A New Proof of the Best Bounds in Wallis' Inequality

A NEW PROOF OF THE BEST BOUNDS IN WALLIS' INEQUALITY

CHAO-PING CHEN AND FENG QI

Abstract. By using some properties of gamma function and psi function and the convolution theorem, a new proof of the following double inequality is given: For all natural number n, we have

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},$$

and the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

1. Introduction

Define $(2m)!! = \prod_{i=1}^{m}(2i)$ and $(2m-1)!! = \prod_{i=1}^{m}(2i-1)$ for any given positive integer m. Then we have

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{1}$$

The inequality (1) is called Wallis' inequality in [7, p. 103] and can be improved to the following

Theorem 1. *For all natural number n, we have*

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{2}$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

In [2, pp. 358–359] and [9], it was twice proved that the function $\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 - x$ is decreasing for $x>0$. This implies that the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ in the lower and upper bounds of inequality (2) are the best possible.

Recently, inequality (2) in Theorem 1 was obtained using different approaches by the authors in [3, 4, 5].

In this short note, we will give a new proof of Theorem 1 by using some properties of gamma and psi functions and the convolution theorem.

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2. Lemmas

The following lemmas regarding to gamma function $\Gamma(x)$ and psi function $\psi = \frac{\Gamma'}{\Gamma}$ are necessary.

Lemma 1 ([6]). *For $x > 0$, we have*

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\bigl(x^{-2}\bigr). \tag{3}$$

Lemma 2 ([1, 8]). *For $x > 0$, we have*

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,\mathrm{d}t, \tag{4}$$

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{r=1}^{n} \frac{(-1)^{r-1}B_r}{2r}x^{-2r} + O\bigl(x^{-2n-2}\bigr), \tag{5}$$

where $\gamma = 0.57721566490153286060651\cdots$ is the Euler's constant. In particular,

$$\psi(x) = \ln x - \frac{1}{2x} + O\bigl(x^{-2}\bigr). \tag{6}$$

Lemma 3. *Let $f_1(t)$ and $f_2(t)$ be piecewise continuous for $t \ge 0$ on any given finite interval and there exist two constants $M > 0$ and $c \ge 0$ such that $|f(t)| \le Me^{ct}$, then we have*

$$\int_0^\infty \left[\int_0^s f_1(u)f_2(t-u)\,\mathrm{d}u\right]e^{-st}\,\mathrm{d}t = \int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u \int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v. \tag{7}$$

Remark 1. Lemma 3 is a convolution theorem of Laplace transform, which can be found in standard textbooks, for example, [1, 10].

3. A new proof of Theorem 1

Since

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\,, \quad 2^n n! = (2n)!!, \tag{8}$$

the double inequality (2) can be rewritten as

$$\frac{1}{4} < \left[\frac{\Gamma(n+1)}{\Gamma\bigl(n+\frac{1}{2}\bigr)}\right]^2 - n \le \frac{4}{\pi} - 1. \tag{9}$$

Let

$$f(x) = \left[\frac{\Gamma(x+1)}{\Gamma\bigl(x+\frac{1}{2}\bigr)}\right]^2 - x, \quad x > 0. \tag{10}$$

Direct computation gives

$$f'(x) = 2\left[\frac{\Gamma(x+1)}{\Gamma\bigl(x+\frac{1}{2}\bigr)}\right]^2\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] - 1 \tag{11}$$

and

$$\begin{aligned} &\frac{\psi(x+1) - \psi\bigl(x+\frac{1}{2}\bigr)}{1+f'(x)}f''(x) \\ &= \psi'(x+1) - \psi'\left(x+\frac{1}{2}\right) + 2\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right]^2 \\ &\triangleq g(x). \end{aligned} \tag{12}$$

Differentiating (4) yields

$$\psi'(x) = \int_0^\infty \frac{te^{-xt}}{1-e^{-t}}\,\mathrm{d}t. \tag{13}$$

From (4) and (13), it follows that

$$g(x) = -\int_0^\infty te^{-xt}h(t)\,\mathrm{d}t + 2\left(\int_0^\infty e^{-xt}h(t)\,\mathrm{d}t\right)^2, \tag{14}$$

where

$$h(x) = \left(e^{t/2}+1\right)^{-1}. \tag{15}$$

By using the convolution theorem, Lemma 3, we have

$$\begin{aligned} g(x) &= -\int_0^\infty te^{-xt}h(t)\,\mathrm{d}t + 2\int_0^\infty \left[\int_0^t h(s)h(t-s)\,\mathrm{d}s\right]\mathrm{d}t \\ &= \int_0^\infty e^{-xt}I(t)\,\mathrm{d}t, \end{aligned} \tag{16}$$

where

$$I(t) = \int_0^\infty \big[2h(s)h(t-s)-h(t)\big]\,\mathrm{d}s. \tag{17}$$

We claim that for $0<s<t$ the following inequality holds:

$$2h(s)h(t-s)-h(t) > 0, \tag{18}$$

which is equivalent to

$$\left(1+e^{s/2}\right)\left(1+e^{(t-s)/2}\right) < 2\left(1+e^{t/2}\right). \tag{19}$$

Let

$$J(t) = \ln\left(1+e^{s/2}\right) + \ln\left(1+e^{(t-s)/2}\right) - \ln\left[2\left(1+e^{t/2}\right)\right], \quad 0<s<t.$$

Calculating straightforwardly yields

$$J'(t) = \frac{e^{t/2}\left[1-e^{s/2}\right]}{2e^{s/2}\left(1+e^{t/2}\right)\left(1+e^{(t-s)/2}\right)} < 0.$$

Therefore we have $J(t) < J(s) = 0$, which means that inequality (18) is valid.

Combining (16), (17) and (18) leads to $g(x) > 0$. From (13), it follows that $\psi'(x) > 0$, and $\psi(x)$ is increasing in $(0,\infty)$. Since $1+f'(x) \geq 0$ by (11), $f''(x)$ and $g(x)$ have the same sign by (12), thus $f''(x) > 0$ and $f'(x)$ is increasing in $(0,\infty)$.

From (3), we have

$$\lim_{x\to\infty} x^{-\frac{1}{2}} \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} = 1, \tag{20}$$

From (6), it follows that

$$\lim_{x\to\infty} x\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] = \frac{1}{2}. \tag{21}$$

Combination of (11), (20) and (21) yields

$$f'(x) < \lim_{x\to\infty} f'(x) = 0,$$

which implies that $f(x)$ is decreasing in $(0,\infty)$. Hence

$$\lim_{n\to\infty}\left\{\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n\right\} < \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n \leq \left[\frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2})}\right]^2 - 1 = \frac{4}{\pi} - 1. \tag{22}$$

We can rewrite $f(x)$ as

$$f(x) = x\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma\big(x+\frac{1}{2}\big)} - 1\right]\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma\big(x+\frac{1}{2}\big)} + 1\right]. \tag{23}$$

Using (3) yields

$$\lim_{n\to\infty}\left\{\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n\right\} = \lim_{x\to\infty} f(x) = \frac{1}{4}. \tag{24}$$

The double inequality (2) follows from (22) and (24), and the constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ are the best possible. The proof is complete.

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(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA
E-mail address: `qifeng@jzit.edu.cn`
URL: `http://rgmia.vu.edu.au/qi.html`