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ON QUASI CONVEX FUNCTIONS AND HADAMARD’S INEQUALITY

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Abstract. In this paper we establish some inequalities of Hadamard’s type involving Godunova-Levin functions, P-functions, quasi-convex functions, J-quasi-convex functions, Wright-convex functions and Wright-quasi-convex functions.

1. Introduction

If \( f : [a, b] \to \mathbb{R} \) is a convex function, then the inequality
\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right)
\]
is known in the literature as Hadamard’s inequality.

For some results which generalize, improve, and extend this famous integral inequality see [1]–[10], [13]–[15], [18]–[21].

Let \( I \) be an interval in \( \mathbb{R} \), and \( a, b \in I \) with \( a < b \). We recall some definitions and theorems from the standpoint of abstract convexity.

Definition 1. (see [8, 11, 12, 13]) We say that \( f : I \to \mathbb{R} \) is a Godunova-Levin function, or that \( f \) belongs to the class \( Q(I) \), if \( f \) is nonnegative and for all \( x, y \in I \) and \( \lambda \in (0, 1) \), we have
\[
f(\lambda x + (1 - \lambda) y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}.
\]

Definition 2. (see [8, 11, 12, 14]) We say that \( f : I \to \mathbb{R} \) is a P-function, or that \( f \) belongs to the class \( P(I) \), if \( f \) is nonnegative and for all \( x, y \in I \) and \( \lambda \in [0, 1] \), we have
\[
f(\lambda x + (1 - \lambda) y) \leq f(x) + f(y).
\]

Dragomir, Pečarić and Persson [8] proved the following two theorems concerning Hadamard type inequalities.

Theorem 1. Let \( f \in Q(I) \cap L_1[a, b] \). Then
\[
f\left(\frac{a+b}{2}\right)(b-a) \leq 4 \int_a^b f(x) dx,
\]
and
\[
\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x) dx \leq \frac{f(a) + f(b)}{2} (b-a).
\]
The constant 4 in (1.2) is the best possible.

**Theorem 2.** Let \( f \in P(I) \cap L_1[a, b] \). Then

\[
(f(a) + f(b)) (b - a) \leq \int_a^b f(x) \, dx \leq 2 \int_a^b f(x) \, dx \leq 2 \int_a^b f(x) \, dx (b - a).
\]

Both inequalities are the best possible.

Recall some other concepts of convexity.

**Definition 3.** (see [16, pp. 228-233]) We say that \( f : I \to \mathbb{R} \) is a quasi-convex function, or that \( f \) belongs to the class \( QC(I) \), if, for all \( x, y \in I \) and \( \lambda \in [0, 1] \), we have

\[
f(\lambda x + (1 - \lambda) y) \leq \max(f(x), f(y)).
\]

**Definition 4.** (see [9]) We say that \( f : I \to \mathbb{R} \) is a \( J \)-quasi-convex function, or that \( f \) belongs to the class \( JQC(I) \), if, for all \( x, y \in I \), we have

\[
f\left(\frac{x + y}{2}\right) \leq \max(f(x), f(y)).
\]

**Definition 5.** (see [9, 17]) We say that \( f : I \to \mathbb{R} \) is a Wright-convex function, or that \( f \) belongs to the class \( WC(I) \), if, for all \( x, y + \delta \in I \) with \( x < y \) and \( \delta > 0 \), we have

\[
f(x + \delta) + f(y) \leq f(y + \delta) + f(x).
\]

**Definition 6.** (see [9]) We say that \( f : I \to \mathbb{R} \) is a Wright-quasi-convex function, or that \( f \) belongs to the class \( WQC(I) \), if, for all \( x, y + \delta \in I \) with \( x < y \) and \( \delta > 0 \) we have

\[
\frac{1}{2} [f(x + \delta) + f(y)] \leq \max(f(x), f(y)).
\]

Dragomir and Pearce [9] proved the following two theorems providing Hadamard type inequalities for the functions involved:

