b) - (x-a)\,f\,(a) + \int_a^b \operatorname{sgn}(x-t)\,f\,(t)\,dt \\ &\leq (x-a)\left[f\,(x) - f\,(a)\right] + (b-x)\left[f\,(b) - f\,(x)\right] \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[f\,(b) - f\,(a)\right],\end{aligned}$$

for all $x \in [a,b]$. The above inequalities are sharp.

Let I_n, l_i $(i = 0, 1, \ldots, n-1)$ be as above and let $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ be intermediate points. Define the sum

$$T_P\,(f, I_n, \xi) := \sum_{i=0}^{n-1}\left[(\xi_i - x_i)\,f\,(x_i) + (x_{i+1} - \xi_i)\,f\,(x_{i+1})\right].$$

We have the following result concerning the approximation of the integral $\int_a^b f(x)dx$ in terms of T_P.

Theorem B. *Let f be defined as in Theorem A, then we have*

$$(1.5)\qquad \int_a^b f(x)dx = T_P\,(f, I_n, \xi) + R_P\,(f, I_n, \xi)\,.$$

The remainder term $R_P\,(f, I_n, \xi)$ satisfies the estimate

$$\begin{aligned}(1.6)\qquad & \left|R_P\,(f, I_n, \xi)\right| \\ &\leq \sum_{i=0}^{n-1}\left[(x_{i+1} - \xi_i)\,f\,(x_{i+1}) - (\xi_i - x_i)\,f\,(x_i)\right] \\ &\qquad + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \operatorname{sgn}(\xi_i - t)\,f\,(t)\,dt \\ &\leq \sum_{i=0}^{n-1}(\xi_i - x_i)\left[f\,(\xi_i) - f\,(x_i)\right] + \sum_{i=0}^{n-1}(x_{i+1} - \xi_i)\left[f\,(x_{i+1}) - f\,(\xi_i)\right] \\ &\leq \sum_{i=0}^{n-1}\left[\frac{1}{2}l_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\left[f\,(x_{i+1}) - f\,(x_i)\right] \\ &\leq \left[\frac{1}{2}\nu\,(l) + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\left[f\,(b) - f\,(a)\right] \\ &\leq \nu\,(l)\left[f\,(b) - f\,(a)\right]\end{aligned}$$

where $\nu\,(l) := \max\{l_i\,|i = 0, 1, \ldots, n-1\}$.

In this paper, we establish weighted generalizations of Theorems A-B, and give several applications for $r-$moments and the expectation of a continuous random variable, the Beta mapping and the Gamma mapping.

2. Some Integral Inequalities

Theorem 1. *Let $g : [a,b] \to \mathbb{R}$ be non-negative and continuous with $g\,(t) > 0$ on (a,b) and let $h : [a,b] \to \mathbb{R}$ be differentiable such that $h'\,(t) = g\,(t)$ on $[a,b]$.*

(a) *Suppose* $f:[a,b]\to\mathbb{R}$ *is a monotonic non-decreasing mapping. Then*

$$
\begin{aligned}
(2.1)\qquad & \left|\int_a^b f(t)g(t)\,dt-\left[(x-h(a))\,f(a)+(h(b)-x)\,f(b)\right]\right| \\
& \le (h(b)-x)\,f(b)-(x-h(a))\,f(a)+\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right)f(t)\,g(t)\,dt \\
& \le (x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right] \\
& \le \left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot\left[f(b)-f(a)\right]
\end{aligned}
$$

for all $x\in[h(a),h(b)]$.

(b) *Suppose* $f:[a,b]\to\mathbb{R}$ *is a monotonic non-increasing mapping. Then*

$$
\begin{aligned}
(2.2)\qquad & \left|\int_a^b f(t)g(t)\,dt-\left[(x-h(a))\,f(a)+(h(b)-x)\,f(b)\right]\right| \\
& \le (x-h(a))\,f(a)-(h(b)-x)\,f(b)+\int_a^b \operatorname{sgn}\left(t-h^{-1}(x)\right)f(t)\,g(t)\,dt \\
& \le (x-h(a))\cdot\left[f(a)-f\left(h^{-1}(x)\right)\right]+(h(b)-x)\cdot\left[f\left(h^{-1}(x)\right)-f(b)\right] \\
& \le \left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot\left[f(a)-f(b)\right]
\end{aligned}
$$

for all $x\in[h(a),h(b)]$.

