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BOUNDING THE ČEBYŠEV FUNCTIONAL FOR SEQUENCES OF VECTORS IN NORMED LINEAR SPACES

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ABSTRACT. Some new bounds for Čebyšev functional for sequences of vectors in normed linear spaces are pointed out.

1. INTRODUCTION

Consider the Čebyšev functional defined for $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, where X is a linear space over the real or complex number field \mathbb{K} :

$$(1.1) \quad T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}) := P_n \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i,$$

where $P_n := \sum_{i=1}^n p_i$.

The following Grüss type inequalities for sequences in normed linear spaces hold.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} , $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$ and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. Then one has the inequalities*

$$(1.2) \quad \|T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x})\| \leq \begin{cases} \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i \right)^2 \right] \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\|, [1]; \\ \frac{1}{2} \sum_{i=1}^n p_i (1 - p_i) \sum_{j=1}^{n-1} |\Delta \alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\|, [3]; \\ \sum_{1 \leq i < j \leq n} p_i p_j (j - i) \left(\sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^q \right)^{1/q}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, [2]. \end{cases}$$

The constant 1 in the first branch, $\frac{1}{2}$ in the second branch and 1 in the third branch are best possible in the sense that they cannot be replaced by smaller constants.

The following particular inequalities for unweighted means hold as well, where $T_n(\boldsymbol{\alpha}, \mathbf{x})$ is defined as follows:

$$T_n(\boldsymbol{\alpha}, \mathbf{x}) := \frac{1}{n} \sum_{i=1}^n \alpha_i x_i - \frac{1}{n} \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \sum_{i=1}^n x_i.$$

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Corollary 1. *With the assumptions of Theorem 1 for X, α and \mathbf{x} , we have*

$$(1.3) \quad \begin{aligned} & \|T_n(\alpha, \mathbf{x})\| \\ & \leq \begin{cases} \frac{1}{12} (n^2 - 1) \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\|, [1]; \\ \frac{1}{2} \cdot \left(1 - \frac{1}{n}\right) \sum_{j=1}^{n-1} |\Delta \alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\|, [3]; \\ \frac{1}{6} \frac{n^2-1}{n} \left(\sum_{j=1}^{n-1} |\Delta \alpha_j|^p\right)^{1/p} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^q\right)^{1/q}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, [2]. \end{cases} \end{aligned}$$

Here the constants $\frac{1}{12}$, $\frac{1}{2}$ and $\frac{1}{6}$ are best possible in the sense that they cannot be replaced by smaller constants.

For applications to estimate the p -moments of guessing mappings, see [1]. For applications in approximating the discrete Fourier transform, the discrete Mellin transform as well as some applications for polynomials and Lipschitzian mappings, see [2] and [3].

For classical results related the Čebyšev functional, see [4], [5], [6], [7], [8], [10] and [12]. For more recent results, see [12], [13], [14], [15], [9] and [11].

2. THE IDENTITIES

The first result is embodied in the following

Theorem 2. *Let $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{a} = (a_1, \dots, a_n)$ be n -tuples of real or complex numbers and $\mathbf{x} = (x_1, \dots, x_n)$ an n -tuple of vectors in the linear space X . If we define*

$$\begin{aligned} P_i & : = \sum_{k=1}^i p_k, \bar{P}_i := P_n - P_i, i \in \{1, \dots, n-1\}, \\ A_i(\mathbf{p}) & : = \sum_{k=1}^i p_k a_k, \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), i \in \{1, \dots, n-1\}, \end{aligned}$$

then we have the identity

$$\begin{aligned} (2.1) \quad T_n(\mathbf{p}; \mathbf{a}, \mathbf{x}) & = \sum_{i=1}^{n-1} \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \cdot \Delta x_i \\ & = P_n \sum_{i=1}^{n-1} P_i \left(\frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta x_i \text{ (if } P_i \neq 0, i \in \{1, \dots, n\}) \\ & = \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta x_i \text{ (if } P_i, \bar{P}_i \neq 0, i \in \{1, \dots, n-1\}); \end{aligned}$$

where $\Delta x_i := x_{i+1} - x_i$ ($i \in \{1, \dots, n-1\}$) is the forward difference.

Proof. We use the following well known summation by parts formula

$$(2.2) \quad \sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l \Big|_p^q - \sum_{l=p}^{q-1} v_{l+1} \Delta d_l,$$

where d_l are real or complex numbers, and v_l are vectors in a linear space, $l = p, \dots, q$ ($q > p; p, q$ are natural numbers).

