On Quasi Convex Functions and Hadamard's Inequality

ON QUASI CONVEX FUNCTIONS AND HADAMARD'S INEQUALITY

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Abstract. In this paper we establish some inequalities of Hadamard's type involving Godunova-Levin functions, P-functions, quasi-convex functions, J-quasi-convex functions, Wright-convex functions and Wright-quasi-convex functions.

1. Introduction

If $f:[a,b]\to\mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known in the literature as Hadamard's inequality.

For some results which generalize, improve, and extend this famous integral inequality see [1]–[10], [13]–[15], [18]–[21].

Let I be an interval in $\mathbb{R}$, and $a,b\in I$ with $a<b$. We recall some definitions and theorems from the standpoint of abstract convexity.

Definition 1. *(see [8, 11, 12, 13]) We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function, or that f belongs to the class $Q(I)$, if f is nonnegative and for all $x,y\in I$ and $\lambda\in(0,1)$, we have*

$$f(\lambda x+(1-\lambda)y)\leq\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}.$$

Definition 2. *(see [8, 11, 12, 14]) We say that $f:I\to\mathbb{R}$ is a $P-$function, or that f belongs to the class $P(I)$, if f is nonnegative and for all $x,y\in I$ and $\lambda\in[0,1]$, we have*

$$f(\lambda x+(1-\lambda)y)\leq f(x)+f(y).$$

Dragomir, Pečarić and Persson [8] proved the following two theorems concerning Hadamard type inequalities.

Theorem 1. *Let $f\in Q(I)\cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)(b-a)\leq 4\int_a^b f(x)dx, \tag{1.2}$$

and

$$\int_a^b\frac{(b-x)(x-a)}{(b-a)^2}f(x)dx\leq\frac{f(a)+f(b)}{2}(b-a). \tag{1.3}$$

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The constant 4 in (1.2) is the best possible.

Theorem 2. *Let $f \in P(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)(b-a) \le 2\int_a^b f(x)dx \le 2\left[f(a)+f(b)\right](b-a). \tag{1.4}$$

Both inequalities are the best possible.

Recall some other cpncepts of convexity.

Definition 3. *(see [16, pp. 228-233]) We say that $f : I \to \mathbb{R}$ is a quasi-convex function, or that f belongs to the class $QC(I)$, if, for all $x, y \in I$ and $\lambda \in [0,1]$, we have*

$$f(\lambda x + (1-\lambda)y) \le \max(f(x), f(y)).$$

Definition 4. *(see [9]) We say that $f : I \to \mathbb{R}$ is a J-quasi-convex function, or that f belongs to the class $JQC(I)$, if, for all $x, y \in I$, we have*

$$f\left(\frac{x+y}{2}\right) \le \max(f(x), f(y)).$$

Definition 5. *(see [9, 17]) We say that $f : I \to \mathbb{R}$ is a Wright-convex function, or that f belongs to the class $WC(I)$, if, for all $x, y+\delta \in I$ with $x < y$ and $\delta > 0$, we have*

$$f(x+\delta) + f(y) \le f(y+\delta) + f(x).$$

Definition 6. *(see [9]) We say that $f : I \to \mathbb{R}$ is a Wright-quasi-convex function, or that f belongs to the class $WQC(I)$, if, for all $x, y+\delta \in I$ with $x < y$ and $\delta > 0$ we have*

$$\frac{1}{2}[f(x+\delta) + f(y)] \le \max(f(x), f(y+\delta)).$$

Dragomir and Pearce [9] proved the following two theorems providing Hadamard type inequalities for the functions involved:

Theorem 3. *Let $f \in JQC(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b f(x)dx + I(a,b)(b-a), \tag{1.5}$$

where

$$\begin{aligned} I(a,b) &:= \frac{1}{2}\int_0^1 \left|f(ta+(1-t)b) - f((1-t)a+tb)\right| dt. \\ &= \frac{1}{2(b-a)}\int_a^b \left|f(x) - f(a+b-x)\right| dx. \end{aligned} \tag{1.6}$$