**Theorem 3.** Let \( f \in JQC(I) \cap L_1[a, b] \). Then

\[
(f(a) + f(b)) (b - a) \leq \int_a^b f(x) \, dx + I(a, b)(b - a),
\]

where

\[
I(a, b) := \frac{1}{2} \int_0^1 |f(ta + (1 - t)b) - f((1 - t)a + tb)| \, dt.
\]

Thus

\[
= \frac{1}{2(b - a)} \int_a^b |f(x) - f(a + b - x)| \, dx.
\]

Further, \( I(a, b) \) satisfies the inequalities

\[
0 \leq I(a, b) \leq \frac{1}{b - a} \min \left\{ \int_a^b |f(x)| \, dx, \frac{1}{\sqrt{2}} \left( (b - a) \int_a^b f^2(x) \, dx - J(a, b) \right)^{\frac{1}{2}} \right\},
\]
where

\[ J(a, b) := (b - a)^2 \int_0^1 f(ta + (1 - t)b)f((1 - t)a + tb)dt \]

\[ = (b - a) \int_a^b f(x)f(a + b - x)dx. \]  

**Theorem 4.** Let \( f \in WQC(I) \cap L_1[a, b] \). Then

\[ \int_a^b f(x)dx \leq \max\{f(a), f(b)\}(b - a). \]  

In this paper, we shall establish some generalizations of Theorem 1-4 for weighted integrals.

**Main Results**

Throughout this section, let \( s : [a, b] \to \mathbb{R} \) be non-negative, integrable and symmetric to \( \frac{a + b}{2} \) and let \( p : [a, b] \to \mathbb{R} \) be non-negative integrable with

\[ p(x) = p\left(\frac{b - a}{2} + x\right) \quad \left(x \in \left[a, \frac{a + b}{2}\right]\right). \]

The following result holds.

**Theorem 5.** Let \( f \in Q(I) \cap L_1[a, b] \). Then

\[ f\left(\frac{a + b}{2}\right) \int_a^b s(x)dx \leq 4 \int_a^b f(x)s(x)dx \]  

and

\[ \int_a^b \frac{(b - x)(x - a)}{(b - a)^2} f(x)s(x)dx \leq \frac{f(a) + f(b)}{2}. \int_a^b s(x)dx. \]

The constant 4 in (1.11) is the best possible.

**Proof.** Since \( f \in Q(I) \cap L_1[a, b] \) and \( g \) is nonnegative, symmetric to \( \frac{a + b}{2} \), we have successively

\[ f\left(\frac{a + b}{2}\right) \int_a^b s(x)dx = \int_a^b f\left(\frac{a + b}{2}\right) s(x)dx = \int_a^b f\left(\frac{x}{2} + \frac{a + b - x}{2}\right) s(x)dx \]

\[ \leq \int_a^b [2f(x) + 2f(a + b - x)]s(x)dx \]

\[ = 2 \left( \int_a^b f(x)s(x)dx + \int_a^b f(a + b - x)s(x)dx \right) \]

\[ = 2 \left( \int_a^b f(x)s(x)dx + \int_a^b f(a + b - x)s(a + b - x)dx \right) \]

\[ = 4 \int_a^b f(x)s(x)dx. \]

This proves (1.11).
Since
\[
\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(a+b-x)s(a+b-x)dx = \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx
\]
and \(s(a+b-x) = s(x)\) for \(x \in [a, b]\), then we have
\[
\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx = \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx + \frac{1}{2} \left[ f(b-a) \int_a^b s(x)dx \right]
\]
\[
\leq \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{1}{2} \left[ f(b-a) + f(b) \right] s(x)dx
\]
\[
= \frac{f(a) + f(b)}{2} \int_a^b s(x)dx.
\]

This proves (1.12).

