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# K. PETR'S FORMULA OF DOUBLE INTEGRAL AND ESTIMATES OF ITS REMAINDER

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ABSTRACT. In the article, K. Petr's formula of single integral is generalized to that of double integral, some important special cases and estimates of its remainder are established.

## 1. INTRODUCTION

In [14, p. 218], K. Petr's formula for single integral is given, which can be modified slightly as follows.

**Theorem A** (K. Petr's formula of single integral). *Let  $f(x)$  be a function defined on  $[a, b] \subset \mathbb{R}$  such that  $f^{(n-1)}(x)$  is absolutely continuous,  $P_n(t)$  a polynomial of degree  $n$  with coefficient  $a_n$  of the term  $t^n$ . Then*

$$\int_a^b f(x) dx = \sum_{k=1}^n \frac{(-1)^{k+1}}{n!a_n} \left[ P_n^{(n-k)}(b)f^{(k-1)}(b) - P_n^{(n-k)}(a)f^{(k-1)}(a) \right] + \frac{(-1)^n}{n!a_n} \int_a^b P_n(x)f^{(n)}(x) dx. \quad (1.1)$$

*Remark 1.* K. Petr's formula stated in [14, p.218] is a special case of (1.1) by letting  $a_n = 1$ .

*Remark 2.* If taking  $P_n(t) = (t - a)^n$  in (1.1), then we have

$$\int_a^b f(x) dx = \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} (b - a)^k f^{(k-1)}(b) + \frac{(-1)^n}{n!} \int_a^b (x - a)^n f^{(n)}(x) dx. \quad (1.2)$$

In this paper, we will generalize K. Petr's formula (1.1) to the following

**Theorem 1** (K. Petr's formula of double integral). *Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be such that  $f^{(j,i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . Let  $P_n(t)$  be a polynomial of degree  $n$  with coefficient  $a_n$  of the term  $t^n$  and  $Q_m(s)$  a polynomial of degree  $m$  with coefficient  $b_m$  of the term  $t^m$ . Then*

$$\int_a^b \int_c^d f(x, y) dx dy = A(f, P_n, Q_m) + B(f, P_n, Q_m) + R(f, P_n, Q_m), \quad (1.3)$$

where

$$\begin{aligned} A(f, P_n, Q_m) &= \sum_{i=1}^m \sum_{j=1}^n \frac{(-1)^{i+j}}{m!n!a_n b_m} P_n^{(n-j)}(a) \left[ Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a, d) - Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a, c) \right] \\ &\quad - \sum_{i=1}^m \sum_{j=1}^n \frac{(-1)^{i+j}}{m!n!a_n b_m} P_n^{(n-j)}(b) \left[ Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b, d) - Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b, c) \right], \end{aligned} \quad (1.4)$$

$$\begin{aligned} B(f, P_n, Q_m) &= \sum_{i=1}^m \frac{(-1)^i}{m!b_m} Q_m^{(m-i)}(c) \int_a^b f^{(0,i-1)}(x, c) dx - \sum_{i=1}^m \frac{(-1)^i}{m!b_m} Q_m^{(m-i)}(d) \int_a^b f^{(0,i-1)}(x, d) dx \\ &\quad + \sum_{j=1}^n \frac{(-1)^j}{n!a_n} P_n^{(n-j)}(a) \int_c^d f^{(j-1,0)}(a, y) dy - \sum_{j=1}^n \frac{(-1)^j}{n!a_n} P_n^{(n-j)}(b) \int_c^d f^{(j-1,0)}(b, y) dy, \end{aligned} \quad (1.5)$$

$$R(f, P_n, Q_m) = \frac{(-1)^{m+n}}{m!n!a_n b_m} \int_a^b \int_c^d P_n(x)Q_m(y)f^{(n,m)}(x, y) dx dy. \quad (1.6)$$

Further, some important special cases of K. Petr's formula (1.3) are given, some estimates of their remainders are established, and an application is discussed.

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*Key words and phrases.* K. Petr's formula, Appell polynomial, harmonic polynomial, single integral, double integral, estimate of remainder.