The above inequalities are sharp.

Proof. (1)

(a) Let $x\in[h(a),h(b)]$. Using integration by parts, we have the following identity

$$
\begin{aligned}
(2.3)\qquad & \int_a^b (x-h(t))\,df(t) \\
& =(x-h(t))\,f(t)\Big|_a^b+\int_a^b f(t)g(t)\,dt \\
& =\int_a^b f(t)g(t)\,dt-\left[(x-h(a))\,f(a)+(h(b)-x)\,f(b)\right].
\end{aligned}
$$

It is well known [3, p. 813] that if $\mu,\nu:[a,b]\to\mathbb{R}$ are such that μ is continuous on $[a,b]$ and ν is monotonic non-decreasing on $[a,b]$, then

$$
(2.4)\qquad \left|\int_a^b \mu(t)\,d\nu(t)\right|\le\int_a^b|\mu(t)|\,d\nu(t).
$$

Now, using identity (2.3) and inequality (2.4), we have

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt - \left[(x-h(a))\,f(a) + (h(b)-x)\,f(b)\right]\right| \qquad (2.5)\\
&\le \int_a^b |x-h(t)|\,df(t)\\
&= \int_a^{h^{-1}(x)} (x-h(t))\,df(t) + \int_{h^{-1}(x)}^b (h(t)-x)\,df(t)\\
&= (x-h(t))\,f(t)\Big|_a^{h^{-1}(x)} + \int_a^{h^{-1}(x)} f(t)\,g(t)\,dt\\
&\qquad + (h(t)-x)\Big|_{h^{-1}(x)}^b - \int_{h^{-1}(x)}^b f(t)\,g(t)\,dt\\
&= (h(b)-x)\,f(b) - (x-h(a))\,f(a) + \int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right) f(t)\,g(t)\,dt
\end{aligned}$$

and the first inequalities in (2.1) are proved.

As f is monotonic non-decreasing on $[a,b]$, we obtain

$$\begin{aligned}
\int_a^{h^{-1}(x)} f(t)\,g(t)\,dt &\le f\left(h^{-1}(x)\right)\int_a^{h^{-1}(x)} g(t)\,dt\\
&= (x-h(a))\,f\left(h^{-1}(x)\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{h^{-1}(x)}^b f(t)\,g(t)\,dt &\ge f\left(h^{-1}(x)\right)\int_{h^{-1}(x)}^b g(t)\,dt\\
&= (h(b)-x)\,f\left(h^{-1}(x)\right),
\end{aligned}$$

then

$$\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right) f(t)\,g(t)\,dt \le (x-h(a))\,f\left(h^{-1}(x)\right) + (x-h(b))\,f\left(h^{-1}(x)\right).$$

Therefore,

$$\begin{aligned}
&(h(b)-x)\,f(b) - (x-h(a))\,f(a) + \int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right) f(t)\,g(t)\,dt \qquad (2.6)\\
&\le (h(b)-x)\,f(b) - (x-h(a))\,f(a)\\
&\qquad + (x-h(a))\,f\left(h^{-1}(x)\right) + (x-h(b))\,f\left(h^{-1}(x)\right)\\
&= (x-h(a))\cdot\left[f\left(h^{-1}(x)\right) - f(a)\right] + (h(b)-x)\cdot\left[f(b) - f\left(h^{-1}(x)\right)\right]
\end{aligned}$$

which proves that the second inequality in (2.1).

As f is monotonic non-decreasing on $[a,b]$, we have

$$f(a) \le f\left(h^{-1}(x)\right) \le f(b)$$

and

$$
\begin{aligned}
(2.7)\qquad & (x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right] \\
&\leq \max\{x-h(a),h(b)-x\}\cdot\left[f\left(h^{-1}(x)\right)-f(a)+f(b)-f\left(h^{-1}(x)\right)\right] \\
&=\left[\frac{h(b)-h(a)}{2}+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)] \\
&=\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)].
\end{aligned}
$$

Thus, by (2.5), (2.6) and (2.7), we obtain (2.1).