Let us consider the function \(f : [a, b] \to \mathbb{R}\) given by
\[
f(x) = \begin{cases} 
1, & a \leq x < \frac{a+b}{2} \\
4, & x = \frac{a+b}{2} \\
1, & \frac{a+b}{2} < x \leq b.
\end{cases}
\]

Then \(f \in Q(I) \cap L_1[a, b]\) (see [8, p. 338]), and this proves that the constant 4 in (1.11) is the best possible as the inequality obviously reduces to an equality in this case. This completes the proof. \(\square\)

**Remark 1.** If we choose \(s(x) \equiv 1\), then Theorem 5 reduces to Theorem 1.

The second result is as follows.

**Theorem 6.** Let \(f \in P(I) \cap L_1[a, b]\). Then
\[
f \left( \frac{a+b}{2} \right) \int_a^b s(x)dx \leq 2 \int_a^b f(x)s(x)dx \leq 2 \left[ f(a) + f(b) \right] \int_a^b s(x)dx.
\]

Both inequalities in (1.13) are sharp.
Proof. Since \( f \in P(I) \cap L_1[a, b] \) and \( s \) is nonnegative, symmetric to \( \frac{a + b}{2} \), we have
\[
\int_a^b f \left( \frac{a + b}{2} \right) s(x) dx = \int_a^b f \left( \frac{a + b}{2} \right) s(x) dx \\
= \int_a^b f \left( \frac{x + a + b - x}{2} \right) s(x) dx \\
\leq \int_a^b [f(x) + f(a + b - x)] s(x) dx \\
= \int_a^b f(x) s(x) dx + \int_a^b f(a + b - x) s(x) dx
\]
\[
= \int_a^b f(x) s(x) dx + \int_a^b f(a + b - x) s(a + b - x) dx \\
= 2 \int_a^b f(x) s(x) dx \\
= 2 \int_a^b f \left( \frac{b - x}{b - a} + \frac{x - a}{b - a} \right) s(x) dx \\
\leq 2 \int_a^b [f(a) + f(b)] s(x) dx \\
= 2 [f(a) + f(b)] \int_a^b s(x) dx.
\]
This proves (1.13).
The functions
\[
f(x) = \begin{cases} 
0, & a \leq x < \frac{a + b}{2}, \\
1, & \frac{a + b}{2} \leq x \leq b,
\end{cases}
\]
and
\[
f(x) = \begin{cases} 
0, & x = a, \\
1, & a < x \leq b,
\end{cases}
\]
can be employed to show that both inequalities in (1.13) are the best possible. This completes the proof. \(\square\)

Remark 2. If we choose \( s(x) \equiv 1 \), then Theorem 6 reduces to Theorem 2.

The following result incorporating the function \( p \) satisfying (1.10) may be stated as well.

Theorem 7. Let \( f \in P(I) \cap L_1[a, b] \). Then
\[
(1.14) \quad f \left( \frac{a + b}{2} \right) \int_a^b p(x) dx \leq 2 \int_a^b f(x) p(x) dx \leq 2 [f(a) + f(b)] \int_a^b p(x) dx.
\]
Inequalities in (1.14) are the best possible.
Proof. By using (1.10), we have the following identities

\[(1.15) \quad \int_a^b p(x)dx = \int_a^{a+b} p(x)dx + \int_{a+b}^b p(x)dx\]
\[= \int_a^{a+b} p(x)dx + \int_{a+b}^b p \left( \frac{b-a}{2} + \left( x - \frac{b-a}{2} \right) \right) dx\]
\[= \int_a^{a+b} p(x)dx + \int_{a+b}^b p \left( x - \frac{b-a}{2} \right) dx\]
\[= 2 \int_a^{a+b} p(x)dx\]

and

\[(1.16) \quad \int_a^{a+b} \left[ f(x) + f \left( \frac{b-a}{2} + x \right) \right] p(x)dx\]
\[= \int_a^{a+b} f(x)p(x)dx + \int_a^{a+b} f \left( \frac{b-a}{2} + x \right) p(x)dx\]
\[= \int_a^{a+b} f(x)p(x)dx + \int_a^{a+b} f \left( \frac{b-a}{2} + x \right) p \left( \frac{b-a}{2} + x \right) dx\]
\[= \int_a^{a+b} f(x)p(x)dx + \int_{a+b}^b f(x)p(x)dx\]
\[= \int_a^b f(x)p(x)dx.\]