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## 2. PROOF OF THEOREM 1

Integrating by part and using Theorem A yields

$$\begin{aligned}
& \int_a^b \int_c^d P_n(x) Q_m(y) f^{(n,m)}(x, y) \, dx \, dy = \int_a^b P_n(x) \left[ \int_c^d Q_m(y) f^{(n,m)}(x, y) \, dy \right] \, dx \\
& = (-1)^m \int_a^b P_n(x) \left\{ m! b_m \int_c^d f^{(n,0)}(x, y) \, dy \right. \\
& \quad \left. + \sum_{i=1}^m (-1)^i \left[ Q_m^{(m-i)}(d) f^{(n,i-1)}(x, d) - Q_m^{(m-i)}(c) f^{(n,i-1)}(x, c) \right] \right\} \, dx \\
& = (-1)^m m! b_m \int_a^b \int_c^d P_n(x) f^{(n,0)}(x, y) \, dx \, dy \\
& \quad + \sum_{i=1}^m (-1)^{m+i} Q_m^{(m-i)}(d) \int_a^b P_n(x) f^{(n,i-1)}(x, d) \, dx \\
& \quad - \sum_{i=1}^m (-1)^{m+i} Q_m^{(m-i)}(c) \int_a^b P_n(x) f^{(n,i-1)}(x, c) \, dx \\
& = (-1)^m m! b_m \int_c^d (-1)^n \left\{ n! a_n \int_a^b f(x, y) \, dx \right. \\
& \quad \left. + \sum_{j=1}^n (-1)^j \left[ P_n^{(n-j)}(b) f^{(j-1,0)}(b, y) - P_n^{(n-j)}(a) f^{(j-1,0)}(a, y) \right] \right\} \, dy \\
& \quad + \sum_{i=1}^m (-1)^{m+i} Q_m^{(m-i)}(d) \left\{ (-1)^n \left[ n! a_n \int_a^b f^{(0,i-1)}(x, d) \, dx \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^n (-1)^j \left[ P_n^{(n-j)}(b) f^{(j-1,i-1)}(b, d) - P_n^{(n-j)}(a) f^{(j-1,i-1)}(a, d) \right] \right] \right\} \\
& \quad - \sum_{i=1}^m (-1)^{m+i} Q_m^{(m-i)}(c) \left\{ (-1)^n \left[ n! a_n \int_a^b f^{(0,i-1)}(x, c) \, dx \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^n (-1)^j \left[ P_n^{(n-j)}(b) f^{(j-1,i-1)}(b, c) - P_n^{(n-j)}(a) f^{(j-1,i-1)}(a, c) \right] \right] \right\} \tag{2.1} \\
& = (-1)^{m+n} \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_n^{(n-j)}(a) \left[ Q_m^{(m-i)}(c) f^{(j-1,i-1)}(a, c) - Q_m^{(m-i)}(d) f^{(j-1,i-1)}(a, d) \right] \\
& \quad + (-1)^{m+n} \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_n^{(n-j)}(b) \left[ Q_m^{(m-i)}(d) f^{(j-1,i-1)}(b, d) - Q_m^{(m-i)}(c) f^{(j-1,i-1)}(b, c) \right] \\
& \quad - (-1)^{m+n} n! a_n \sum_{i=1}^m (-1)^i Q_m^{(m-i)}(c) \int_a^b f^{(0,i-1)}(x, c) \, dx \\
& \quad + (-1)^{m+n} n! a_n \sum_{i=1}^m (-1)^i Q_m^{(m-i)}(d) \int_a^b f^{(0,i-1)}(x, d) \, dx \\
& \quad + (-1)^{m+n} m! b_m \sum_{j=1}^n (-1)^j P_n^{(n-j)}(b) \int_c^d f^{(j-1,0)}(b, y) \, dy \\
& \quad - (-1)^{m+n} m! b_m \sum_{j=1}^n (-1)^j P_n^{(n-j)}(a) \int_c^d f^{(j-1,0)}(a, y) \, dy \\
& \quad + (-1)^{m+n} m! n! a_n b_m \int_a^b \int_c^d f(x, y) \, dx \, dy.
\end{aligned}$$

Rearranging (2.1) leads to

$$\begin{aligned}
& \int_a^b \int_c^d f(x, y) \, dx \, dy \\
&= \sum_{i=1}^m \sum_{j=1}^n \frac{(-1)^{i+j}}{m!n!a_n b_m} P_n^{(n-j)}(a) \left[ Q_m^{(m-i)}(d) f^{(j-1, i-1)}(a, d) - Q_m^{(m-i)}(c) f^{(j-1, i-1)}(a, c) \right] \\
&\quad - \sum_{i=1}^m \sum_{j=1}^n \frac{(-1)^{i+j}}{m!n!a_n b_m} P_n^{(n-j)}(b) \left[ Q_m^{(m-i)}(d) f^{(j-1, i-1)}(b, d) - Q_m^{(m-i)}(c) f^{(j-1, i-1)}(b, c) \right] \\
&\quad + \sum_{i=1}^m \frac{(-1)^i}{m!b_m} Q_m^{(m-i)}(c) \int_a^b f^{(0, i-1)}(x, c) \, dx - \sum_{i=1}^m \frac{(-1)^i}{m!b_m} Q_m^{(m-i)}(d) \int_a^b f^{(0, i-1)}(x, d) \, dx \\
&\quad + \sum_{j=1}^n \frac{(-1)^j}{n!a_n} P_n^{(n-j)}(a) \int_c^d f^{(j-1, 0)}(a, y) \, dy - \sum_{j=1}^n \frac{(-1)^j}{n!a_n} P_n^{(n-j)}(b) \int_c^d f^{(j-1, 0)}(b, y) \, dy \\
&\quad + \frac{(-1)^{m+n}}{m!n!a_n b_m} \int_a^b \int_c^d P_n(x) Q_m(y) f^{(n, m)}(x, y) \, dx \, dy.
\end{aligned} \tag{2.2}$$

