New Bounds for the Čebyšev Functional

NEW BOUNDS FOR THE ČEBYŠEV FUNCTIONAL

P. CERONE AND S.S. DRAGOMIR

Abstract. In this paper some new inequalities for the Čebyšev functional are presented. They have applications in a variety of branches of applied mathematics.

1. Introduction

Over the last five years, the development of Grüss type inequalities has experienced a surge, having been stimulated by their applications in different branches of Applied Mathematics including: in perturbed quadrature rules (see for example [13], [14]) and in the approximation of integral transforms (see [15], [16]) and the references therein.

The main aim of the present paper is to point out other such results that may be used as new tools in obtaining perturbed version of classical quadrature or new approximations for different kinds of integral operators encountered in various branches of Applied Mathematics. Now for some preliminaries.

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.1}$$

In 1934, G. Grüss [4] showed that

$$|T(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided m, M, n, N are real numbers with the property

$$-\infty < m \le f \le M < \infty, \ -\infty < n \le g \le N < \infty \ \text{ a.e. on } [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller one. Another less well known inequality for $T(f,g)$ was derived in 1882 by Čebyšev [3] under the assumption that f', g' exist and are continuous in $[a,b]$ and is given by

$$|T(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)|$.

The constant $\frac{1}{12}$ cannot be improved in the general case.

Čebyšev's inequality (1.4) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty [a,b]$.

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In 1970, A.M. Ostrowski [5] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{1.5}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.3) with $g:[a,b]\to\mathbb{R}$ being absolutely continuous and $g'\in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

Finally, let us recall a result by Lupaş (see for example [1, p. 210]), which states that:

$$|T(f,g;a,b)| \leq \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.6}$$

provided f,g are absolutely continuous and $f',g'\in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here also.

For other Grüss type integral inequalities, see the books [1], [2], and the papers [6]-[11], where further references are given.

2. Bounds for the Čebyšev Functional

The following lemma holds and it will prove useful for procuring specifically an inequality for the Čebyšev functional in terms of the Hilbertian norm. The lemma is also of intrinsic interest in its own right.

Lemma 1. *Assume that the function $f:[a,b]\to\mathbb{R}$ is Lebesgue integrable and $\int_a^b f(x)\,dx=0$. Define $F(x)=\int_a^x f(t)\,dt$, $e(x)=x$, $x\in[a,b]$ and assume that $F,f,ef\in L_2[a,b]$. Then we have the inequality*

$$\begin{aligned}\int_a^b F^2(x)\,dx &\leq 4\cdot\frac{\int_a^b f^2(t)\,dt\int_a^b t^2f^2(t)\,dt-\left(\int_a^b tf^2(t)\,dt\right)^2}{\int_a^b f^2(t)\,dt}\\ &\leq 4\int_a^b t^2f^2(t)\,dt.\end{aligned} \tag{2.1}$$

Proof. For a given $\lambda\in\mathbb{R}$ we have, on integrating the Lebesgue integral by parts,

$$\begin{aligned}\int_a^b [F(x)]^2\,dx &= \int_a^b [F(x)]^2\,d(x-\lambda)\\ &= F^2(x)(x-\lambda)\Big|_a^b-\int_a^b(x-\lambda)\frac{d}{dx}\left(F^2(x)\right)dx\\ &= 2\int_a^b(\lambda-x)f(x)F(x)\,dx.\end{aligned} \tag{2.2}$$

Using the Cauchy-Schwartz-Buniakowski inequality for integrals, we have

$$\int_a^b(\lambda-x)f(x)F(x)\,dx \leq \left[\int_a^b(\lambda-x)^2f^2(x)\,dx\right]^{\frac{1}{2}}\left(\int_a^b[F(x)]^2\,dx\right)^{\frac{1}{2}}. \tag{2.3}$$