Since

\[0 \leq \frac{2(x-a)}{b-a}, \frac{a+b-2x}{b-a} \leq 1\]

and

\[\frac{2(x-a)}{b-a} + \frac{a+b-2x}{b-a} = 1\]
for \( x \in [a, \frac{a+b}{2}] \), it follows from \( f \in P(I) \cap L_1[a, b] \) and the identities (1.15) and (1.16), that

\[
f \left( \frac{a+b}{2} \right) \int_a^b p(x)dx = 2f \left( \frac{a+b}{2} \right) \int_a^{\frac{a+b}{2}} p(x)dx
\]

\[
= 2 \int_a^{\frac{a+b}{2}} f \left[ \frac{2(x-a)}{b-a} x + \frac{a+b-2x}{b-a} \left( \frac{b-a}{2} + x \right) \right] p(x)dx
\]

\[
\leq 2 \int_a^{\frac{a+b}{2}} \left[ f(x) + f \left( \frac{b-a}{2} + x \right) \right] p(x)dx
\]

\[
= 2 \int_a^b f(x)p(x)dx
\]

\[
= 2 \int_a^b f \left( \frac{b-x}{b-a} + \frac{x-a}{b-a} \right) p(x)dx
\]

\[
\leq 2 \int_a^b (f(a) + f(b))p(x)dx
\]

\[
= 2(f(a) + f(b)) \int_a^b p(x)dx.
\]

This proves (1.14). The functions

\[
f(x) = \begin{cases} 
0, & a \leq x < \frac{a+b}{2}, \\
1, & \frac{a+b}{2} \leq x \leq b,
\end{cases}
\]

and

\[
f(x) = \begin{cases} 
0, & x = a, \\
1, & a < x \leq b,
\end{cases}
\]

can be employed to show that both inequalities are the best possible. This completes the proof. \(\square\)

**Remark 3.** If we choose \( p(x) \equiv 1 \), then Theorem 7 reduces to Theorem 2.

We may now state the following result for quasi-convex functions.

**Theorem 8.** Let \( f \in QC(I) \cap L_1[a, b] \). Then

\[
(1.17) \quad f \left( \frac{a+b}{2} \right) \int_a^b s(x)dx \leq \int_a^b f(x)s(x)dx + I_1(a, b),
\]

where

\[
I_1(a, b) = \frac{1}{2} \int_a^b |f(x) - f(a + b - x)| s(x)dx.
\]

Further, \( I_1(a, b) \) satisfies the inequalities

\[
(1.18) \quad 0 \leq I_1(a, b)
\]

\[
\leq \min \left\{ \int_a^b |f(x)| s(x)dx, \frac{1}{\sqrt{2}} \left( \int_a^b f^2(x)dx - \int_a^b f(x)f(a + b - x)dx \right)^{\frac{1}{2}} \left( \int_a^b s^2(x)dx \right)^{\frac{1}{2}} \right\}.
\]
Proof. We shall use the fact that max\{c, d\} = \frac{1}{2}(c + d + |d - c|) for c, d ∈ \mathbb{R}. Since \(f \in QC(I) \cap L_1[a, b]\) and s is nonnegative, symmetric to \(\frac{a + b}{2}\), we have

\[
f\left(\frac{a + b}{2}\right) \int_a^b s(x)dx = \int_a^b f\left(\frac{x + a + b - x}{2}\right) s(x)dx
\]

\[
\leq \int_a^b \max\{f(x), f(a + b - x)\} \cdot s(x)dx
\]

\[
= \frac{1}{2} \left[ \int_a^b f(x) s(x)dx + \int_a^b f(a + b - x) s(x)dx \right]
\]

\[
+ \frac{1}{2} \int_a^b |f(x) - f(a + b - x)| s(x)dx
\]

\[
= \int_a^b f(x) s(x)dx + \frac{1}{2} \int_a^b |f(x) - f(a + b - x)| s(x)dx.
\]

This proves the inequality (1.17).