If we choose in (2.2), $p = 1, q = n, d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = x_i$ ($i \in \{1, \dots, n-1\}$), then we get

$$\begin{aligned}
 & \sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot \Delta x_i \\
 = & [P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})] \cdot x_i \Big|_1^n - \sum_{i=1}^{n-1} \Delta (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot x_{i+1} \\
 = & [P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p})] \cdot x_n - [P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p})] \cdot x_1 \\
 & - \sum_{i=1}^{n-1} [P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p})] \cdot x_{i+1} \\
 = & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - \sum_{i=1}^{n-1} (p_{i+1} A_n(\mathbf{p}) - P_n p_{i+1} a_{i+1}) \cdot x_{i+1} \\
 = & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - A_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} x_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} a_{i+1} x_{i+1} \\
 = & P_n \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i x_i \\
 = & T_n(\mathbf{p}; \mathbf{a}, \mathbf{x}),
 \end{aligned}$$

which produce the first identity in (2.1).

The second and the third identities are obvious and we omit the details. \square

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

Lemma 1. *Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{a} = (a_1, \dots, a_n)$ be n -tuples of real or complex numbers. Then we have the equality*

$$(2.3) \quad \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j,$$

for each $i \in \{1, \dots, n-1\}$.

Proof. Define, for $i \in \{1, \dots, n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$\begin{aligned}
(2.4) \quad K(i) &= \sum_{j=1}^i P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \\
&= \sum_{j=1}^i P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j \\
&= \bar{P}_i \sum_{j=1}^i P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j.
\end{aligned}$$

Using the summation by parts formula, we have

$$\begin{aligned}
(2.5) \quad \sum_{j=1}^i P_j \cdot \Delta a_j &= P_j \cdot a_j \Big|_1^{i+1} - \sum_{j=1}^i (P_{j+1} - P_j) \cdot a_{j+1} \\
&= P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^i p_{j+1} \cdot a_{j+1} \\
&= P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j
\end{aligned}$$

and

$$\begin{aligned}
(2.6) \quad \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j &= \bar{P}_j \cdot a_j \Big|_{i+1}^n - \sum_{j=i+1}^{n-1} (\bar{P}_{j+1} - \bar{P}_j) \cdot a_{j+1} \\
&= \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) \cdot a_{j+1} \\
&= -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1}.
\end{aligned}$$

Using (2.5) and (2.6) we have

$$\begin{aligned}
K(i) &= \bar{P}_i \left(P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j \right) + P_i \left(\sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_{i+1} a_{i+1} \right) \\
&= \bar{P}_i P_{i+1} a_{i+1} - P_i \bar{P}_{i+1} a_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} \\
&= [(P_n - P_i) P_{i+1} - P_i (P_n - P_{i+1})] a_{i+1} \\
&\quad + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j
\end{aligned}$$

$$\begin{aligned}
 &= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
 &= (P_i + \bar{P}_i) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
 &= P_i \sum_{j=i+1}^{n-1} p_j \cdot a_j - \bar{P}_i \sum_{j=1}^i p_j \cdot a_j = P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
 &= \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix};
 \end{aligned}$$

and the identity is proved. \square

We are able now to state and prove the second identity for the Čebyšev functional

Theorem 3. *With the assumptions of Theorem 2, we have the equality*

$$(2.7) \quad T_n(\mathbf{p}; \mathbf{a}, \mathbf{x}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \cdot \Delta x_i.$$

The proof is obvious by Theorem 2 and Lemma 1.

Remark 1. *The identity (2.7), for n -tuples of real numbers, was stated without a proof in paper [12]. It also may be found for the same sequences in [9, p. 281], again without a proof. In the second place mentioned above there is a misprint for the index of \bar{P} which, instead of $\max\{i, j\} + 1$, should be $\max\{i, j\}$.*

3. SOME NEW INEQUALITIES

The following result holds

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} , $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. Then one has the inequalities*

$$(3.1) \quad \|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq \begin{cases} \max_{1 \leq i \leq n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \cdot \sum_{j=1}^{n-1} \|\Delta x_j\|; \\ \left(\sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right|^q \right)^{1/q} \cdot \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^p \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \cdot \max_{1 \leq j \leq n-1} \|\Delta x_j\|. \end{cases}$$

All the inequalities in (3.1) are sharp in the sense that the constants 1 cannot be replaced by smaller constants.

Proof. Using the first identity in (2.1), we have

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq \sum_{i=1}^n \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \|\Delta x_i\|.$$

Using Hölder's inequality, we deduce the desired result (3.1).

Let prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $C > 0$, we have that

$$(3.2) \quad \|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq C \left(\sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right|^q \right)^{1/q} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^p \right)^{1/p}$$

for $p > 1, \frac{1}{p} + \frac{1}{q} = 1, n \geq 2$.