The proof of Theorem 1 is complete.

### 3. SOME IMPORTANT SPECIAL CASES OF K. PETR'S FORMULA FOR DOUBLE INTEGRAL

**Definition 1** ([2]). Let  $P_k(t)$  be a polynomial satisfying

$$P'_k(t) = P_{k-1}(t), \quad P_0(t) = 1, \quad k = 1, 2, \dots, \tag{3.1}$$

then we call  $P_k(t)$  an Appell polynomial or a harmonic polynomial.

**Proposition 1.** Let  $P_n(t)$  be an Appell polynomial of degree  $n$  and the coefficient of the term  $t^n$  equal  $a_n$ . Then  $a_n = \frac{1}{n!}$ .

**Theorem 2** (Harmonic K. Petr's formula of double integral). Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be a function such that  $f^{(j, i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . If  $P_n(t)$  and  $Q_m(s)$  are two harmonic polynomials, then

$$\int_a^b \int_c^d f(x, y) \, dx \, dy = A(f, P_n, Q_m) + B(f, P_n, Q_m) + R(f, P_n, Q_m), \tag{3.2}$$

where

$$\begin{aligned}
A(f, P_n, Q_m) &= \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_j(a) \left[ Q_i(d) f^{(j-1, i-1)}(a, d) - Q_i(c) f^{(j-1, i-1)}(a, c) \right] \\
&\quad - \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_j(b) \left[ Q_i(d) f^{(j-1, i-1)}(b, d) - Q_i(c) f^{(j-1, i-1)}(b, c) \right],
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
B(f, P_n, Q_m) &= \sum_{i=1}^m (-1)^i Q_i(c) \int_a^b f^{(0, i-1)}(x, c) \, dx - \sum_{i=1}^m (-1)^i Q_i(d) \int_a^b f^{(0, i-1)}(x, d) \, dx \\
&\quad + \sum_{j=1}^n (-1)^j P_j(a) \int_c^d f^{(j-1, 0)}(a, y) \, dy - \sum_{j=1}^n (-1)^j P_j(b) \int_c^d f^{(j-1, 0)}(b, y) \, dy,
\end{aligned} \tag{3.4}$$

$$R(f, P_n, Q_m) = (-1)^{m+n} \int_a^b \int_c^d P_n(x) Q_m(y) f^{(n, m)}(x, y) \, dx \, dy. \tag{3.5}$$

*Proof.* Since  $P_n(t)$  and  $Q_m(s)$  are harmonic polynomials, then

$$a_n = \frac{1}{n!}, \quad b_m = \frac{1}{m!}, \quad P_n^{(n-j)}(t) = P_j(t), \quad Q_m^{(m-i)}(s) = Q_i(s). \tag{3.6}$$

Substituting (3.6) into Theorem 1 yields Theorem 2.  $\square$

**Theorem 3.** Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be such that  $f^{(j,i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . Then for  $0 \leq \lambda \leq 1$  and  $0 \leq \mu \leq 1$ , we have

$$\begin{aligned}
\int_a^b \int_c^d f(x, y) dx dy &= \sum_{i=1}^m \sum_{j=1}^n \frac{(1-\lambda)^j (c-d)^i (b-a)^j}{i!j!} [\mu^i f^{(j-1, i-1)}(a, d) - (\mu-1)^i f^{(j-1, i-1)}(a, c)] \\
&\quad - \sum_{i=1}^m \sum_{j=1}^n \frac{\lambda^j (c-d)^i (a-b)^j}{i!j!} [\mu^i f^{(j-1, i-1)}(b, d) - (\mu-1)^i f^{(j-1, i-1)}(b, c)] \\
&\quad + \sum_{i=1}^m \frac{(1-\mu)^i (d-c)^i}{i!} \int_a^b f^{(0, i-1)}(x, c) dx - \sum_{i=1}^m \frac{\mu^i (c-d)^i}{i!} \int_a^b f^{(0, i-1)}(x, d) dx \\
&\quad + \sum_{j=1}^n \frac{(1-\lambda)^j (b-a)^j}{j!} \int_c^d f^{(j-1, 0)}(a, y) dy - \sum_{j=1}^n \frac{\lambda^j (a-b)^j}{j!} \int_c^d f^{(j-1, 0)}(b, y) dy \\
&\quad + R(f, a, b, c, d),
\end{aligned} \tag{3.7}$$

