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ON QUASI CONVEX FUNCTIONS AND HADAMARD'S INEQUALITY

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ABSTRACT. In this paper we establish some inequalities of Hadamard's type involving Godunova-Levin functions, P-functions, quasi-convex functions, J-quasi-convex functions, Wright-convex functions and Wright-quasi-convex functions.

1. INTRODUCTION

If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then the inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

is known in the literature as Hadamard's inequality.

For some results which generalize, improve, and extend this famous integral inequality see [1]–[10], [13]–[15], [18]–[21].

Let I be an interval in \mathbb{R} , and $a, b \in I$ with $a < b$. We recall some definitions and theorems from the standpoint of abstract convexity.

Definition 1. (see [8, 11, 12, 13]) We say that $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function, or that f belongs to the class $Q(I)$, if f is nonnegative and for all $x, y \in I$ and $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}.$$

Definition 2. (see [8, 11, 12, 14]) We say that $f : I \rightarrow \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is nonnegative and for all $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y).$$

Dragomir, Pečarić and Persson [8] proved the following two theorems concerning Hadamard type inequalities.

Theorem 1. Let $f \in Q(I) \cap L_1[a, b]$. Then

$$(1.2) \quad f\left(\frac{a+b}{2}\right)(b-a) \leq 4 \int_a^b f(x)dx,$$

and

$$(1.3) \quad \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)dx \leq \frac{f(a)+f(b)}{2}(b-a).$$

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The constant 4 in (1.2) is the best possible.

Theorem 2. Let $f \in P(I) \cap L_1[a, b]$. Then

$$(1.4) \quad f\left(\frac{a+b}{2}\right)(b-a) \leq 2 \int_a^b f(x)dx \leq 2[f(a) + f(b)](b-a).$$

Both inequalities are the best possible.

Recall some other concepts of convexity.

Definition 3. (see [16, pp. 228-233]) We say that $f : I \rightarrow \mathbb{R}$ is a quasi-convex function, or that f belongs to the class $QC(I)$, if, for all $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y)).$$

Definition 4. (see [9]) We say that $f : I \rightarrow \mathbb{R}$ is a J -quasi-convex function, or that f belongs to the class $JQC(I)$, if, for all $x, y \in I$, we have

$$f\left(\frac{x+y}{2}\right) \leq \max(f(x), f(y)).$$

Definition 5. (see [9, 17]) We say that $f : I \rightarrow \mathbb{R}$ is a Wright-convex function, or that f belongs to the class $WC(I)$, if, for all $x, y + \delta \in I$ with $x < y$ and $\delta > 0$, we have

$$f(x + \delta) + f(y) \leq f(y + \delta) + f(x).$$

Definition 6. (see [9]) We say that $f : I \rightarrow \mathbb{R}$ is a Wright-quasi-convex function, or that f belongs to the class $WQC(I)$, if, for all $x, y + \delta \in I$ with $x < y$ and $\delta > 0$ we have

$$\frac{1}{2}[f(x + \delta) + f(y)] \leq \max(f(x), f(y + \delta)).$$

Dragomir and Pearce [9] proved the following two theorems providing Hadamard type inequalities for the functions involved:

Theorem 3. Let $f \in JQC(I) \cap L_1[a, b]$. Then

$$(1.5) \quad f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(x)dx + I(a, b)(b-a),$$

where

$$(1.6) \quad \begin{aligned} I(a, b) &:= \frac{1}{2} \int_0^1 |f(ta + (1-t)b) - f((1-t)a + tb)| dt. \\ &= \frac{1}{2(b-a)} \int_a^b |f(x) - f(a+b-x)| dx. \end{aligned}$$

Further, $I(a, b)$ satisfies the inequalities

$$(1.7) \quad \begin{aligned} 0 &\leq I(a, b) \\ &\leq \frac{1}{b-a} \min \left\{ \int_a^b |f(x)| dx, \frac{1}{\sqrt{2}} \left((b-a) \int_a^b f^2(x) dx - J(a, b) \right)^{\frac{1}{2}} \right\}, \end{aligned}$$

where

$$(1.8) \quad \begin{aligned} J(a, b) &:= (b-a)^2 \int_0^1 f(ta + (1-t)b) f((1-t)a + tb) dt \\ &= (b-a) \int_a^b f(x) f(a+b-x) dx. \end{aligned}$$

Theorem 4. *Let $f \in WQC(I) \cap L_1[a, b]$. Then*

$$(1.9) \quad \int_a^b f(x) dx \leq \max\{f(a), f(b)\}(b-a).$$

In this paper, we shall establish some generalizations of Theorem 1-4 for weighted integrals.