Combining (2.2) with (2.3) and dividing by $\left(\int_a^b[F(x)]^2\,dx\right)^{\frac{1}{2}}\geq 0$ (since we may assume that $f\neq 0$), we deduce

$$\left(\int_a^b[F(x)]^2\,dx\right)^{\frac{1}{2}}\leq 2\left[\int_a^b(\lambda-x)^2f^2(x)\,dx\right]^{\frac{1}{2}}$$

for any $\lambda \in \mathbb{R}$, which is clearly equivalent to

$$\int_a^b [F(x)]^2 \, dx \leq 4 \int_a^b (\lambda - x)^2 f^2(x) \, dx, \quad \lambda \in \mathbb{R}. \tag{2.4}$$

Taking the infimum in (2.4) for $\lambda \in \mathbb{R}$, we deduce

$$\int_a^b [F(x)]^2 \, dx \leq 4 \inf_{\lambda \in \mathbb{R}} g(\lambda), \tag{2.5}$$

where

$$g(\lambda) := \int_a^b (\lambda - x)^2 f^2(x) \, dx.$$

Now, observe that

$$g(\lambda) = \lambda^2 \int_a^b f^2(x) \, dx - 2\lambda \int_a^b x f^2(x) \, dx + \int_a^b x^2 f^2(x) \, dx.$$

Since

$$\inf_{\lambda \in \mathbb{R}} g(\lambda) = \frac{\int_a^b f^2(x) \, dx \int_a^b x^2 f^2(x) \, dx - \left(\int_a^b x f^2(x) \, dx \right)^2}{\int_a^b f^2(x) \, dx},$$

then by (2.5) we deduce the desired inequality (2.1). □

If $a > 0$, we may point out the following inequality that may be easier to apply in practice.

Corollary 1. *Assume that f satisfies the assumptions in Lemma 1 and $0 < a < b$. Then we have the inequality*

$$\int_a^b F^2(x) \, dx \leq \frac{(b-a)^2}{ab} \cdot \frac{\left(\int_a^b t f^2(t) \, dt \right)^2}{\int_a^b f^2(t) \, dt} \leq \frac{(b-a)^2}{ab} \cdot \int_a^b f^2(t) \, t^2 dt. \tag{2.6}$$

Proof. We use the following integral version of Cassels' inequality (see for example [12])

$$\frac{\int_a^b p(t) \, l^2(t) \, dt \int_a^b p(t) \, h^2(t) \, dt}{\left(\int_a^b p(t) \, l(t) \, h(t) \, dt \right)^2} \leq \frac{(M+m)^2}{4mM}, \tag{2.7}$$

provided

$$0 < m \leq \frac{h(t)}{l(t)} \leq M < \infty \text{ for } a.e.t \text{ on } [a,b]$$

and $p \geq 0$ a.e. on $[a,b]$.

Applying (2.7) for $p(t) = f^2(t)$, $l(t) = 1$, $h(t) = t$, $t \in [a,b]$, we get

$$\frac{\int_a^b f^2(t) \, dt \int_a^b f^2(t) \, t^2 dt}{\left(\int_a^b t f^2(t) \, dt \right)^2} \leq \frac{(a+b)^2}{4ab} \tag{2.8}$$

giving

$$\int_a^b f^2(t) \, dt \int_a^b t^2 f^2(t) \, dt - \left(\int_a^b t f^2(t) \, dt \right)^2 \leq \frac{(b-a)^2}{4ab} \left(\int_a^b t f^2(t) \, dt \right)^2. \tag{2.9}$$

Using (2.1) and (2.9), we deduce the first inequality in (2.6).

The last inequality is obvious by Schwartz's inequality. □

The following Grüss type inequality holds for the Čebyšev functional $T(f,g)$.