Further, $I(a,b)$ satisfies the inequalities

$$\begin{aligned} 0 &\le I(a,b) \\ &\le \frac{1}{b-a}\min\left\{\int_a^b |f(x)|\,dx, \frac{1}{\sqrt{2}}\left((b-a)\int_a^b f^2(x)dx - J(a,b)\right)^{\frac{1}{2}}\right\}, \end{aligned} \tag{1.7}$$

where

$$J(a,b) := (b-a)^2 \int_0^1 f(ta+(1-t)b)f((1-t)a+tb)dt \tag{1.8}$$
$$= (b-a)\int_a^b f(x)f(a+b-x)dx.$$

Theorem 4. *Let $f \in WQC(I) \cap L_1[a,b]$. Then*

$$\int_a^b f(x)dx \le \max\{f(a), f(b)\}(b-a). \tag{1.9}$$

In this paper, we shall establish some generalizations of Theorem 1-4 for weighted integrals.

Main Results

Throughout this section, let $s : [a,b] \to \mathbb{R}$ be non-negative, integrable and symmetric to $\frac{a+b}{2}$ and let $p : [a,b] \to \mathbb{R}$ be non-negative integrable with

$$p(x) = p\left(\frac{b-a}{2} + x\right) \qquad \left(x \in \left[a, \frac{a+b}{2}\right]\right). \tag{1.10}$$

The following result holds.

Theorem 5. *Let $f \in Q(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx \le 4\int_a^b f(x)s(x)dx \tag{1.11}$$

and

$$\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx \le \frac{f(a)+f(b)}{2} \cdot \int_a^b s(x)dx. \tag{1.12}$$

The constant 4 in (1.11) is the best possible.

Proof. Since $f \in Q(I) \cap L_1[a,b]$ and g is nonnegative, symmetric to $\frac{a+b}{2}$, we have successively

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx &= \int_a^b f\left(\frac{a+b}{2}\right)s(x)dx = \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right)s(x)dx \\
&\le \int_a^b [2f(x) + 2f(a+b-x)]s(x)dx \\
&= 2\left(\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(x)dx\right) \\
&= 2\left(\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx\right) \\
&= 4\int_a^b f(x)s(x)dx.
\end{aligned}$$

This proves (1.11).

Since

$$\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(a+b-x)s(a+b-x)dx = \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx$$

and $s(a+b-x) = s(x)$ for $x \in [a,b]$, then we have

$$\begin{aligned}
&\int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx \\
&= \int_a^b \frac{1}{2}\left[\frac{(b-x)(x-a)}{(b-a)^2} f(x)s(x)dx + \frac{(b-x)(x-a)}{(b-a)^2} f(a+b-x)s(a+b-x)\right] dx \\
&= \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{f(x)+f(a+b-x)}{2} s(x)dx \\
&= \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{1}{2}\left[f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right] s(x)dx \\
&\le \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} \cdot \frac{1}{2}\left[\frac{b-a}{b-x}f(a) + \frac{b-a}{x-a}f(b) + \frac{b-a}{x-a}f(a) + \frac{b-a}{b-x}f(b)\right] s(x)dx \\
&= \frac{f(a)+f(b)}{2}\int_a^b s(x)dx.
\end{aligned}$$

This proves (1.12).

Let us consider the function $f : [a,b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & a \le x < \frac{a+b}{2}, \\ 4, & x = \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} < x \le b. \end{cases}$$

Then $f \in Q(I) \cap L_1[a,b]$ (see [8, p. 338]), and this proves that the constant 4 in (1.11) is the best possible as the inequality obviously reduces to an equality in this case. This completes the proof. □

Remark 1. *If we choose $s(x) \equiv 1$, then Theorem 5 reduces to Theorem 1.*

The second result is as follows.

Theorem 6. *Let $f \in P(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx \le 2\int_a^b f(x)s(x)dx \le 2\left[f(a)+f(b)\right]\int_a^b s(x)dx. \tag{1.13}$$

Both inequalities in (1.13) are sharp.

Proof. Since $f \in P(I) \cap L_1[a,b]$ and s is nonnegative, symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx &= \int_a^b f\left(\frac{a+b}{2}\right)s(x)dx \\ &= \int_a^b f\left(\frac{x}{2}+\frac{a+b-x}{2}\right)s(x)dx \\ &\le \int_a^b \left[f(x)+f(a+b-x)\right]s(x)dx \\ &= \int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(x)dx \end{aligned}$$

$$\begin{aligned} &= \int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx \\ &= 2\int_a^b f(x)s(x)dx \\ &= 2\int_a^b f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)s(x)dx \\ &\le 2\int_a^b \left[f(a)+f(b)\right]s(x)dx \\ &= 2\left[f(a)+f(b)\right]\int_a^b s(x)dx. \end{aligned}$$

This proves (1.13).

