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# HADAMARD INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

KUEI-LIN TSENG, GOU-SHENG YANG, AND SEVER S. DRAGOMIR

ABSTRACT. In this paper, we establish several inequalities of Hadamard's type for Wright-Convex functions.

## 1. INTRODUCTION

If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hadamard inequality ([5]).

For some results which generalize, improve, and extend this famous integral inequality see [1] – [8], [10] – [15].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

**Theorem 1.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function, and  $H$  is defined on  $[0, 1]$  by*

$$(1.2) \quad H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

*then  $H$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have*

$$(1.3) \quad f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx$$

In [10], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1).

**Theorem 2.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function, and  $F$  is defined on  $[0, 1]$  by*

$$(1.4) \quad F(t) = \frac{1}{2(b-a)} \int_a^b \left[ f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] dx,$$

*then  $F$  is convex, increasing on  $[0, 1]$ , and for all  $t \in [0, 1]$ , we have*

$$(1.5) \quad \frac{1}{b-a} \int_a^b f(x) dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2}.$$

We recall the definition of a Wright-convex function:

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**Definition 1.** (see [9, p. 223]). We say that  $f : [a, b] \rightarrow \mathbb{R}$  is a Wright-convex function, if, for all  $x, y + \delta \in [a, b]$  with  $x < y$  and  $\delta \geq 0$ , we have

$$(1.6) \quad f(x + \delta) + f(y) \leq f(y + \delta) + f(x).$$

Let  $C([a, b])$  be the set of all convex functions on  $[a, b]$  and  $W([a, b])$  be the set of all Wright-convex functions on  $[a, b]$ . Then  $C([a, b]) \subsetneq W([a, b])$ . That is, a convex function must be a Wright-convex function but not conversely (see [9, p. 224]).

In this paper, we shall establish several inequalities of Hadamard's type for Wright-convex functions.

## 2. MAIN RESULTS

In order to prove our main theorems, we need the following lemma:

**Lemma 1.** If  $f : [a, b] \rightarrow \mathbb{R}$ , then the following statements are equivalent:

- (1)  $f \in W([a, b])$ ;
- (2) for all  $s, t, u, v \in [a, b]$  with  $s \leq t \leq u \leq v$  and  $t + u = s + v$ , we have

$$(2.1) \quad f(t) + f(u) \leq f(s) + f(v).$$

*Proof.* Suppose  $f \in W([a, b])$ . If  $s, t, u, v \in [a, b]$ , and  $s \leq t \leq u \leq v$ , where  $t + u = s + v$ , then we can write  $x = s$ ,  $x + \delta = t$ ,  $y = u$ ,  $y + \delta = v$ , it follows from (1.6) that

$$f(t) + f(u) \leq f(s) + f(v).$$

Conversely, if  $x, y + \delta \in [a, b]$ ,  $x < y$  and  $\delta \geq 0$ . We may have

$$x \leq x + \delta \leq y \leq y + \delta$$

or

$$x \leq y \leq x + \delta \leq y + \delta.$$

In either case we have, by (2.1), that

$$f(x + \delta) + f(y) \leq f(x) + f(y + \delta).$$

Thus  $f \in W([a, b])$ . ■

**Theorem 3.** Let  $f \in W([a, b]) \cap L_1[a, b]$ . Then (1.1) holds.

*Proof.* For (2.1), we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{1}{(b-a)} \int_a^{\frac{a+b}{2}} \left[ f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) \right] dx \\ &\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx \quad \left( x \leq \frac{a+b}{2} \leq \frac{a+b}{2} \leq a+b-x \right) \\ &= \frac{1}{b-a} \left[ \int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \right] \\ &= \frac{1}{b-a} \int_a^b f(x) dx, \end{aligned}$$

and

$$\begin{aligned}
\frac{f(a) + f(b)}{2} &= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(a) + f(b)] dx \\
&\geq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx \quad (a \leq x \leq a+b-x \leq b) \\
&= \frac{1}{b-a} \left[ \int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \right] \\
&= \frac{1}{b-a} \int_a^b f(x) dx,
\end{aligned}$$

This completes the proof. ■

**Theorem 4.** Let  $f \in W([a, b]) \cap L_1[a, b]$  and let  $H$  be defined as in (1.2). Then  $H \in W([0, 1])$  is increasing on  $[0, 1]$ , and (1.3) holds for all  $t \in [0, 1]$ .

*Proof.* If  $s, t, u, v \in [0, 1]$  and  $s \leq t \leq u \leq v$ ,  $t+u = s+v$ , then for  $x \in [a, \frac{a+b}{2}]$  we have

$$\begin{aligned}
b &\geq sx + (1-s) \frac{a+b}{2} \\
&\geq tx + (1-t) \frac{a+b}{2} \\
&\geq ux + (1-u) \frac{a+b}{2} \\
&\geq vx + (1-v) \frac{a+b}{2} \geq a,
\end{aligned}$$

and if  $x \in [\frac{a+b}{2}, b]$ , then

$$\begin{aligned}
a &\leq sx + (1-s) \frac{a+b}{2} \\
&\leq tx + (1-t) \frac{a+b}{2} \\
&\leq ux + (1-u) \frac{a+b}{2} \\
&\leq vx + (1-v) \frac{a+b}{2} \leq b,
\end{aligned}$$