Theorem 1. *Assume that* $f:[a,b]\to\mathbb{R}$ *is a measurable function on* $[a,b]$ *and such that* $\bar{f}:=f-\frac{1}{b-a}\int_a^b f(t)\,dt$, $e\bar{f}\in L_2[a,b]$. *If* $g:[a,b]\to\mathbb{R}$ *is absolutely continuous and* $g'\in L_2[a,b]$, *then we have the inequality*

$$|T(f,g)| \le \frac{2}{b-a}\|g'\|_2\left[\frac{\int_a^b \bar{f}^2(t)\,dt\int_a^b t^2\bar{f}^2(t)\,dt-\left(\int_a^b t\bar{f}^2(t)\,dt\right)^2}{\int_a^b \bar{f}^2(t)\,dt}\right]^{\frac{1}{2}} \le \frac{2}{b-a}\|g'\|_2\left\|e\bar{f}\right\|_2. \tag{2.10}$$

Proof. Denote

$$\overline{F}(x)=\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(u)\,du=\int_a^x \bar{f}(t)\,dt,\quad x\in[a,b], \tag{2.11}$$

where, as above,

$$\bar{f}(t):=f(t)-\frac{1}{b-a}\int_a^b f(u)\,du,\quad t\in[a,b].$$

We observe that, on integrating by parts, we have

$$\frac{1}{b-a}\int_a^b \overline{F}(x)\,g'(x)\,dx=\frac{1}{b-a}\overline{F}(x)\,g(x)\Big|_a^b-\frac{1}{b-a}\int_a^b g(x)\,\overline{F}(t)\,dx = -T(f,g). \tag{2.12}$$

Taking the modulus in (2.12) and using the Cauchy-Buniakowski-Schwartz inequality, we have

$$|T(f,g)| \le \frac{1}{b-a}\int_a^b \left|\overline{F}(x)\right||g'(x)|\,dx \le \frac{1}{b-a}\left(\int_a^b |g'(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_a^b \left|\overline{F}(x)\right|^2 dx\right)^{\frac{1}{2}}. \tag{2.13}$$

Applying Lemma 1, we deduce the desired inequality (2.10). □

The following corollary holds.

Corollary 2. *With the assumptions of Theorem 1 and if* $0<a<b$, *then we have the inequality*

$$|T(f,g)| \le \frac{1}{\sqrt{ab}}\|g'\|_2\frac{\int_a^b t\bar{f}^2(t)\,dt}{\left(\int_a^b \bar{f}^2(t)\,dt\right)^{\frac{1}{2}}} \le \frac{1}{\sqrt{ab}}\|g'\|_2\left\|e\bar{f}\right\|_2. \tag{2.14}$$

3. Further Bounds

We observe that, indeed, for a Lebesgue integrable function f, $\overline{F}$ as defined by (2.11) is an absolutely continuous function and

$$\overline{F}(a) = \overline{F}(b) = 0. \tag{3.1}$$

We also must note that

$$\begin{aligned} \overline{F}(t) &= \frac{(b-t)\int_a^t f(s)\,ds - (t-a)\int_t^b f(s)\,ds}{b-a} \\ &= \frac{(b-t)\int_a^b f(s)\,ds - (b-a)\int_t^b f(s)\,ds}{b-a}. \end{aligned} \tag{3.2}$$

The following properties for $\overline{F}$ may also be stated.

(1) If $f \in L[a,b]$, then $\overline{F} \in BV[a,b]$ and

$$\bigvee_a^b \left(\overline{F}\right) = \int_a^b \left|\overline{f}(t)\right| dt. \tag{3.3}$$

(2) If $f \in L_\infty[a,b]$, then $\overline{F} \in Lip_K[a,b]$, where

$$K := \left\|\overline{f}\right\|_{[a,b],\infty}. \tag{3.4}$$

The following identity is useful in the sequel.

Lemma 2. *Assume that $f \in L[a,b]$ and $g \in C[a,b]$. Then for any $\gamma \in \mathbb{R}$, one has the identity*

$$T(f,g) = \frac{1}{b-a}\int_a^b (g(t)-\gamma)\, d\left(\overline{F}(t)\right). \tag{3.5}$$

Proof. If $f \in L[a,b]$, then obviously $\overline{F} \in AC[a,b]$ and since $g \in C[a,b]$, the Stieltjes integral in the right hand side of (3.5) exists.