The functions

$$f(x)=\begin{cases} 0, & a \le x < \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \le x \le b, \end{cases}$$

and

$$f(x)=\begin{cases} 0, & x = a, \\ 1, & a < x \le b, \end{cases}$$

can be employed to show that both inequalities in (1.13) are the best possible. This completes the proof. □

Remark 2. *If we choose $s(x) \equiv 1$, then Theorem 6 reduces to Theorem 2.*

The following result incorporating the function p satisfying (1.10) may be stated as well.

Theorem 7. *Let $f \in P(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \le 2\int_a^b f(x)p(x)dx \le 2\left[f(a)+f(b)\right]\int_a^b p(x)dx. \tag{1.14}$$

Inequalities in (1.14) are the best possible.

Proof. By using (1.10), we have the following identities

$$\begin{aligned}
\int_a^b p(x)dx &= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p(x)dx \\
&= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p\left(\frac{b-a}{2} + \left(x - \frac{b-a}{2}\right)\right) dx \\
&= \int_a^{\frac{a+b}{2}} p(x)dx + \int_{\frac{a+b}{2}}^b p\left(x - \frac{b-a}{2}\right) dx \\
&= 2\int_a^{\frac{a+b}{2}} p(x)dx
\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2} + x\right)\right] p(x)dx \\
&= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2} + x\right) p(x)dx \\
&= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2} + x\right) p\left(\frac{b-a}{2} + x\right) dx \\
&= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_{\frac{a+b}{2}}^b f(x)p(x)dx \\
&= \int_a^b f(x)p(x)dx.
\end{aligned} \tag{1.16}$$

Since

$$0 \le \frac{2(x-a)}{b-a}, \frac{a+b-2x}{b-a} \le 1$$

and

$$\frac{2(x-a)}{b-a} + \frac{a+b-2x}{b-a} = 1$$

for $x \in [a, \frac{a+b}{2}]$, it follows from $f \in P(I) \cap L_1[a,b]$ and the identities (1.15) and (1.16), that

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx &= 2f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} p(x)dx \\
&= 2\int_a^{\frac{a+b}{2}} f\left[\frac{2(x-a)}{b-a}x + \frac{a+b-2x}{b-a}\left(\frac{b-a}{2}+x\right)\right] p(x)dx \\
&\le 2\int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2}+x\right)\right] p(x)dx \\
&= 2\int_a^b f(x)p(x)dx \\
&= 2\int_a^b f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) p(x)dx \\
&\le 2\int_a^b (f(a)+f(b))p(x)dx \\
&= 2(f(a)+f(b))\int_a^b p(x)dx.
\end{aligned}$$

This proves (1.14). The functions

$$f(x) = \begin{cases} 0, & a \le x < \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \le x \le b, \end{cases}$$

and

$$f(x) = \begin{cases} 0, & x = a, \\ 1, & a < x \le b, \end{cases}$$

can be employed to show that both inequalities are the best possible. This completes the proof. □

Remark 3. *If we choose $p(x) \equiv 1$, then Theorem 7 reduces to Theorem 2.*

We may now state the following result for quasi-convex functions.

Theorem 8. *Let $f \in QC(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx \le \int_a^b f(x)s(x)dx + I_1(a,b), \tag{1.17}$$

where

$$I_1(a,b) = \frac{1}{2}\int_a^b |f(x) - f(a+b-x)|\, s(x)dx.$$

Further, $I_1(a,b)$ satisfies the inequalities

$$\begin{aligned}
0 &\le I_1(a,b) \\
&\le \min\left\{\int_a^b |f(x)|\, s(x)dx\ ,\right. \\
&\left.\frac{1}{\sqrt{2}}\left(\int_a^b f^2(x)dx - \int_a^b f(x)f(a+b-x)dx\right)^{\frac{1}{2}}\left(\int_a^b s^2(x)dx\right)^{\frac{1}{2}}\right\}.
\end{aligned} \tag{1.18}$$