If we choose $n = 2$, then we get

$$T_2(\mathbf{p}; \mathbf{a}, \mathbf{x}) = p_1 p_2 (a_2 - a_1) (x_2 - x_1).$$

Also, for $n = 2$,

$$\left(\sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right|^q \right)^{1/q} = |p_1 p_2| |a_2 - a_1|$$

and

$$\left(\sum_{j=1}^{n-1} \|\Delta x_j\|^p \right)^{1/p} = \|x_2 - x_1\|.$$

Then by (3.2), holding for $n = 2, p_1, p_2 > 0, a_1 \neq a_2, x_2 \neq x_1$, we deduce $C \geq 1$, proving that 1 is the best possible constant in that inequality. \square

The following corollary for the uniform distribution of the probability \mathbf{p} holds.

Corollary 2. *With the assumptions of Theorem 4 for \mathbf{a} and \mathbf{x} , we have the inequalities*

$$\begin{aligned} & \|T_n(\mathbf{a}, \mathbf{x})\| \\ & \leq \frac{1}{n^2} \times \begin{cases} \max_{1 \leq i \leq n-1} \left| \det \begin{pmatrix} i & n \\ \sum_{k=1}^i a_k & \sum_{k=1}^n a_k \end{pmatrix} \right| \cdot \sum_{j=1}^{n-1} \|\Delta x_j\|; \\ \left(\sum_{i=1}^{n-1} \left| \det \begin{pmatrix} i & n \\ \sum_{k=1}^i a_k & \sum_{k=1}^n a_k \end{pmatrix} \right|^q \right)^{1/q} \cdot \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^p \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \left| \det \begin{pmatrix} i & n \\ \sum_{k=1}^i a_k & \sum_{k=1}^n a_k \end{pmatrix} \right| \cdot \max_{1 \leq j \leq n-1} \|\Delta x_j\|. \end{cases} \end{aligned}$$

The following result may be stated as well.

Theorem 5. *With the assumptions of Theorem 4 and if $P_i \neq 0$ ($i = 1, \dots, n$), then we have the inequalities*

$$(3.3) \quad \|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq |P_n| \times \begin{cases} \max_{1 \leq i \leq n-1} \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right| \cdot \sum_{i=1}^{n-1} |P_i| \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} |P_i| \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} |P_i| \|\Delta x_i\| \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right| \cdot \max_{1 \leq i \leq n-1} \|\Delta x_i\|. \end{cases}$$

All the inequalities in (3.3) are sharp in the sense that the constant 1 cannot be replaced by a smaller constant.

Proof. Follows by the second identity in (2.1) and taking into account that

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq |P_n| \sum_{i=1}^{n-1} \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right| \cdot |P_i| \|\Delta x_i\|.$$

Using Hölder's weighted inequality, we easily deduce (3.3).

The sharpness of the constant may be shown in a similar manner. We omit the details. \square

The following corollary containing the unweighted inequalities holds.

Corollary 3. *With the above assumptions for \mathbf{a} and \mathbf{x} one, has*

$$(3.4) \quad \|T_n(\mathbf{a}, \mathbf{x})\| \leq \frac{1}{n} \times \begin{cases} \max_{1 \leq i \leq n-1} \left| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right| \cdot \sum_{i=1}^{n-1} i \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} i \|\Delta x_i\|^p \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right| \cdot \max_{1 \leq i \leq n-1} \|\Delta x_i\|. \end{cases}$$

The inequalities in (3.4) are sharp in the sense mentioned above.

Another type of inequalities may be stated if one uses the third identity in (2.1).

Theorem 6. *With the assumptions in Theorem 4 and if $P_i, \bar{P}_i \neq 0, i \in \{1, \dots, n-1\}$, then we have the inequalities*

$$(3.5) \quad \|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq \begin{cases} \max_{1 \leq i \leq n-1} \left| \frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right| \cdot \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \left| \frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \|\Delta x_i\|^p \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \left| \frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right| \cdot \max_{1 \leq i \leq n-1} \|\Delta x_i\|. \end{cases}$$

In particular, if $p_i = \frac{1}{n}, i \in \{1, \dots, n\}$, then we have

$$(3.6) \quad \|T_n(\mathbf{a}, \mathbf{x})\| \leq \frac{1}{n^2} \cdot \begin{cases} \max_{1 \leq i \leq n-1} \left| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right| \cdot \sum_{i=1}^{n-1} i(n-i) \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right|^q \right)^{1/q} \\ \times \left(\sum_{i=1}^{n-1} i(n-i) \|\Delta x_i\|^p \right)^{1/p} \text{ for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right| \cdot \max_{1 \leq i \leq n-1} \|\Delta x_i\|. \end{cases}$$

The inequalities in (3.5) and (3.6) are sharp in the above mentioned sense.

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by the Theorem 3.

The following result holds.