Let

$$
\begin{aligned}
&g(t)\equiv 1,\ t\in[a,b] \\
&h(t)=t,\ t\in[a,b] \\
&f(t)=\begin{cases}0, & t\in[a,b) \\ 1, & t=b\end{cases}
\end{aligned}
$$

and $x=\frac{a+b}{2}$. Then

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))\,f(a)+(h(b)-x)\,f(b)]\right| \\
&=(h(b)-x)\,f(b)-(x-h(a))\,f(a)+\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right)f(t)\,g(t)\,dt \\
&=(x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right] \\
&=\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)] \\
&=\frac{b-a}{2}
\end{aligned}
$$

which proves that the inequalities (2.1) are sharp.

(b) If f is replaced by $-f$ in (a), then (2.2) is obtained from (2.1).

This completes the proof. ∎

Remark 1. *If we choose $g(t)\equiv 1, h(t)=t$ on $[a,b]$, then the inequalities (2.1) reduce to (1.4).*

Corollary 1. *If we choose $x=\frac{h(a)+h(b)}{2}$, then we get*

$$
\begin{aligned}
(2.8)\qquad & \left|\int_a^b f(t)g(t)\,dt-\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \\
&\leq \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)] \\
&\qquad\qquad +\int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)-t\right)f(t)\,g(t)\,dt \\
&\leq \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)]
\end{aligned}
$$

where f and g are defined as in (a) of Theorem 1, and

$$\begin{aligned}(2.9)\qquad &\left|\int_a^b f(t)g(t)\,dt-\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)] \\ &\qquad +\int_a^b \operatorname{sgn}\left(t-h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)f(t)\,g(t)\,dt \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)]\end{aligned}$$

where f and g are defined as in (b) of Theorem 1.

The inequalities (2.8) and (2.9) are the "*weighted trapezoid*" inequalities.

Note that the trapezoid inequality (2.8) and (2.9) are, in a sense, the best possible inequalities we can obtain from (2.1) and (2.2). Moreover, the constant $\frac{1}{2}$ is the best possible for both inequalities in (2.8) and (2.9), respectively.

Remark 2. *The following inequality is well-known in the literature as the Fejér inequality (see for example* [9]*):*

$$(2.10)\qquad f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt\le\int_a^b f(t)g(t)\,dt\le\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt,$$

where $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to\mathbb{R}$ is positive integrable and symmetric to $\frac{a+b}{2}$.

Using the above results and (2.8) – (2.9), we obtain the following error bound of the second inequality in (2.10):

$$\begin{aligned}(2.11)\qquad 0&\le\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)] \\ &\qquad +\int_a^b \operatorname{sgn}\left(h^{-1}(\frac{h(a)+h(b)}{2})-t\right)f(t)\,g(t)\,dt \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)]\end{aligned}$$

provided that f is monotonic non-decreasing on $[a,b]$.

$$\begin{aligned}(2.12)\qquad 0&\le\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)] \\ &\qquad +\int_a^b \operatorname{sgn}\left(t-h^{-1}(\frac{h(a)+h(b)}{2})\right)f(t)\,g(t)\,dt \\ &\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)]\end{aligned}$$

provided that f is monotonic non-increasing on $[a,b]$.

3. Applications for Quadrature Formula

Throughout this section, let g and h be defined as in Theorem 1.

Let $f : [a, b] \to \mathbb{R}$, and let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$ and $\xi_i \in [h(x_i), h(x_{i+1})]$ $(i = 0, 1, \ldots, n-1)$ be intermediate points. Put $l_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t)\, dt$ and define the sum

$$T_P(f, g, h, I_n, \xi) := \sum_{i=0}^{n-1} \left[(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1}) \right].$$

We have the following result concerning the approximation of the integral $\int_a^b f(t) g(t)\, dt$ in terms of T_P.

Theorem 2. *Let $\nu(l) := \max\{l_i \,|i = 0, 1, \ldots, n-1\}$, f be defined as in Theorem 1 and let*

$$\int_a^b f(t) g(t)\, dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi). \tag{3.1}$$

Then, the remainder term $R_P(f, g, h, I_n, \xi)$ satisfies the following estimates:

(a) *Suppose f is monotonic non-decreasing on $[a, b]$, then*

$$\begin{aligned}
&|R_P(f, g, h, I_n, \xi)| \\
&\leq \sum_{i=0}^{n-1} \left[(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i) \right] \\
&\qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}(\xi_i) - t \right) f(t) g(t)\, dt \\
&\leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) \left[f\left(h^{-1}(\xi_i) \right) - f(x_i) \right] \\
&\qquad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) \left[f(x_{i+1}) - f\left(h^{-1}(\xi_i) \right) \right] \\
&\leq \sum_{i=0}^{n-1} \left[\frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot \left[f(x_{i+1}) - f(x_i) \right] \\
&\leq \left[\frac{1}{2} \nu(l) + \max_{i=0,1,\ldots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot \left[f(b) - f(a) \right] \\
&\leq \nu(l) \left[f(b) - f(a) \right].
\end{aligned} \tag{3.2}$$