Proof. We shall use the fact that $\max\{c,d\} = \frac{1}{2}(c+d+|d-c|)$ for $c,d \in \mathbb{R}$. Since $f \in QC(I) \cap L_1[a,b]$ and s is nonnegative, symmetric to $\dfrac{a+b}{2}$, we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b s(x)dx &= \int_a^b f\left(\frac{a+b}{2}\right)s(x)dx \\
&= \int_a^b f\left(\frac{x}{2}+\frac{a+b-x}{2}\right)s(x)dx \\
&\le \int_a^b \max\{f(x), f(a+b-x)\}\cdot s(x)dx \\
&= \int_a^b \frac{1}{2}[f(x)+f(a+b-x)+|f(x)-f(a+b-x)|]s(x)dx \\
&= \frac{1}{2}\left[\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(x)dx\right] \\
&\qquad + \frac{1}{2}\int_a^b |f(x)-f(a+b-x)|\, s(x)dx \\
&= \int_a^b f(x)s(x)dx + \frac{1}{2}\int_a^b |f(x)-f(a+b-x)|\, s(x)dx.
\end{aligned}$$

This proves the inequality (1.17).

Since s is symmetric, it follows that

$$\begin{aligned}
0 \le I_1(a,b) &\le \frac{1}{2}\left[\int_a^b |f(x)|\, s(x)dx + \int_a^b |f(a+b-x)|\, s(x)dx\right] \\
&= \frac{1}{2}\left[\int_a^b |f(x)|\, s(x)dx + \int_a^b |f(a+b-x)|\, s(a+b-x)dx\right] \\
&= \int_a^b |f(x)|\, s(x)dx.
\end{aligned} \tag{1.19}$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_1(a,b) &= \frac{1}{2}\int_a^b |f(x)-f(a+b-x)|\, s(x)dx \\
&\le \frac{1}{2}\left(\int_a^b (f(x)-f(a+b-x))^2 dx\right)^{\frac{1}{2}}\left(\int_a^b s^2(x)dx\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\int_a^b (f(x)^2+f^2(a+b-x)-2f(x)f(a+b-x))dx\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_a^b s^2(x)dx\right)^{\frac{1}{2}}
\end{aligned} \tag{1.20}$$

$$= \frac{1}{2}\left(2\int_a^b f(x)^2dx - 2\int_a^b f(x)f(a+b-x)dx\right)^{\frac{1}{2}}\left(\int_a^b s^2(x)dx\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2}}\left(\int_a^b f^2(x)dx - \int_a^b f(x)f(a+b-x)dx\right)^{\frac{1}{2}}\left(\int_a^b s^2(x)dx\right)^{\frac{1}{2}}.$$

The inequality (1.18) then follows from (1.19) and (1.20). This completes the proof. □

Similarly, we have the following theorem:

Theorem 9. *Let $f \in JQC(I) \cap L_1[a,b]$. Then the inequalities (1.17) and (1.18) also hold.*

Remark 4. *If we choose $s(x) \equiv 1$, then Theorem 9 reduces to Theorem 3.*

The corresponding result for the mapping p reads as:

Theorem 10. *Let $f \in QC(I) \cap L_1(a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \le \int_a^b f(x)p(x)dx + I_2(a,b), \tag{1.21}$$

where

$$I_2(a,b) = \frac{1}{2}\int_a^b \left|f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right)\right| p\left(\frac{x+a}{2}\right)dx.$$

Further,

$$\begin{aligned} 0 &\le I_2(a,b) \\ &\le \min\left\{\int_a^b |f(x)|\,p(x)dx\ ,\ \frac{1}{\sqrt{2}}\left(\int_a^b f^2(x)dx\right.\right. \\ &\qquad \left.\left. - \int_a^b f\left(\frac{x+a}{2}\right)f\left(\frac{x+b}{2}\right)dx\right)^{\frac{1}{2}}\left(\int_a^b p^2(x)dx\right)^{\frac{1}{2}}\right\}. \end{aligned} \tag{1.22}$$

Proof. We have

$$0 \le \frac{2(x-a)}{b-a}, \frac{a+b-2x}{b-a} \le 1$$

and

$$\frac{2(x-a)}{b-a} + \frac{a+b-2x}{b-a} = 1$$

for $x \in \left[a, \frac{a+b}{2}\right]$. By $f \in QC(I) \cap L_1[a,b]$ and the identities (1.15) and (1.16), we may state that