Since s is symmetric, it follows that

\[
0 \leq I_1(a, b) \leq \frac{1}{2} \left[ \int_a^b |f(x)| s(x)dx + \int_a^b |f(a + b - x)| s(x)dx \right]
\]

\[
= \frac{1}{2} \left[ \int_a^b |f(x)| s(x)dx + \int_a^b |f(a + b - x)| s(a + b - x)dx \right]
\]

\[
= \int_a^b |f(x)| s(x)dx.
\]

On the other hand, by the Cauchy-Schwarz inequality, we have

\[
I_1(a, b) = \frac{1}{2} \int_a^b |f(x) - f(a + b - x)| s(x)dx
\]

\[
\leq \frac{1}{2} \left( \int_a^b (f(x) - f(a + b - x))^2 dx \right)^{\frac{1}{2}} \left( \int_a^b s^2(x)dx \right)^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \left( \int_a^b (f(x)^2 + f^2(a + b - x) - 2f(x)f(a + b - x)) dx \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_a^b s^2(x)dx \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{2} \left( 2 \int_a^b f(x)^2 \, dx - 2 \int_a^b f(x)f(a + b - x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b s^2(x) \, dx \right)^{\frac{1}{2}} \\
= \frac{1}{\sqrt{2}} \left( \int_a^b f^2(x) \, dx - \int_a^b f(x)f(a + b - x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b s^2(x) \, dx \right)^{\frac{1}{2}}.
\]

The inequality (1.18) then follows from (1.19) and (1.20). This completes the proof. \[\square\]

Similarly, we have the following theorem:

**Theorem 9.** Let \( f \in JQC(I) \cap L_{1}[a, b] \). Then the inequalities (1.17) and (1.18) also hold.

**Remark 4.** If we choose \( s(x) \equiv 1 \), then Theorem 9 reduces to Theorem 3.

The corresponding result for the mapping \( p \) reads as:

**Theorem 10.** Let \( f \in QC(I) \cap L_{1}(a, b) \). Then

\[
(1.21) \quad f \left( \frac{a + b}{2} \right) \int_a^b p(x) \, dx \leq \int_a^b f(x)p(x) \, dx + I_2(a, b),
\]

where

\[
I_2(a, b) = \frac{1}{2} \int_a^b \left| f \left( \frac{x + a}{2} \right) - f \left( \frac{x + b}{2} \right) \right| p \left( \frac{x + a}{2} \right) \, dx.
\]

Further,

\[
(1.22) \quad 0 \leq I_2(a, b) \leq \min \left\{ \int_a^b |f(x)|p(x) \, dx , \frac{1}{\sqrt{2}} \left( \int_a^b f^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b p^2(x) \, dx \right)^{\frac{1}{2}} \right\}.
\]

**Proof.** We have

\[
0 \leq \frac{2(x - a)}{b - a} \cdot \frac{a + b - 2x}{b - a} \leq 1
\]

and

\[
\frac{2(x - a)}{b - a} + \frac{a + b - 2x}{b - a} = 1
\]
for $x \in \left[ a, \frac{a+b}{2} \right]$. By $f \in \text{QC}(I) \cap L_1[a,b]$ and the identities (1.15) and (1.16), we may state that