where

$$R(f, a, b, c, d) = \frac{(-1)^{m+n}}{m!n!} \int_a^b \int_c^d [x - (\lambda a + (1-\lambda)b)]^n [y - (\mu c + (1-\mu)d)]^m f^{(n,m)}(x, y) dx dy. \tag{3.8}$$

*Proof.* Letting

$$P_n(x) = [x - (\lambda a + (1-\lambda)b)]^n, \quad Q_m(y) = [y - (\mu c + (1-\mu)d)]^m \tag{3.9}$$

in Theorem 1, then  $a_n = 1$  and  $b_m = 1$ . Further, by direct computation, we have

$$P_n^{(n-j)}(x) = \frac{n!}{j!} [x - (\lambda a + (1-\lambda)b)]^j, \quad Q_m^{(m-i)}(y) = \frac{m!}{i!} [y - (\mu c + (1-\mu)d)]^i. \tag{3.10}$$

Therefore, Theorem 3 follows easily.  $\square$

**Definition 2** ([1, 23.1.1]). Bernoulli's polynomials  $B_k(x)$  for  $k$  being nonnegative integers are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x), \quad |x| < 2\pi, \quad t \in \mathbb{R}, \tag{3.11}$$

where  $B_k(0) = B_k$  is called Bernoulli's numbers.

**Definition 3** ([1, 23.1.1]). Euler's polynomials  $E_k(x)$  for  $k$  being nonnegative integers are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x), \quad |x| < \pi, \quad t \in \mathbb{R}, \tag{3.12}$$

where  $2^k E_k(\frac{1}{2}) = E_k$  is called Euler's numbers.

*Remark 3.* Notice that Bernoulli's numbers and polynomials and Euler's numbers and polynomials have been generalized by the authors in [4, 5, 6, 7, 9] recently.

**Lemma 1** ([1, 23.1.5] and [8]). The following identities hold

$$B'_k(x) = kB_{k-1}(x), \quad E'_k(x) = kE_{k-1}(x), \quad k = 1, 2, \dots \tag{3.13}$$

**Lemma 2** ([1, 23.1.6] and [8]). The following identities hold

$$B_i(t+1) - B_i(t) = it^{i-1}, \quad E_i(t+1) + E_i(t) = 2t^i, \quad i = 0, 1, \dots \tag{3.14}$$

**Lemma 3** ([1, 23.1.20] and [8]). The following identities hold

$$B_k(0) = (-1)^k B_k(1) = B_k, \quad k = 0, 1, 2, \dots, \tag{3.15}$$

$$E_i(0) = -E_i(1) = -\frac{2}{i+1} (2^{i+1} - 1) B_{i+1}, \quad i = 1, 2, \dots \tag{3.16}$$

**Theorem 4.** Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be such that  $f^{(j,i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . Then

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dx dy &= \sum_{i=1}^m \sum_{j=1}^n \frac{2(2^{i+1} - 1)(a-b)^j (c-d)^i}{(i+1)! j!} \\ &\times B_{i+1} B_j \left[ f^{(j-1, i-1)}(a, d) + f^{(j-1, i-1)}(a, c) - (-1)^j \left[ f^{(j-1, i-1)}(b, d) + f^{(j-1, i-1)}(b, c) \right] \right] \\ &+ \sum_{i=1}^m \frac{2(1 - 2^{i+1})(c-d)^i}{(i+1)!} B_{i+1} \int_a^b \left[ f^{(0, i-1)}(x, c) + f^{(0, i-1)}(x, d) \right] dx \\ &+ \sum_{j=1}^n \frac{(a-b)^j}{j!} B_j \int_c^d \left[ f^{(j-1, 0)}(a, y) + (-1)^{j+1} f^{(j-1, 0)}(b, y) \right] dy \\ &+ R(f, B_n, E_m), \end{aligned} \quad (3.17)$$

where

$$R(f, B_n, E_m) = \frac{(a-b)^n (c-d)^m}{m! n!} \int_a^b \int_c^d B_n \left( \frac{x-a}{b-a} \right) E_m \left( \frac{y-c}{d-c} \right) f^{(n, m)}(x, y) dx dy. \quad (3.18)$$