Reducing to a Lebesgue integral, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b (g(t)-\gamma)\, d\left(\overline{F}(t)\right) &= \frac{1}{b-a}\int_a^b (g(t)-\gamma)\frac{d}{dt}\left(\overline{F}(t)\right) dt \\ &= \frac{1}{b-a}\int_a^b (g(t)-\gamma)\,\overline{f}(t)\,dt \\ &= T(f,g). \end{aligned}$$

□

The following corollary containing some particular cases of interest also holds.

Corollary 3. *Under the assumptions of Lemma 2, we have specifically*

$$T(f,g) = \frac{1}{b-a}\int_a^b g(t)\,d\left(\overline{F}(t)\right) \tag{3.6}$$
$$= \frac{1}{b-a}\int_a^b \left(g(t) - \frac{m+M}{2}\right) d\left(\overline{F}(t)\right) \quad (m, M \in \mathbb{R})$$
$$= \frac{1}{b-a}\int_a^b \left[g(t) - g\left(\frac{a+b}{2}\right)\right] d\left(\overline{F}(t)\right)$$
$$= \frac{1}{b-a}\int_a^b \overline{g}(t)\,d\left(\overline{F}(t)\right).$$

We may now state the first result relating to Lemma 2.

Theorem 2. *Let $f \in L[a,b]$ and $g \in C[a,b]$. Then one has the inequality*

$$|T(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \frac{1}{b-a}\int_a^b \left|\overline{f}(t)\right| dt. \tag{3.7}$$

Proof. If $f \in L[a,b]$, then obviously $\overline{F} \in AC[a,b]$ and

$$\frac{d\overline{F}(t)}{dt} = \overline{f}(t) \quad \text{for a.e.} \quad t \in [a,b]. \tag{3.8}$$

Since any absolutely continuous function is of bounded variation, it follows that $\overline{F} \in BV[a,b]$ and its total variation is

$$\bigvee_a^b \left(\overline{F}\right) = \int_a^b \left|\frac{d\overline{F}(t)}{dt}\right| dt = \int_a^b \left|\overline{f}(t)\right| dt. \tag{3.9}$$

It is known that if $p \in C[c,d]$ and $v \in BV[c,d]$, then

$$\left|\int_a^b p(s)\,dv(s)\right| \le \|p\|_\infty \bigvee_a^b (v). \tag{3.10}$$

Using the property, we deduce that

$$|T(f,g)| = \frac{1}{b-a}\left|\int_a^b (g(t)-\gamma)\,d\left(\overline{F}(t)\right)\right| \tag{3.11}$$
$$\le \frac{1}{b-a}\|g-\gamma\|_\infty \cdot \bigvee_a^b \left(\overline{F}\right)$$
$$= \|g-\gamma\|_\infty \frac{1}{b-a}\int_a^b \left|\overline{f}(t)\right| dt$$

for any $\gamma \in \mathbb{R}$.

Taking the infimum over $\gamma \in \mathbb{R}$, we deduce the desired inequality (3.7). □

There are a number of bounds that are coarser than (3.7) but may prove to be more useful in practical applications. They arise from taking particular choices of γ in the identity (3.5) and the result (3.7).

Corollary 4. *If $f \in L[a,b]$ and $g \in C[a,b]$, then*

$$|T(f,g)| \le \|g\|_\infty \frac{1}{b-a}\int_a^b \left|\overline{f}(t)\right| dt. \tag{3.12}$$

The constant 1 cannot be replaced by a smaller constant.

Proof. The inequality is obvious from (3.7).