$$
\int_a^b f(x) p(x) dx
= 2 \int_a^{a+b/2} f(x) p(x) dx
= 2 \int_a^{a+b/2} f \left[ \frac{2(x-a)}{b-a} x + \frac{a+b-2x}{b-a} \left( \frac{b-a}{2} + x \right) \right] p(x) dx
\leq 2 \int_a^{a+b/2} \max \left\{ f(x), f \left( \frac{b-a}{2} + x \right) \right\} p(x) dx
= \int_a^{a+b/2} \left[ f(x) + f \left( \frac{b-a}{2} + x \right) \right] p(x) dx
\leq \int_a^{a+b/2} f(x) p(x) dx + \int_a^{a+b/2} f \left( \frac{b-a}{2} + x \right) p \left( \frac{b-a}{2} + x \right) dx
+ \int_a^{a+b/2} \left| f(x) - f \left( \frac{b-a}{2} + x \right) \right| p(x) dx
= \int_a^b f(x) p(x) dx + \int_a^{a+b/2} \left| f(x) - f \left( \frac{b-a}{2} + x \right) \right| p(x) dx
= \int_a^b f(x) p(x) dx + \int_a^b \left| f \left( \frac{x+a}{2} \right) - f \left( \frac{x+b}{2} \right) \right| p \left( \frac{x+a}{2} \right) dx.
$$

This proves (1.21).

A similar argument as in the proof of the inequality (1.18) implies the inequality (1.22). This completes the proof. □

**Corollary 1.** Let $f \in \text{QC}(I) \cap L_1[a,b]$. Then

$$
\left( \frac{a+b}{2} \right) (b-a) \leq \int_a^b \left| f(x) - f(a+b-x) \right| dx,
\int_a^b \left| f \left( \frac{x+a}{2} \right) - f \left( \frac{x+b}{2} \right) \right| dx.
$$

Proof. This follows from Theorem 8 and Theorem 10 by choosing $s(x) = p(x) = 1$. □

**Theorem 11.** Let $f \in \text{WC}(I) \cap L_1[a,b]$. Then

$$
\int_a^b f(x) s(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b s(x) dx.
$$

The inequality is the best possible.
Proof. Since \( f \in WC(I) \cap L_1[a,b] \) and \( s \) is nonnegative symmetric to \( \frac{a+b}{2} \), we have

\[
\int_a^b f(x)s(x)dx = \frac{1}{2} \left[ \int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx \right]
\]

\[
= \frac{1}{2} \int_a^b [f(x) + f(a+b-x)]s(x)dx
\]

\[
= \frac{1}{2} \int_a^b [f(a + (x-a)) + f(a+b-x)]s(x)dx
\]

\[
\leq \frac{1}{2} \int_a^b [f((a + b - x) + (x-a)) + f(a)]s(x)dx
\]

\[
= \frac{f(a) + f(b)}{2} \int_a^b s(x)dx.
\]

This proves the inequality (1.23), which reduces to an equality for \( f(x) \equiv 1 \).
This completes the proof. \( \square \)

Finally, we may state

**Theorem 12.** Let \( f \in WQC(I) \cap L_1[a,b] \). Then

(1.24) \[
\int_a^b f(x)s(x)dx \leq \max\{f(a), f(b)\} \int_a^b s(x)dx.
\]

The inequality is the best possible.

**Proof.** Since \( f \in WQC(I) \cap L_1[a,b] \) and \( s \) is nonnegative symmetric to \( \frac{a+b}{2} \), we have

\[
\int_a^b f(x)s(x)dx = \frac{1}{2} \left[ \int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx \right]
\]

\[
= \int_a^b \frac{1}{2} [f(x) + f(a+b-x)]s(x)dx
\]

\[
= \int_a^b \frac{1}{2} [f(a + (x-a)) + f(a+b-x)]s(x)dx
\]

\[
\leq \int_a^b \max\{f(a), f((a + b - x) + (x-a))\}s(x)dx
\]

\[
= (\max\{f(a), f(b)\}) \int_a^b s(x)dx.
\]

This proves the inequality (1.24), which reduces to an equality for \( f(x) \equiv 1 \).
This completes the proof. \( \square \)

**Remark 5.** If we choose \( g(x) \equiv 1 \), then Theorem 12 reduces to Theorem 4.

**References**


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