where

$$\begin{aligned}
&\left[ tx + (1-t) \frac{a+b}{2} \right] + \left[ ux + (1-u) \frac{a+b}{2} \right] \\
&= \left[ sx + (1-s) \frac{a+b}{2} \right] + \left[ vx + (1-v) \frac{a+b}{2} \right].
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&f\left(tx + (1-t) \frac{a+b}{2}\right) + f\left(ux + (1-u) \frac{a+b}{2}\right) \\
&\leq f\left(sx + (1-s) \frac{a+b}{2}\right) + f\left(vx + (1-v) \frac{a+b}{2}\right).
\end{aligned}$$

for all  $x \in [a, b]$ . Integrating this inequality over  $x$  on  $[a, b]$ , and dividing both sides by  $b - a$ , yields

$$H(t) + H(u) \leq H(s) + H(v).$$

Hence,  $H \in W([0, 1])$ .

Next, if  $0 \leq s \leq t \leq 1$  and  $x \in [a, \frac{a+b}{2}]$ , then

$$\begin{aligned} tx + (1-t) \frac{a+b}{2} &\leq sx + (1-s) \frac{a+b}{2} \\ &\leq s(a+b-x) + (1-s) \frac{a+b}{2} \\ &\leq t(a+b-x) + (1-t) \frac{a+b}{2}, \end{aligned}$$

where

$$\begin{aligned} &\left[ sx + (1-s) \frac{a+b}{2} \right] + \left[ s(a+b-x) + (1-s) \frac{a+b}{2} \right] \\ &= \left[ tx + (1-t) \frac{a+b}{2} \right] + \left[ t(a+b-x) + (1-t) \frac{a+b}{2} \right]. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} H(s) &= \frac{1}{b-a} \int_a^b f \left( sx + (1-s) \frac{a+b}{2} \right) dx \\ &= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left[ f \left( sx + (1-s) \frac{a+b}{2} \right) + f \left( s(a+b-x) + (1-s) \frac{a+b}{2} \right) \right] dx \\ &\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left[ f \left( tx + (1-t) \frac{a+b}{2} \right) + f \left( t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right] dx \\ &= \frac{1}{b-a} \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) dx \\ &= H(t). \end{aligned}$$

Thus,  $H$  is increasing on  $[0, 1]$ , and (1.3) holds for all  $t \in [0, 1]$ .

This completes the proof. ■

**Theorem 5.** Let  $f \in W([a, b]) \cap L_1[a, b]$  and let  $F$  be defined as in (1.4). Then  $F \in W([0, 1])$  is increasing on  $[0, 1]$ , and (1.5) holds for all  $t \in [0, 1]$ .

*Proof.* If  $s, t, u, v \in [0, 1]$  and  $s \leq t \leq u \leq v$ ,  $t + u = s + v$ , then

$$\begin{aligned} a &\leq \left( \frac{1+v}{2} \right) a + \left( \frac{1-v}{2} \right) x \\ &\leq \left( \frac{1+u}{2} \right) a + \left( \frac{1-u}{2} \right) x \\ &\leq \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) x \\ &\leq \left( \frac{1+s}{2} \right) a + \left( \frac{1-s}{2} \right) x \leq b \text{ for all } x \in [a, b], \end{aligned}$$

and

$$\begin{aligned}
a &\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \\
&\leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x \\
&\leq \left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x \\
&\leq \left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x \leq b \text{ for all } x \in [a, b],
\end{aligned}$$

where

$$\begin{aligned}
&\left[\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] \\
&= \left[\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right] \\
&= \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right].
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&f\left(\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \\
&\quad + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right) \\
&\leq f\left(\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right) + f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \\
&\quad + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) + f\left(\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right),
\end{aligned}$$

for all  $x \in [a, b]$ . Integrating this inequality over  $x$  on  $[a, b]$ , and dividing both sides by  $2(b-a)$ , we have

$$F(t) + F(u) \leq F(s) + F(v),$$

hence,  $F \in W([0, 1])$ .

Next, if  $0 \leq s \leq t \leq 1$  and  $x \in [a, b]$ , then

$$\begin{aligned}
\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x &\leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \\
&\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x) \\
&\leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x),
\end{aligned}$$

and

$$\begin{aligned} \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x) &\leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x) \\ &\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \\ &\leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \end{aligned}$$

where

$$\begin{aligned} &\left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right] \\ &= \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)\right], \end{aligned}$$

and

$$\begin{aligned} &\left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right] \\ &= \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right]. \end{aligned}$$

Thus

$$\begin{aligned} F(s) &= \frac{1}{2(b-a)} \int_a^b \left[ f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \right. \\ &\quad \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) \right] dx \\ &= \frac{1}{4(b-a)} \int_a^b \left\{ \left[ f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right) \right. \right. \\ &\quad \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right) \right] \\ &\quad \left. + \left[ f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right) \right. \right. \\ &\quad \left. + f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) \right] \right\} dx \\ &\leq \frac{1}{4(b-a)} \int_a^b \left\{ \left[ f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right. \right. \\ &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)\right) \right] \\ &\quad \left. + \left[ f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x)\right) \right. \right. \\ &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] \right\} dx. \end{aligned}$$

Hence,  $F$  is increasing on  $[0, 1]$  and (1.5) holds for all  $t \in [0, 1]$ .

This completes the proof. ■

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