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \\
&= 2f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} p(x)dx \\
&= 2\int_a^{\frac{a+b}{2}} f\left[\frac{2(x-a)}{b-a}x + \frac{a+b-2x}{b-a}\left(\frac{b-a}{2}+x\right)\right]p(x)dx \\
&\le 2\int_a^{\frac{a+b}{2}} \max\left\{f(x), f\left(\frac{b-a}{2}+x\right)\right\}p(x)dx \\
&= \int_a^{\frac{a+b}{2}} \left[f(x) + f\left(\frac{b-a}{2}+x\right) + \left|f(x) - f\left(\frac{b-a}{2}+x\right)\right|\right]p(x)dx \\
&= \int_a^{\frac{a+b}{2}} f(x)p(x)dx + \int_a^{\frac{a+b}{2}} f\left(\frac{b-a}{2}+x\right)p\left(\frac{b-a}{2}+x\right)dx \\
&\quad + \int_a^{\frac{a+b}{2}} \left|f(x) - f\left(\frac{b-a}{2}+x\right)\right|p(x)dx \\
&= \int_a^b f(x)p(x)dx + \int_a^{\frac{a+b}{2}} \left|f(x) - f\left(\frac{b-a}{2}+x\right)\right|p(x)dx \\
&= \int_a^b f(x)p(x)dx + \int_a^b \left|f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right)\right|p\left(\frac{x+a}{2}\right)dx.
\end{aligned}$$

This proves (1.21).

A similar argument as in the proof of the inequality (1.18) implies the inequality (1.22). This completes the proof. □

Corollary 1. *Let $f \in QC(I) \cap L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b f(x)dx + \frac{1}{2}\min\left\{\int_a^b |f(x) - f(a+b-x)|\,dx,\right.$$
$$\left.\int_a^b \left|f\left(\frac{x+a}{2}\right) - f\left(\frac{x+b}{2}\right)\right|dx\right\}.$$

Proof. This follows from Theorem 8 and Theorem 10 by choosing $s(x) = p(x) = 1$. □

Theorem 11. *Let $f \in WC(I) \cap L_1[a,b]$. Then*

$$\int_a^b f(x)s(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b s(x)dx. \tag{1.23}$$

The inequality is the best possible.

Proof. Since $f \in WC(I) \cap L_1[a,b]$ and s is nonnegative symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned}
\int_a^b f(x)s(x)dx &= \frac{1}{2}\left[\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx\right] \\
&= \frac{1}{2}\int_a^b \left[f(x) + f(a+b-x)\right] s(x)dx \\
&= \frac{1}{2}\int_a^b \left[f(a+(x-a)) + f(a+b-x)\right] s(x)dx \\
&\leq \frac{1}{2}\int_a^b \left[f((a+b-x)+(x-a)) + f(a)\right] s(x)dx \\
&= \frac{f(a)+f(b)}{2}\int_a^b s(x)dx.
\end{aligned}$$

This proves the inequality (1.23), which reduces to an equality for $f(x) \equiv 1$. This completes the proof. □

Finally, we may state

Theorem 12. *Let $f \in WQC(I) \cap L_1[a,b]$. Then*

$$\int_a^b f(x)s(x)dx \leq \max\{f(a), f(b)\}\int_a^b s(x)dx. \tag{1.24}$$

The inequality is the best possible.

Proof. Since $f \in WQC(I) \cap L_1[a,b]$ and s is nonnegative symmetric to $\frac{a+b}{2}$, we have

$$\begin{aligned}
\int_a^b f(x)s(x)dx &= \frac{1}{2}\left[\int_a^b f(x)s(x)dx + \int_a^b f(a+b-x)s(a+b-x)dx\right] \\
&= \int_a^b \frac{1}{2}(f(x) + f(a+b-x))s(x)dx \\
&= \int_a^b \frac{1}{2}\left[f(a+(x-a)) + f(a+b-x)\right] s(x)dx \\
&\leq \int_a^b \max\{f(a), f((a+b-x)+(x-a))\}s(x)dx \\
&= (\max\{f(a), f(b)\})\int_a^b s(x)dx.
\end{aligned}$$

This proves the inequality (1.24), which reduces to an equality for $f(x) \equiv 1$. This completes the proof. □

Remark 5. *If we choose $g(x) \equiv 1$, then Theorem 12 reduces to Theorem 4.*

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