Theorem 7. *With the assumptions in Theorem 4, we have the inequalities*

$$(3.7) \quad \|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq \begin{cases} \max_{1 \leq i, j \leq n-1} \left\{ |P_{\min\{i,j\}}| |\bar{P}_{\max\{i,j\}}| \right\} \cdot \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |P_{\min\{i,j\}}|^q |\bar{P}_{\max\{i,j\}}|^q \right)^{1/q} \\ \times \left(\sum_{i=1}^{n-1} |\Delta a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta x_i\|^p \right)^{1/p} \text{ for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |P_{\min\{i,j\}}| |\bar{P}_{\max\{i,j\}}| \\ \times \max_{1 \leq i \leq n-1} |\Delta a_i| \max_{1 \leq i \leq n-1} \|\Delta x_i\|. \end{cases}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (2.7) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_\infty := \max_{1 \leq i, j \leq n-1} \left\{ \frac{\min \{i, j\}}{n} \left(1 - \frac{\max \{i, j\}}{n} \right) \right\}, n \geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4} (a + b)^2, \quad a, b \in R;$$

we deduce

$$\begin{aligned} \min \{i, j\} \cdot (n - \max \{i, j\}) &\leq \frac{1}{4} (n + \min \{i, j\} - \max \{i, j\})^2 \\ &= \frac{1}{4} (n - |i - j|)^2, \quad 1 \leq i, j \leq n - 1. \end{aligned}$$

Consequently, we observe that

$$k_\infty \leq \frac{1}{4n^2} \max_{1 \leq i, j \leq n-1} \left\{ (n - |i - j|)^2 \right\} = \frac{1}{4}.$$

We may state now the following corollary of Theorem 7.

Corollary 4. *Let $(X, \|\cdot\|)$ be a normed linear space, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{X}^n$. Then we have the inequality*

$$(3.8) \quad \|T_n(\mathbf{a}, \mathbf{x})\| \leq k_\infty \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\| \leq \frac{1}{4} \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\|.$$

The constant $\frac{1}{4}$ cannot be replaced in general by a smaller constant.

Remark 2. *The inequality (3.8) is better than the second inequality in Corollary 1.*

Consider now, for $q > 1$, the number

$$k_q := \frac{1}{n^2} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [\min \{i, j\} \cdot (n - \max \{i, j\})]^q \right)^{1/q}.$$

We observe, by the symmetry of the terms under the sums symbol, we have that

$$k_q = \frac{1}{n^2} \left(2 \sum_{1 \leq i < j \leq n-1} i^q (n - j)^q + \sum_{i=1}^{n-1} i^q (n - i)^q \right)^{1/q},$$

that may be computed exactly if $q = 2$ or another natural number.

Since, as above,

$$[\min \{i, j\} \cdot (n - \max \{i, j\})]^q \leq \frac{1}{4^q} (n - |i - j|)^{2q}$$

we deduce

$$\begin{aligned} k_q &\leq \frac{1}{4n^2} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (n - |i - j|)^{2q} \right)^{1/q} \\ &\leq \frac{1}{4n^2} \left[(n - 1)^2 n^{2q} \right]^{1/q} = \frac{1}{4} (n - 1)^{2/q}. \end{aligned}$$

Consequently, we may state the following corollary as well.

Corollary 5. *With the assumption in Corollary 4, we have the inequalities*

$$\begin{aligned} \|T_n(\mathbf{a}, \mathbf{x})\| &\leq k_q \left(\sum_{i=1}^{n-1} |\Delta a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta x_i\|^p \right)^{1/p} \\ &\leq \frac{1}{4} (n-1)^{2/q} \left(\sum_{i=1}^{n-1} |\Delta a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta x_i\|^p \right)^{1/p}; \end{aligned}$$

provided $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. The constant $\frac{1}{4}$ cannot be replaced in general by a smaller constant.

Finally, if we denote

$$k_1 := \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [\min\{i, j\} \cdot (n - \max\{i, j\})],$$

then we observe, for $\mathbf{u} = (\frac{1}{n}, \dots, \frac{1}{n})$, $\mathbf{e} = (1, 2, \dots, n)$, that

$$k_1 = T_n(\mathbf{u}; \mathbf{e}, \mathbf{e}) = \frac{1}{n} \sum_{i=1}^n i^2 - \left(\frac{1}{n} \sum_{i=1}^n i \right)^2 = \frac{1}{12} (n^2 - 1),$$

and by Theorem 7, we deduce the inequality

$$\|T_n(\mathbf{a}, \mathbf{x})\| \leq \frac{1}{12} (n^2 - 1) \max_{1 \leq j \leq n-1} |\Delta a_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\|.$$

Note that, the above inequality, has been discovered with a different method in [1]. The constant $\frac{1}{12}$ is best possible.

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