MAIN RESULTS

Throughout this section, let $s : [a, b] \rightarrow \mathbb{R}$ be non-negative, integrable and symmetric to $\frac{a+b}{2}$ and let $p : [a, b] \rightarrow \mathbb{R}$ be non-negative integrable with

$$(1.10) \quad p(x) = p\left(\frac{b-a}{2} + x\right) \quad \left(x \in \left[a, \frac{a+b}{2}\right]\right).$$

The following result holds.

Theorem 5. *Let $f \in Q(I) \cap L_1[a, b]$. Then*

$$(1.11) \quad f\left(\frac{a+b}{2}\right) \int_a^b s(x) dx \leq 4 \int_a^b f(x) s(x) dx$$

and

$$(1.12) \quad \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x) s(x) dx \leq \frac{f(a) + f(b)}{2} \cdot \int_a^b s(x) dx.$$

The constant 4 in (1.11) is the best possible.

Proof. Since $f \in Q(I) \cap L_1[a, b]$ and g is nonnegative, symmetric to $\frac{a+b}{2}$, we have successively

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b s(x) dx &= \int_a^b f\left(\frac{a+b}{2}\right) s(x) dx = \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) s(x) dx \\ &\leq \int_a^b [2f(x) + 2f(a+b-x)] s(x) dx \\ &= 2 \left(\int_a^b f(x) s(x) dx + \int_a^b f(a+b-x) s(x) dx \right) \\ &= 2 \left(\int_a^b f(x) s(x) dx + \int_a^b f(a+b-x) s(a+b-x) dx \right) \\ &= 4 \int_a^b f(x) s(x) dx. \end{aligned}$$

This proves (1.11).

Since

$$\begin{aligned} \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(a+b-x) s(a+b-x) dx \\ = \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x) s(x) dx \end{aligned}$$

and $s(a+b-x) = s(x)$ for $x \in [a, b]$, then we have

$$\begin{aligned} & \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x) s(x) dx \\ &= \int_a^b \frac{1}{2} \left[\frac{(b-x)(x-a)}{(b-a)^2} f(x) s(x) dx + \frac{(b-x)(x-a)}{(b-a)^2} f(a+b-x) s(a+b-x) \right] dx \\ &= \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{f(x) + f(a+b-x)}{2} s(x) dx \\ &= \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{1}{2} \left[f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right) \right] s(x) dx \\ &\leq \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{1}{2} \left[\frac{b-a}{b-x} f(a) + \frac{b-a}{x-a} f(b) + \frac{b-a}{x-a} f(a) + \frac{b-a}{b-x} f(b) \right] s(x) dx \\ &= \frac{f(a) + f(b)}{2} \int_a^b s(x) dx. \end{aligned}$$

This proves (1.12).

Let us consider the function $f : [a, b] \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & a \leq x < \frac{a+b}{2}, \\ 4, & x = \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} < x \leq b. \end{cases}$$

Then $f \in Q(I) \cap L_1[a, b]$ (see [8, p. 338]), and this proves that the constant 4 in (1.11) is the best possible as the inequality obviously reduces to an equality in this case. This completes the proof. \square

Remark 1. If we choose $s(x) \equiv 1$, then Theorem 5 reduces to Theorem 1.

The second result is as follows.

Theorem 6. Let $f \in P(I) \cap L_1[a, b]$. Then

$$(1.13) \quad f\left(\frac{a+b}{2}\right) \int_a^b s(x) dx \leq 2 \int_a^b f(x) s(x) dx \leq 2[f(a) + f(b)] \int_a^b s(x) dx.$$

Both inequalities in (1.13) are sharp.