*Proof.* Taking

$$P_n(x) = B_n \left( \frac{x-a}{b-a} \right), \quad Q_m(y) = E_m \left( \frac{y-c}{d-c} \right) \quad (3.19)$$

in Theorem 1, then  $a_n = \frac{1}{(b-a)^n}$  and  $b_m = \frac{1}{(d-c)^m}$ . Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x) = \frac{n!}{j!} (b-a)^{j-n} B_j \left( \frac{x-a}{b-a} \right), \quad Q_m^{(m-i)}(y) = \frac{m!}{i!} (d-c)^{i-m} E_i \left( \frac{y-c}{d-c} \right), \quad (3.20)$$

Theorem 4 follows.  $\square$

**Theorem 5.** Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be such that  $f^{(j,i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . Then

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dx dy &= \sum_{i=1}^m \sum_{j=1}^n \frac{(a-b)^j (c-d)^i}{i! j!} B_i B_j \\ &\times \left[ (-1)^i f^{(j-1, i-1)}(a, d) - f^{(j-1, i-1)}(a, c) - (-1)^j \left[ (-1)^i f^{(j-1, i-1)}(b, d) - f^{(j-1, i-1)}(b, c) \right] \right] \\ &+ \sum_{i=1}^m \frac{(c-d)^i}{i!} B_i \int_a^b \left[ f^{(0, i-1)}(x, c) + (-1)^{i+1} f^{(0, i-1)}(x, d) \right] dx \\ &+ \sum_{j=1}^n \frac{(a-b)^j}{j!} B_j \int_c^d \left[ f^{(j-1, 0)}(a, y) + (-1)^{j+1} f^{(j-1, 0)}(b, y) \right] dy \\ &+ R(f, B_n, B_m), \end{aligned} \quad (3.21)$$

where

$$R(f, B_n, B_m) = \frac{(a-b)^n (c-d)^m}{m! n!} \int_a^b \int_c^d B_n \left( \frac{x-a}{b-a} \right) B_m \left( \frac{y-c}{d-c} \right) f^{(n, m)}(x, y) dx dy. \quad (3.22)$$

*Proof.* Setting

$$P_n(x) = B_n \left( \frac{x-a}{b-a} \right), \quad Q_m(y) = B_m \left( \frac{y-c}{d-c} \right) \quad (3.23)$$

in Theorem 1, then  $a_n = \frac{1}{(b-a)^n}$  and  $b_m = \frac{1}{(d-c)^m}$ . Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x) = \frac{n!}{j!} (b-a)^{j-n} B_j \left( \frac{x-a}{b-a} \right), \quad Q_m^{(m-i)}(y) = \frac{m!}{i!} (d-c)^{i-m} B_i \left( \frac{y-c}{d-c} \right), \quad (3.24)$$

Theorem 5 follows.  $\square$

**Theorem 6.** Let  $D = [a, b] \times [c, d] \subset \mathbb{R}^2$  and  $f : D \rightarrow \mathbb{R}$  be such that  $f^{(j,i)}(x, y)$  is continuous on  $D$  for  $0 \leq j \leq n$  and  $0 \leq i \leq m$ . Then

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dx dy &= - \sum_{i=1}^m \sum_{j=1}^n \frac{4(2^{i+1}-1)(2^{j+1}-1)(a-b)^j(c-d)^i}{(i+1)!(j+1)!} B_{i+1} B_{j+1} \\ &\quad \times \left[ f^{(j-1, i-1)}(b, d) + f^{(j-1, i-1)}(b, c) + f^{(j-1, i-1)}(a, d) + f^{(j-1, i-1)}(a, c) \right] \\ &\quad + \sum_{i=1}^m \frac{2(c-d)^i(1-2^{i+1})}{(i+1)!} B_{i+1} \int_a^b \left[ f^{(0, i-1)}(x, c) + f^{(0, i-1)}(x, d) \right] dx \\ &\quad + \sum_{j=1}^n \frac{2(a-b)^j(1-2^{j+1})}{(j+1)!} B_{j+1} \int_c^d \left[ f^{(j-1, 0)}(a, y) + f^{(j-1, 0)}(b, y) \right] dy \\ &\quad + R(f, E_n, E_m), \end{aligned} \quad (3.25)$$

where

$$R(f, E_n, E_m) = \frac{(a-b)^n(c-d)^m}{m!n!} \int_a^b \int_c^d E_n\left(\frac{x-a}{b-a}\right) E_m\left(\frac{y-c}{d-c}\right) f^{(n,m)}(x, y) dx dy. \quad (3.26)$$