(b) *Suppose f is monotonic non-increasing on $[a, b]$, then*

$$\begin{aligned}
&|R_P(f, g, h, I_n, \xi)| \\
&\leq \sum_{i=0}^{n-1} \left[(\xi_i - h(x_i)) f(x_i) - (h(x_{i+1}) - \xi_i) f(x_{i+1}) \right] \\
&\qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(t - h^{-1}(\xi_i) \right) f(t) g(t)\, dt
\end{aligned} \tag{3.3}$$

$$\leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) \left[f(x_i) - f\left(h^{-1}(\xi_i)\right) \right] + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) \left[f\left(h^{-1}(\xi_i)\right) - f(x_{i+1}) \right]$$

$$\leq \sum_{i=0}^{n-1} \left[\frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(x_i) - f(x_{i+1})]$$

$$\leq \left[\frac{1}{2} \nu(l) + \max_{i=0,1,\ldots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(a) - f(b)]$$

$$\leq \nu(l) [f(a) - f(b)].$$

Proof. (1)

(a) Apply Theorem 1 on the intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(t) g(t)\, dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right|$$

$$\leq (h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i) + \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}(\xi_i) - t\right) f(t) g(t)\, dt$$

$$\leq (\xi_i - h(x_i)) \cdot \left[f\left(h^{-1}(\xi_i)\right) - f(x_i) \right] + (h(x_{i+1}) - \xi_i) \cdot \left[f(x_{i+1}) - f\left(h^{-1}(\xi_i)\right) \right]$$

$$\leq \left[\frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(x_{i+1}) - f(x_i)]$$

for all $i \in \{0, 1, \ldots, n-1\}$.

Using this and the generalized triangle inequality, we have

$$
\begin{aligned}
(3.4)\quad & \left|R_P\left(f,g,h,I_n,\xi\right)\right| \\
&\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)g\left(t\right)dt-\left[\left(\xi_i-h(x_i)\right)f\left(x_i\right)+\left(h(x_{i+1})-\xi_i\right)f\left(x_{i+1}\right)\right]\right| \\
&\le \sum_{i=0}^{n-1}\left[\left(h\left(x_{i+1}\right)-\xi_i\right)f\left(x_{i+1}\right)-\left(\xi_i-h\left(x_i\right)\right)f\left(x_i\right)\right] \\
&\qquad +\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\operatorname{sgn}\left(h^{-1}\left(\xi_i\right)-t\right)f\left(t\right)g\left(t\right)dt \\
&\le \sum_{i=0}^{n-1}\left(\xi_i-h\left(x_i\right)\right)\left[f\left(h^{-1}\left(\xi_i\right)\right)-f\left(x_i\right)\right] \\
&\qquad +\sum_{i=0}^{n-1}\left(h\left(x_{i+1}\right)-\xi_i\right)\left[f\left(x_{i+1}\right)-f\left(h^{-1}\left(\xi_i\right)\right)\right] \\
&\le \sum_{i=0}^{n-1}\left[\frac{1}{2}l_i+\left|\xi_i-\frac{h\left(x_i\right)+h\left(x_{i+1}\right)}{2}\right|\right]\left[f\left(x_{i+1}\right)-f\left(x_i\right)\right] \\
&\le \left[\frac{1}{2}\nu\left(l\right)+\max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{h\left(x_i\right)+h\left(x_{i+1}\right)}{2}\right|\right]\left[f\left(b\right)-f\left(a\right)\right].
\end{aligned}
$$

Next, we observe that

$$
(3.5)\qquad \left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\le\frac{1}{2}l_i\ \ (i=0,1,\ldots,n-1);
$$

and then

$$
(3.6)\qquad \max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\le\frac{1}{2}\nu\left(l\right).
$$

Thus, by (3.4), (3.5) and (3.6), we obtain (3.2).

(b) The proof is similar as (a) and we omit the details.