Proof. Since $f \in P(I) \cap L_1[a, b]$ and s is nonnegative, symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) \int_a^b s(x) dx &= \int_a^b f\left(\frac{a+b}{2}\right) s(x) dx \\
 &= \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) s(x) dx \\
 &\leq \int_a^b [f(x) + f(a+b-x)] s(x) dx \\
 &= \int_a^b f(x) s(x) dx + \int_a^b f(a+b-x) s(x) dx \\
 &= \int_a^b f(x) s(x) dx + \int_a^b f(a+b-x) s(a+b-x) dx \\
 &= 2 \int_a^b f(x) s(x) dx \\
 &= 2 \int_a^b f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) s(x) dx \\
 &\leq 2 \int_a^b [f(a) + f(b)] s(x) dx \\
 &= 2 [f(a) + f(b)] \int_a^b s(x) dx.
 \end{aligned}$$

This proves (1.13).

The functions

$$f(x) = \begin{cases} 0, & a \leq x < \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \leq x \leq b, \end{cases}$$

and

$$f(x) = \begin{cases} 0, & x = a, \\ 1, & a < x \leq b, \end{cases}$$

can be employed to show that both inequalities in (1.13) are the best possible. This completes the proof. \square

Remark 2. If we choose $s(x) \equiv 1$, then Theorem 6 reduces to Theorem 2.

The following result incorporating the function p satisfying (1.10) may be stated as well.

Theorem 7. Let $f \in P(I) \cap L_1[a, b]$. Then

$$(1.14) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq 2 \int_a^b f(x) p(x) dx \leq 2 [f(a) + f(b)] \int_a^b p(x) dx.$$

Inequalities in (1.14) are the best possible.

Proof. By using (1.10), we have the following identities

$$\begin{aligned}
 (1.15) \quad \int_a^b p(x)dx &= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p(x)dx \\
 &= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p\left(\frac{b-a}{2} + \left(x - \frac{b-a}{2}\right)\right) dx \\
 &= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p\left(x - \frac{b-a}{2}\right) dx \\
 &= 2 \int_a^{\frac{a+b}{2}} p(x)dx
 \end{aligned}$$

and

$$\begin{aligned}
 (1.16) \quad &\int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2} + x\right) \right] p(x)dx \\
 &= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2} + x\right) p(x)dx \\
 &= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2} + x\right) p\left(\frac{b-a}{2} + x\right) dx \\
 &= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_{\frac{a+b}{2}}^b f(x)p(x)dx \\
 &= \int_a^b f(x)p(x)dx.
 \end{aligned}$$

Since

$$0 \leq \frac{2(x-a)}{b-a}, \frac{a+b-2x}{b-a} \leq 1$$

and

$$\frac{2(x-a)}{b-a} + \frac{a+b-2x}{b-a} = 1$$

for $x \in [a, \frac{a+b}{2}]$, it follows from $f \in P(I) \cap L_1[a, b]$ and the identities (1.15) and (1.16), that

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx &= 2f\left(\frac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} p(x)dx \\
 &= 2 \int_a^{\frac{a+b}{2}} f\left[\frac{2(x-a)}{b-a}x + \frac{a+b-2x}{b-a}\left(\frac{b-a}{2} + x\right)\right] p(x)dx \\
 &\leq 2 \int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2} + x\right)\right] p(x)dx \\
 &= 2 \int_a^b f(x)p(x)dx \\
 &= 2 \int_a^b f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) p(x)dx \\
 &\leq 2 \int_a^b (f(a) + f(b))p(x)dx \\
 &= 2(f(a) + f(b)) \int_a^b p(x)dx.
 \end{aligned}$$

This proves (1.14). The functions

$$f(x) = \begin{cases} 0, & a \leq x < \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \leq x \leq b, \end{cases}$$

and

$$f(x) = \begin{cases} 0, & x = a, \\ 1, & a < x \leq b, \end{cases}$$

can be employed to show that both inequalities are the best possible. This completes the proof. \square

Remark 3. If we choose $p(x) \equiv 1$, then Theorem 7 reduces to Theorem 2.

We may now state the following result for quasi-convex functions.