*Proof.* Letting

$$P_n(x) = E_n\left(\frac{x-a}{b-a}\right), \quad Q_m(y) = E_m\left(\frac{y-c}{d-c}\right) \quad (3.27)$$

in Theorem 1, then  $a_n = \frac{1}{(b-a)^n}$  and  $b_m = \frac{1}{(d-c)^m}$ . Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x) = \frac{n!}{j!} (b-a)^{j-n} E_j\left(\frac{x-a}{b-a}\right), \quad Q_m^{(m-i)}(y) = \frac{m!}{i!} (d-c)^{i-m} E_i\left(\frac{y-c}{d-c}\right), \quad (3.28)$$

Theorem 6 follows.  $\square$

*Remark 4.* If taking the following harmonic polynomials

$$P_n(x) = \frac{(x-b)^n}{n!}, \quad Q_m(y) = \frac{(y-d)^m}{m!}, \quad (3.29)$$

$$P_n(x) = \frac{(x-a)^n}{n!} B_n\left(\frac{x-a}{b-a}\right), \quad Q_m(y) = \frac{(y-b)^m}{m!} E_m\left(\frac{y-c}{d-c}\right), \quad (3.30)$$

$$P_n(x) = \frac{(x-a)^n}{n!} B_n\left(\frac{x-a}{b-a}\right), \quad Q_m(y) = \frac{(y-b)^m}{m!} B_m\left(\frac{y-c}{d-c}\right), \quad (3.31)$$

$$P_n(x) = \frac{(x-a)^n}{n!} E_n\left(\frac{x-a}{b-a}\right), \quad Q_m(y) = \frac{(y-b)^m}{m!} E_m\left(\frac{y-c}{d-c}\right) \quad (3.32)$$

in Theorem 2, then the theorems in this section can be obtained again.

#### 4. ESTIMATES OF REMAINDERS

In this section, we will give some estimates of the remainders mentioned above.

**Lemma 4** ([1, 23.1.12]). *We have the following*

$$\int_0^1 B_n(x) B_m(x) dx = (-1)^{n-1} \frac{m!n!}{(m+n)!} B_{m+n}, \quad m, n = 1, 2, \dots, \quad (4.1)$$

$$\int_0^1 E_n(x) E_m(x) dx = 4(-1)^n (2^{m+n+2} - 1) \frac{m!n!}{(m+n+2)!} B_{m+n+2}, \quad m, n = 0, 1, \dots \quad (4.2)$$

**Theorem 7.** *Under conditions of Theorem 1, the remainder (1.6) can be estimated as follows*

$$|R(f, P_n, Q_m)| \leq \frac{1}{m!n! |a_n b_m|} \max_{x \in [a, b]} \{|P_n(x)|\} \max_{y \in [c, d]} \{|Q_m(y)|\} \left| \int_a^b \int_c^d f^{(n,m)}(x, y) dx dy \right|, \quad (4.3)$$

$$|R(f, P_n, Q_m)| \leq \frac{1}{m!n! |a_n b_m|} \max_{(x,y) \in [a,b] \times [c,d]} \left\{ |f^{(n,m)}(x, y)| \right\} \left| \int_a^b P_n(x) dx \right| \left| \int_c^d Q_m(y) dy \right|, \quad (4.4)$$

$$|R(f, P_n, Q_m)| \leq \frac{(b-a)(d-c)}{m!n! |a_n b_m|} \max_{x \in [a, b]} \{|P_n(x)|\} \max_{y \in [c, d]} \{|Q_m(y)|\} \max_{(x,y) \in [a,b] \times [c,d]} \left\{ |f^{(n,m)}(x, y)| \right\}, \quad (4.5)$$

$$|R(f, P_n, Q_m)| \leq \frac{1}{m!n! |a_n b_m|} \left[ \int_a^b |P_n(x)|^q dx \int_c^d |Q_m(y)|^q dy \right]^{\frac{1}{q}} \left[ \int_a^b \int_c^d |f^{(n,m)}(x, y)|^p dx dy \right]^{\frac{1}{p}}, \quad (4.6)$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* The estimates (4.3), (4.4) and (4.5) follows from standard arguments. The estimate (4.6) follows from Hölder's inequality of double integral.  $\square$