This completes the proof. ∎

Remark 3. *If we choose $g\left(t\right)\equiv 1, h\left(t\right)=t$ on $[a,b]$, then the inequalities (3.2) reduce to (1.6).*

Now, let $\xi_i=\frac{h(x_i)+h(x_{i+1})}{2}$ $(i=0,1,\ldots,n-1)$ and let $T_{PW}\left(f,g,h,I_n\right)$ and $R_P\left(f,g,h,I_n\right)$ be defined as

$$
T_{PW}\left(f,g,h,I_n\right)=T_P\left(f,g,h,I_n,\xi\right)=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(x_i\right)+f\left(x_{i+1}\right)\right]\int_{x_i}^{x_{i+1}}g\left(t\right)dt
$$

and

$$
\begin{aligned}
R_{PW}\left(f,g,h,I_n\right)&=R_P\left(f,g,h,I_n,\xi\right) \\
&=\int_a^b f\left(t\right)g\left(t\right)dt-\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(x_i\right)+f\left(x_{i+1}\right)\right]\int_{x_i}^{x_{i+1}}g\left(t\right)dt.
\end{aligned}
$$

If we consider the *weighted trapezoidal formula* $T_{PW}\left(f,g,h,I_n\right)$, then we have the following corollary:

Corollary 2. *Let f, g, h be defined as in Theorem 2 and let* $\xi_i = \frac{h(x_i)+h(x_{i+1})}{2}$ $(i = 0, 1, \ldots, n-1)$. *Then*

$$\int_a^b f(t)\, g(t)\, dt = T_{PW}(f, g, h, I_n) + R_{PW}(f, g, h, I_n)$$

where the remainder satisfies the following estimates:

(a) *Suppose f is monotonic non-decreasing on $[a, b]$, then*

$$\begin{aligned}(3.7)\qquad & |R_{PW}(f, g, h, I_n)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t)\, dt \right) [f(x_{i+1}) - f(x_i)] \\ & \qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}\left(\frac{h(x_i) + h(x_{i+1})}{2} \right) - t \right) f(t)\, g(t)\, dt \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t)\, dt \right) \cdot [f(x_{i+1}) - f(x_i)] \\ & \leq \frac{\nu(l)}{2} \cdot [f(b) - f(a)].\end{aligned}$$

(b) *Suppose f is monotonic non-increasing on $[a, b]$, then*

$$\begin{aligned}(3.8)\qquad & |R_{PW}(f, g, h, I_n)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t)\, dt \right) [f(x_i) - f(x_{i+1})] \\ & \qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(t - h^{-1}\left(\frac{h(x_i) + h(x_{i+1})}{2} \right) \right) f(t)\, g(t)\, dt \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t)\, dt \right) \cdot [f(x_i) - f(x_{i+1})] \\ & \leq \frac{\nu(l)}{2} \cdot [f(a) - f(b)].\end{aligned}$$

Remark 4. *In Corollary 2, suppose f is monotonic on $[a, b]$,*

$$x_i = h^{-1}\left[h(a) + \frac{i(h(b) - h(a))}{n} \right] \qquad (i = 0, 1, \ldots, n),$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b) - h(a)}{n} = \frac{1}{n} \int_a^b g(t)\, dt. \qquad (i = 0, 1, \ldots, n-1).$$

If we want to approximate the integral $\int_a^b f(t)\, g(t)\, dt$ by $T_{PW}(f, g, h, I_n)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbb{N}$ points for the partation I_n, where

$$n_\varepsilon := \left[\frac{1}{2\varepsilon} \int_a^b g(t)\, dt \cdot |f(b) - f(a)| \right] + 1$$

and $[r]$ denotes the Gaussian integer of r $(r \in \mathbb{R})$.

4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density mapping $g : [a, b] \to \mathbb{R}$ with $g(t) > 0$ on (a, b), $h : [a, b] \to \mathbb{R}$ with $h'(t) = g(t)$ for $t \in (a, b)$ and the $r-$moment

$$E_r(X) := \int_a^b t^r g(t)\, dt,$$

which is assumed to be finite.

Theorem 3. *The inequalities*

$$\left| E_r(X) - \frac{a^r + b^r}{2} \right| \le \frac{1}{2}(b^r - a^r) + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right) - t \right) t^r g(t)\, dt \le \frac{1}{2}(b^r - a^r) \quad \text{as } r \ge 0 \tag{4.1}$$

and

$$\left| E_r(X) - \frac{a^r + b^r}{2} \right| \le \frac{1}{2}(a^r - b^r) + \int_a^b \operatorname{sgn}\left(t - h^{-1}\left(\frac{1}{2}\right) \right) t^r g(t)\, dt \le \frac{1}{2}(a^r - b^r) \quad \text{as } r < 0, \tag{4.2}$$

hold.