Theorem 8. Let $f \in QC(I) \cap L_1[a, b]$. Then

$$(1.17) \quad f\left(\frac{a+b}{2}\right) \int_a^b s(x)dx \leq \int_a^b f(x)s(x)dx + I_1(a, b),$$

where

$$I_1(a, b) = \frac{1}{2} \int_a^b |f(x) - f(a+b-x)| s(x)dx.$$

Further, $I_1(a, b)$ satisfies the inequalities

$$\begin{aligned}
 (1.18) \quad 0 &\leq I_1(a, b) \\
 &\leq \min \left\{ \int_a^b |f(x)| s(x)dx, \right. \\
 &\quad \left. \frac{1}{\sqrt{2}} \left(\int_a^b f^2(x)dx - \int_a^b f(x)f(a+b-x)dx \right)^{\frac{1}{2}} \left(\int_a^b s^2(x)dx \right)^{\frac{1}{2}} \right\}.
 \end{aligned}$$

Proof. We shall use the fact that $\max\{c, d\} = \frac{1}{2}(c + d + |d - c|)$ for $c, d \in \mathbb{R}$. Since $f \in QC(I) \cap L_1[a, b]$ and s is nonnegative, symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) \int_a^b s(x) dx &= \int_a^b f\left(\frac{a+b}{2}\right) s(x) dx \\
&= \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) s(x) dx \\
&\leq \int_a^b \max\{f(x), f(a+b-x)\} \cdot s(x) dx \\
&= \int_a^b \frac{1}{2} [f(x) + f(a+b-x) + |f(x) - f(a+b-x)|] s(x) dx \\
&= \frac{1}{2} \left[\int_a^b f(x) s(x) dx + \int_a^b f(a+b-x) s(x) dx \right] \\
&\quad + \frac{1}{2} \int_a^b |f(x) - f(a+b-x)| s(x) dx \\
&= \int_a^b f(x) s(x) dx + \frac{1}{2} \int_a^b |f(x) - f(a+b-x)| s(x) dx.
\end{aligned}$$

This proves the inequality (1.17).

Since s is symmetric, it follows that

$$\begin{aligned}
(1.19) \quad 0 \leq I_1(a, b) &\leq \frac{1}{2} \left[\int_a^b |f(x)| s(x) dx + \int_a^b |f(a+b-x)| s(x) dx \right] \\
&= \frac{1}{2} \left[\int_a^b |f(x)| s(x) dx + \int_a^b |f(a+b-x)| s(a+b-x) dx \right] \\
&= \int_a^b |f(x)| s(x) dx.
\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(1.20) \quad I_1(a, b) &= \frac{1}{2} \int_a^b |f(x) - f(a+b-x)| s(x) dx \\
&\leq \frac{1}{2} \left(\int_a^b (f(x) - f(a+b-x))^2 dx \right)^{\frac{1}{2}} \left(\int_a^b s^2(x) dx \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left(\int_a^b (f(x)^2 + f^2(a+b-x) - 2f(x)f(a+b-x)) dx \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_a^b s^2(x) dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(2 \int_a^b f(x)^2 dx - 2 \int_a^b f(x) f(a+b-x) dx \right)^{\frac{1}{2}} \left(\int_a^b s^2(x) dx \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \left(\int_a^b f^2(x) dx - \int_a^b f(x) f(a+b-x) dx \right)^{\frac{1}{2}} \left(\int_a^b s^2(x) dx \right)^{\frac{1}{2}}.
\end{aligned}$$

The inequality (1.18) then follows from (1.19) and (1.20). This completes the proof. \square

Similarly, we have the following theorem:

Theorem 9. *Let $f \in JQC(I) \cap L_1[a, b]$. Then the inequalities (1.17) and (1.18) also hold.*

Remark 4. *If we choose $s(x) \equiv 1$, then Theorem 9 reduces to Theorem 3.*

The corresponding result for the mapping p reads as:

Theorem 10. *Let $f \in QC(I) \cap L_1(a, b]$. Then*

$$(1.21) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq \int_a^b f(x) p(x) dx + I_2(a, b),$$

where

$$I_2(a, b) = \frac{1}{2} \int_a^b \left| f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right) \right| p\left(\frac{x+a}{2}\right) dx.$$

Further,

$$\begin{aligned}
(1.22) \quad &0 \leq I_2(a, b) \\
&\leq \min \left\{ \int_a^b |f(x)| p(x) dx, \frac{1}{\sqrt{2}} \left(\int_a^b f^2(x) dx \right. \right. \\
&\quad \left. \left. - \int_a^b f\left(\frac{x+a}{2}\right) f\left(\frac{x+b}{2}\right) dx \right)^{\frac{1}{2}} \left(\int_a^b p^2(x) dx \right)^{\frac{1}{2}} \right\}.
\end{aligned}$$