To prove the sharpness of the constant $C = 1$, assume that (3.12) holds with a constant $C > 0$. That is,

$$|T(f,g)| \leq C \|g\|_\infty \frac{1}{b-a} \int_a^b \left|\overline{f}(t)\right| dt. \tag{3.13}$$

Consider $[a,b] = [-1,1]$ and define the functions

$$f_0(x) = \begin{cases} -1 & \text{if } x \in [-1,0); \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x \in (0,1], \end{cases} \quad , \ g_n(x) = \begin{cases} -1 & \text{if } x \in \left[-1, -\frac{1}{n}\right); \\ nx & \text{if } x \in \left[-\frac{1}{n}, \frac{1}{n}\right]; \\ 1 & \text{if } x \in \left(\frac{1}{n}, 1\right]. \end{cases} \tag{3.14}$$

Obviously, $g_n \in C[-1,1]$. We also have

$$\int_{-1}^1 f_0(x) g_n(x) = 2\int_0^1 g_n(x)\, dx = 2\left(1 - \frac{1}{2n}\right)$$

and thus

$$T(f_0, g_n) = 1 - \frac{1}{2n}, \quad \|g_n\|_\infty = 1, \quad \frac{1}{2}\int_{-1}^1 \left| f_0(t) - \frac{1}{2}\int_{-1}^1 f_0(s)\, ds \right| dt = 1,$$

giving from (3.13) that

$$1 - \frac{1}{2n} \leq C.$$

Letting $n \to \infty$, we deduce $C \geq 1$, and the sharpness of the constant is proved. □

Corollary 5. *If $f \in L[a,b]$, $g \in C[a,b]$ and there exists the real constants m, M such that*

$$m \leq g(x) \leq M \quad \text{for any } x \in [a,b], \tag{3.15}$$

then

$$|T(f,g)| \leq \frac{1}{2}(M-m) \frac{1}{b-a} \int_a^b \left|\overline{f}(t)\right| dt. \tag{3.16}$$

The constant $\frac{1}{2}$ is best possible.

Proof. The inequality follows by (3.7) on taking into account that if g satisfies (3.15), then

$$\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m).$$

The sharpness of the constant may be proved in a similar way as in the proof of that sharpness of (3.12) by selecting the same examples (3.14). We omit the details. □

Remark 1. *The inequality (3.16) was obtained in a different way in* [13]. *For generalisations, best constants and discrete versions, see also* [14].

Corollary 6. *Let* $f \in L[a,b]$. *If* $g : [a,b] \to \mathbb{R}$ *is of* $r - H-$*Hölder type with* $r \in (0,1]$, $H > 0$, *so that*

$$|g(t) - g(s)| \leq H|t-s|^r \quad \textit{for each} \quad t,s \in [a,b], \tag{3.17}$$

then we have the bound:

$$|T(f,g)| \leq \frac{1}{2^r} H(b-a)^{r-1} \int_a^b |\overline{f}(t)| \, dt. \tag{3.18}$$

Proof. Since g is of $r - H-$Hölder type, then

$$\sup_{t \in [a,b]} \left| g(t) - g\left(\frac{a+b}{2}\right) \right| \leq H \sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right|^r = \frac{1}{2^r} H(b-a)^r.$$

Using (3.7) with $\gamma = g\left(\frac{a+b}{2}\right)$, we deduce (3.18). □

The following corollary also holds.

Corollary 7. *If* $f \in L[a,b]$ *and* $g \in C[a,b]$, *then*

$$|T(f,g)| \leq \|\overline{g}\|_\infty \frac{1}{b-a} \int_a^b |\overline{f}(t)| \, dt. \tag{3.19}$$

The proof is obvious by (3.7) with $\gamma = \frac{1}{b-a} \int_a^b g(s) \, ds$.
The following theorem also holds.

Theorem 3. *Let* $f \in L_p[a,b]$ *and* $g \in L_q[a,b]$, *with* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ *(for* $p = \infty$ *we choose* $q = 1$*). Then we have the inequality*

$$|T(f,g)| \leq \frac{1}{b-a} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \left\|\overline{f}\right\|_p. \tag{3.20}$$

Proof. Follows by Lemma 2 and by Hölder's inequality. □

Similar particular inequalities for different choices of γ may be stated. We omit any further details.

The application of these bounds to problems in applied mathematics for the Čebyšev functional is left to future work and the pursuit of the interested readers.

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