Proof. If we put $f(t) = t^r \ (t \in [a, b])$, $h(t) = \int_a^t g(x)\, dx \ (t \in [a, b])$and $x = \frac{h(a)+h(b)}{2} = \frac{1}{2}$ in Corollary 1, then we obtain (4.1) and (4.2). This completes the proof. ∎

The following corollary which is a special case of Theorem 3.

Corollary 3. *The inequalities*

$$\left| E(X) - \frac{a+b}{2} \right| \le \frac{b-a}{2} + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right) - t \right) t g(t)\, dt \le \frac{b-a}{2} \tag{4.3}$$

hold where $E(X)$ is the expectation of the random variable X.

5. Inequalities for Beta Mapping and Gamma Mapping

The following two mappings are well-known in the literature as the *Beta mapping* and the *Gamma mapping*, respectively:

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x > 0,\ y > 0.$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0.$$

The following inequality which is an applicaton of Theorem 1 for the Beta mapping holds:

Theorem 4. *Let p, $q > 0$. Then we have the inequality*

$$\begin{aligned} &|\beta(p+1,q+1)-x| \\ &\le x+\int_a^b \operatorname{sgn}\left[t-((p+1)x)^{\frac{1}{p+1}}\right] t^p(1-t)^q\,dt \\ &\le x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)x)^{\frac{1}{p+1}}\right]^q \\ &\le \frac{1}{2(p+1)}+\left|x-\frac{1}{2(p+1)}\right| \end{aligned} \tag{5.1}$$

for all $x\in\left[0,\frac{1}{p+1}\right]$.

Proof. If we put $a=0$, $b=1$, $f(t)=(1-t)^q$, $g(t)=t^p$ and $h(t)=\frac{t^{p+1}}{p+1}$ $(t\in[0,1])$ in Theorem 1, we obtain the inequality (5.1) for all $x\in\left[0,\frac{1}{p+1}\right]$. This completes the proof. ∎

The following remark which is an applicaton of Theorem 4 for the Gamma mapping holds:

Remark 5. *Taking into account that* $\beta(p+1,q+1)=\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$, *the inequality (5.1) is equivalent to*

$$\begin{aligned} \left|\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}-x\right| &\le x+\int_a^b \operatorname{sgn}\left[t-((p+1)x)^{\frac{1}{p+1}}\right] t^p(1-t)^q\,dt \\ &\le x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)x)^{\frac{1}{p+1}}\right]^q \\ &\le \frac{1}{2(p+1)}+\left|x-\frac{1}{2(p+1)}\right| \end{aligned}$$

i.e.,

$$\begin{aligned} &|(p+1)\Gamma(p+1)\Gamma(q+1)-x(p+1)\Gamma(p+q+2)| \\ &\le \left[x+\int_a^b \operatorname{sgn}\left[t-((p+1)x)^{\frac{1}{p+1}}\right] t^p(1-t)^q\,dt\right](p+1)\Gamma(p+q+2) \\ &\le \left[x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)x)^{\frac{1}{p+1}}\right]^q\right](p+1)\Gamma(p+q+2) \\ &\le \left[\frac{1}{2}+\left|x(p+1)-\frac{1}{2}\right|\right]\cdot\Gamma(p+q+2) \end{aligned}$$

and as $(p+1)\Gamma(p+1)=\Gamma(p+2)$, *we get*

$$\begin{aligned} &|\Gamma(p+2)\Gamma(q+1)-x(p+1)\Gamma(p+q+2)| \\ &\le \left[x+\int_a^b \operatorname{sgn}\left[t-((p+1)x)^{\frac{1}{p+1}}\right] t^p(1-t)^q\,dt\right](p+1)\Gamma(p+q+2) \\ &\le \left[x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)x)^{\frac{1}{p+1}}\right]^q\right](p+1)\Gamma(p+q+2) \\ &\le \left[\frac{1}{2}+\left|x(p+1)-\frac{1}{2}\right|\right]\cdot\Gamma(p+q+2). \end{aligned} \tag{5.2}$$

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