**Theorem 8.** *Under conditions of Theorem 3, we have the following estimates for remainder (3.8)*

$$|R(f, a, b, c)| \leq \frac{(b-a)^n(d-c)^m}{m!n!} \left| \int_a^b \int_c^d f^{(n,m)}(x, y) dx dy \right|, \quad (4.7)$$

$$|R(f, a, b, c)| \leq \frac{(b-a)^{n+1}(d-c)^{m+1} |\lambda^{n+1} - (1-\lambda)^{n+1}| |\mu^{m+1} - (1-\mu)^{m+1}|}{(m+1)!(n+1)!} \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}, \quad (4.8)$$

$$|R(f, P_n, Q_m)| \leq \frac{(b-a)^{n+\frac{1}{q}}(d-c)^{m+\frac{1}{q}} [\lambda^{nq+1} + (1-\lambda)^{nq+1}]^{\frac{1}{q}} [\mu^{mq+1} + (1-\mu)^{mq+1}]^{\frac{1}{q}}}{m!n![(nq+1)(mq+1)]^{\frac{1}{q}}} \\ \times \left( \int_a^b \int_c^d \left| f^{(n,m)}(x, y) \right|^p dx dy \right)^{\frac{1}{p}}, \quad (4.9)$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Estimates (4.7) and (4.8) follows from taking  $P_n(x) = [x - (\lambda a + (1-\lambda)b)]^n$  and  $Q_m(y) = [y - (\mu c + (1-\mu)d)]^m$  in (4.3) and (4.4) and using  $a_n = 1$  and  $b_m = 1$ . Estimate (4.9) follows from Hölder's inequality of double integral.  $\square$

**Theorem 9.** *Under conditions of Theorem 4, we can estimate (3.18) as*

$$|R(f, B_n, E_m)| \leq \frac{(b-a)^n(d-c)^m}{m!n!} \max_{x \in [a,b]} \left\{ \left| B_n \left( \frac{x-a}{b-a} \right) \right| \right\} \max_{y \in [c,d]} \left\{ \left| E_m \left( \frac{y-c}{d-c} \right) \right| \right\} \\ \times \left| \int_a^b \int_c^d f^{(n,m)}(x, y) dx dy \right|, \quad (4.10)$$

$$|R(f, B_n, E_m)| \leq \frac{n!(b-a)^{n+1}(d-c)^{m+1} [(2n^2 + 3n + 1) |B_{2n}| + (2^{2n+3} - 2) |B_{2n+2}|]}{m!(2n+2)!} \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}. \quad (4.11)$$

*Proof.* The estimate (4.10) is straightforward.

Taking  $m = n$  in (4.1) and (4.2) of Lemma 4 yields

$$\int_0^1 B_n^2(x) dx = (-1)^{n-1} \frac{(n!)^2}{(2n)!} B_{2n} = \frac{(n!)^2}{(2n)!} |B_{2n}|, \\ \int_0^1 E_m^2(x) dx = \frac{4(-1)^n (4^{n+1} - 1)(n!)^2}{(2n+2)!} B_{2n+2} = \frac{4(4^{n+1} - 1)(n!)^2}{(2n+2)!} |B_{2n+2}|. \quad (4.12)$$

From (4.12), we have

$$|R(f, B_n, E_m)| \leq \frac{(b-a)^n(d-c)^m}{m!n!} \int_a^b \int_c^d \left| B_n \left( \frac{x-a}{b-a} \right) \right| \left| E_m \left( \frac{y-c}{d-c} \right) \right| \left| f^{(n,m)}(x, y) \right| dx dy \\ \leq \frac{(b-a)^n(d-c)^m}{2m!n!} \int_a^b \int_c^d \left[ B_n^2 \left( \frac{x-a}{b-a} \right) + E_m^2 \left( \frac{y-c}{d-c} \right) \right] \left| f^{(n,m)}(x, y) \right| dx dy \\ \leq \frac{(b-a)^{n+1}(d-c)^{m+1}}{2m!n!} \left[ \int_0^1 B_n^2(x) dx + \int_0^1 E_m^2(y) dy \right] \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\} \\ \leq \frac{n!(b-a)^{n+1}(d-c)^{m+1} [(2n^2 + 3n + 1) |B_{2n}| + (2^{2n+3} - 2) |B_{2n+2}|]}{m!(2n+2)!} \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}. \quad (4.13)$$

The proof is complete.  $\square$



**Theorem 10.** Under conditions of Theorem 5, the remainder (3.22) can be estimated as

$$|R(f, B_n, B_m)| \leq \frac{(b-a)^n (d-c)^m}{m!n!} \max_{x \in [a,b]} \left\{ \left| B_n \left( \frac{x-a}{b-a} \right) \right| \right\} \max_{y \in [c,d]} \left\{ \left| B_m \left( \frac{y-c}{d-c} \right) \right| \right\} \\ \times \left| \int_a^b \int_c^d f^{(n,m)}(x, y) dx dy \right|, \quad (4.14)$$

$$|R(f, B_n, B_m)| \leq \frac{(b-a)^{n+1} (d-c)^{m+1}}{2} \left[ \frac{n!}{m!(2n)!} |B_{2n}| + \frac{m!}{n!(2m)!} |B_{2m}| \right] \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}. \quad (4.15)$$