Proof. We have

$$0 \leq \frac{2(x-a)}{b-a}, \frac{a+b-2x}{b-a} \leq 1$$

and

$$\frac{2(x-a)}{b-a} + \frac{a+b-2x}{b-a} = 1$$

for $x \in [a, \frac{a+b}{2}]$. By $f \in QC(I) \cap L_1[a, b]$ and the identities (1.15) and (1.16), we may state that

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \\
&= 2f\left(\frac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} p(x) dx \\
&= 2 \int_a^{\frac{a+b}{2}} f\left[\frac{2(x-a)}{b-a}x + \frac{a+b-2x}{b-a}\left(\frac{b-a}{2} + x\right)\right] p(x) dx \\
&\leq 2 \int_a^{\frac{a+b}{2}} \max\left\{f(x), f\left(\frac{b-a}{2} + x\right)\right\} p(x) dx \\
&= \int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2} + x\right) + \left|f(x) - f\left(\frac{b-a}{2} + x\right)\right|\right] p(x) dx \\
&= \int_a^{\frac{a+b}{2}} f(x)p(x) dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2} + x\right) p\left(\frac{b-a}{2} + x\right) dx \\
&\quad + \int_a^{\frac{a+b}{2}} \left|f(x) - f\left(\frac{b-a}{2} + x\right)\right| p(x) dx \\
&= \int_a^b f(x)p(x) dx + \int_a^{\frac{a+b}{2}} \left|f(x) - f\left(\frac{b-a}{2} + x\right)\right| p(x) dx \\
&= \int_a^b f(x)p(x) dx + \int_a^b \left|f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right)\right| p\left(\frac{x+a}{2}\right) dx.
\end{aligned}$$

This proves (1.21).

A similar argument as in the proof of the inequality (1.18) implies the inequality (1.22). This completes the proof. \square

Corollary 1. *Let $f \in QC(I) \cap L_1[a, b]$. Then*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)(b-a) &\leq \int_a^b f(x) dx + \frac{1}{2} \min \left\{ \int_a^b |f(x) - f(a+b-x)| dx, \right. \\
&\quad \left. \int_a^b \left|f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right)\right| dx \right\}.
\end{aligned}$$

Proof. This follows from Theorem 8 and Theorem 10 by choosing $s(x) = p(x) = 1$. \square

Theorem 11. *Let $f \in WC(I) \cap L_1[a, b]$. Then*

$$(1.23) \quad \int_a^b f(x)s(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b s(x) dx.$$

The inequality is the best possible.

Proof. Since $f \in WC(I) \cap L_1[a, b]$ and s is nonnegative symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned} \int_a^b f(x)s(x)dx &= \frac{1}{2} \left[\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx \right] \\ &= \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] s(x)dx \\ &= \frac{1}{2} \int_a^b [f(a+(x-a)) + f(a+b-x)] s(x)dx \\ &\leq \frac{1}{2} \int_a^b [f((a+b-x) + (x-a)) + f(a)] s(x)dx \\ &= \frac{f(a) + f(b)}{2} \int_a^b s(x)dx. \end{aligned}$$

This proves the inequality (1.23), which reduces to an equality for $f(x) \equiv 1$. This completes the proof. \square

Finally, we may state

Theorem 12. *Let $f \in WQC(I) \cap L_1[a, b]$. Then*

$$(1.24) \quad \int_a^b f(x)s(x)dx \leq \max\{f(a), f(b)\} \int_a^b s(x)dx.$$

The inequality is the best possible.

Proof. Since $f \in WQC(I) \cap L_1[a, b]$ and s is nonnegative symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned} \int_a^b f(x)s(x)dx &= \frac{1}{2} \left[\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx \right] \\ &= \int_a^b \frac{1}{2} (f(x) + f(a+b-x))s(x)dx \\ &= \int_a^b \frac{1}{2} [f(a+(x-a)) + f(a+b-x)] s(x)dx \\ &\leq \int_a^b \max\{f(a), f((a+b-x) + (x-a))\} s(x)dx \\ &= (\max\{f(a), f(b)\}) \int_a^b s(x)dx. \end{aligned}$$

This proves the inequality (1.24), which reduces to an equality for $f(x) \equiv 1$. This completes the proof. \square

Remark 5. *If we choose $g(x) \equiv 1$, then Theorem 12 reduces to Theorem 4.*

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