*Proof.* The proof of (4.14) is easy. The proof of (4.15) is similar to that of (4.11).  $\square$

**Theorem 11.** Under conditions of Theorem 6, the remainder (3.26) can be estimated as

$$|R(f, E_n, E_m)| \leq \frac{(b-a)^{n+1} (d-c)^{m+1}}{m!n!} \max_{x \in [a,b]} \left\{ \left| E_n \left( \frac{x-a}{b-a} \right) \right| \right\} \max_{y \in [c,d]} \left\{ \left| E_m \left( \frac{y-c}{d-c} \right) \right| \right\} \\ \times \left| \int_a^b \int_c^d f^{(n,m)}(x, y) dx dy \right|, \quad (4.16)$$

$$|R(f, E_n, E_m)| \leq 4(b-a)^{n+1} (d-c)^{m+1} \left[ \frac{n!(4^{n+1}-1)}{m!(2n+2)!} |B_{2n+2}| + \frac{m!(4^{m+1}-1)}{n!(2m+2)!} |B_{2m+2}| \right] \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}, \quad (4.17)$$

$$|R(f, E_n, E_m)| \leq \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1} (d-c)^{m+1}}{(n+2)!(m+2)!} |B_{n+2}| |B_{m+2}| \\ \times \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}. \quad (4.18)$$

*Proof.* The estimate of (4.16) is evident. The proof of (4.17) is similar to (4.11).

Using  $\int_0^1 E_n(x) dx = \frac{E_{n+1}(1) - E_{n+1}(0)}{n+1}$  in [1, 23.1.11] and (3.16) in Lemma 3, we have

$$|R(f, E_n, E_m)| \\ \leq \frac{(b-a)^n (d-c)^m}{m!n!} \left| \int_a^b E_n \left( \frac{y-c}{d-c} \right) dx \right| \left| \int_c^d E_m \left( \frac{y-c}{d-c} \right) dy \right| \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\} \\ = \frac{(b-a)^{n+1} (d-c)^{m+1}}{m!n!} \int_0^1 E_n(x) dx \int_0^1 E_m(y) dy \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\} \\ = \frac{(b-a)^{n+1} (d-c)^{m+1}}{m!n!} \left| \frac{E_{n+1}(1) - E_{n+1}(0)}{n+1} \right| \left| \frac{E_{m+1}(1) - E_{m+1}(0)}{m+1} \right| \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\} \\ = \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1} (d-c)^{m+1}}{(n+2)!(m+2)!} |B_{n+2}| |B_{m+2}| \max_{(x,y) \in [a,b] \times [c,d]} \left\{ \left| f^{(n,m)}(x, y) \right| \right\}. \quad (4.19)$$

The proof is complete.  $\square$

## 5. EXAMPLES

Now we apply K. Petr's formula to the function  $e^{x+y}$ .

*Example 1.* If taking  $[a, b] \times [c, d] = [0, 1] \times [0, 1]$  and  $f(x, y) = e^{x+y}$  in Theorem 3, then we have

$$\int_0^1 \int_0^1 e^{x+y} dx dy = - \sum_{i=1}^m \sum_{j=1}^n \frac{1}{i!j!} + (e-1) \left[ \sum_{i=1}^m \frac{1}{i!} + \sum_{j=1}^n \frac{1}{j!} \right] \\ + \frac{1}{m!n!} \int_0^1 (1-x)^n e^x dx \int_0^1 (1-y)^m e^y dy. \quad (5.1)$$

*Example 2.* If taking  $[a, b] \times [c, d] = [0, 1] \times [0, 1]$  and  $f(x, y) = e^{x+y}$  in Theorem 4, then we have

$$\begin{aligned} \int_0^1 \int_0^1 e^{x+y} dx dy &= (e+1) \sum_{i=1}^m \sum_{j=1}^n \frac{2(-1)^{i+j}(2^{i+1}-1)B_{i+1}B_j}{(i+1)!j!} [1+(-1)^{j+1}e] \\ &+ (e^2-1) \sum_{i=1}^m \frac{2(-1)^i(1-2^{i+1})B_{i+1}}{(i+1)!} \\ &+ (e-1) \sum_{j=1}^n \frac{(-1)^j B_j}{j!} [1+(-1)^{j+1}e] dy \\ &+ \frac{(-1)^{m+n}}{m!n!} \int_0^1 B_n(x)e^x dx \int_0^1 E_m(y)e^y dy. \end{aligned} \tag{5.2}$$

By exploiting Theorem 5 and Theorem 6, we can obtain more other expansions of the double integral  $\int_0^1 \int_0^1 e^{x+y} dx dy$ .

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