Pre-Grüss Type Inequalities in Inner Product Spaces

PRE-GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

S.S. DRAGOMIR, J. E. PEČARIĆ, AND B. TEPEŠ

Abstract. Some pre-Grüss type inequalities in real or complex inner product spaces and applications for integrals are given.

1. Introduction

Let f, g be two functions defined and integrable on $[a, b]$. Assume that

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a, b]$, where φ, Φ, γ, Γ are given real constants. Then the following inequality is well known in the literature as the Grüss inequality ([4, pp. 296])

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| .$$

In this inequality, G. Grüss has proven that, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, and is achieved for

$$f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2} \right).$$

Recently, S. S. Dragomir has proved the following Grüss' type inequality in real or complex inner product spaces [1].

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}, (\mathbb{K}{=}\mathbb{R},\mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0,$$

hold, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.1}$$

The constant $\frac{1}{4}$ is best possible in sense that it cannot be replaced by a smaller constant.

In [2], by using the following lemmas

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Lemma 1. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta,\ \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0$$

if and only if

$$\left\| x - \frac{\delta + \Delta}{2} e \right\| \leq \frac{1}{2} |\Delta - \delta|,$$

and

Lemma 2. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x,\ e\rangle|^2 = \inf_{\lambda \in K} \|x - \lambda e\|^2,$$

the author gave an alternative proof for (1.1) and also obtained the following refinement of it, namely

Theorem 2. *Let $(H, \langle ., .\rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that either the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0,\ \ \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0,$$

or equivalently,

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

hold, then we have the inequality

$$\begin{aligned} &|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}} [\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}} \\ &\leq \left(\frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right). \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Further, as a generalization for orthonormal families of vectors in inner product spaces, S.S. Dragomir proved, in [3], the following reverse of Bessel's inequality:

Theorem 3. *Let $\{e_i\}$, $i \in I$ be a family of orthonormal vectors in H, F a finite part of I, $\varphi_i,\ \Phi_i \in \mathbb{K}$, $i \in F$ and and x is vector in H such that either the condition*

$$\operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x,\ x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

or equivalently,

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}},$$

holds, then we have the following reverse of Bessel's inequality

$$\|x\|^2 - \sum_{i \in F} |\langle x,\ e_i\rangle|^2 \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \varphi_i|^2 - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x,\ e_i\rangle \right|^2. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

The corresponding Grüss' type inequality is embodied in the following theorem:

Theorem 4. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$ $(i \in F)$, and $x, y \in H$. If either*

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} \Phi_i e_i - x, x - \sum_{i=1}^{n} \phi_i e_i \right\rangle \geq 0,$$
$$\operatorname{Re}\left\langle \sum_{i=1}^{n} \Gamma_i e_i - y, y - \sum_{i=1}^{n} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$
$$\left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold true, then

$$\begin{aligned} 0 &\leq \left| \langle x, y\rangle - \sum_{i=1}^{n} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\ &\leq \frac{1}{4}\left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\qquad - \sum_{i\in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i\rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i\rangle \right| \\ &\left(\leq \frac{1}{4}\left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The main aim of this paper is to provide some similar inequalities which, providing refinements of the usual Grüss' inequality, are known in the literature as pre-Grüss type inequalities. Applications for Lebesgue integrals in general measure spaces are also given.

2. Pre-Grüss Inequalities in Inner Product Spaces

We start with the following result:

Theorem 5. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}, (\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e \in H$, $\|e\| = 1$. If φ, Φ are real or complex numbers and x, y are vectors in H such that either the condition*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0,$$

or equivalently,

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2}|\Phi - \varphi|, \tag{2.1}$$

holds true, then we have the inequalities

$$|\langle x,y\rangle - \langle x,e\rangle\ \langle e,y\rangle| \le \frac{1}{2}|\Phi-\varphi|\cdot\sqrt{\left(\|y\|^2 - |\langle y,\,e\rangle|^2\right)} \tag{2.2}$$

and

$$|\langle x,y\rangle - \langle x,e\rangle\ \langle e,y\rangle| \le \frac{1}{2}|\Phi-\varphi|\cdot\|y\| - \left(\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle\right)^{\frac{1}{2}}\cdot|\langle y,\,e\rangle|\,. \tag{2.3}$$

Proof. It is obvious that:

$$\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle = \langle x-\langle x,e\rangle e, y-\langle y,e\rangle e\rangle\,.$$

Using Schwarz's inequality in inner product spaces $|\langle u,\,v\rangle| \le \|u\|\cdot\|v\|$ for the vectors $x-\langle x,e\rangle e$ and $y-\langle y,e\rangle e$, we deduce:

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \le \left(\|x\|^2 - |\langle x,\,e\rangle|^2\right)\cdot\left(\|y\|^2 - |\langle y,\,e\rangle|^2\,.\right) \tag{2.4}$$

Now, the inequality (2.2) is a simple consequence of (1.1) for $x=y$, or of Lemma 2 and (2.1).

Since (see for instance [1]),

$$\begin{aligned} &\|x\|^2 - |\langle x,\,e\rangle|^2 \\ = \ & \operatorname{Re}\left((\Phi - \langle x,\,e\rangle)\cdot(\langle e,\,x\rangle - \bar{\varphi})\right) - \operatorname{Re}\langle \Phi e - x,\,x-\varphi e\rangle\,, \end{aligned} \tag{2.5}$$

then making use of the elementary inequality $4\operatorname{Re}\left(a\bar{b}\right) \le |a+b|^2$ with $a,b \in \mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$, we can state that

$$\operatorname{Re}\left((\Phi - \langle x,\,e\rangle)\cdot(\langle e,\,x\rangle - \bar{\varphi})\right) \le \frac{1}{4}|\Phi-\varphi|^2\,. \tag{2.6}$$

Using (2.5) and (2.6) we have

$$\|x\|^2 - |\langle x,\,e\rangle|^2 \le \left(\frac{1}{2}|\Phi-\varphi|\right)^2 - \left(\left(\operatorname{Re}\langle \Phi e - x,\,x-\varphi e\rangle\right)^{\frac{1}{2}}\right)^2. \tag{2.7}$$

Taking into account the inequalities (2.4) and (2.7), we get that

$$\begin{aligned} &|\langle x-\langle x,\,e\rangle e,\,y-\langle y,\,e\rangle e\rangle|^2 \\ &\le \left(\left(\frac{1}{2}|\Phi-\varphi|\right)^2 - \left(\left(\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle\right)^{\frac{1}{2}}\right)^2\right)\cdot\left(\|y\|^2 - |\langle y,e\rangle|^2\right). \end{aligned}$$

Finally, using the elementary inequality for positive real numbers:

$$\left(m^2-n^2\right)\cdot\left(p^2-q^2\right) \le (mp-nq)^2 \tag{2.8}$$

we have:

$$\begin{aligned} &\left(\left(\frac{1}{2}|\Phi-\varphi|\right)^2 - \left(\left(\operatorname{Re}\langle \Phi e - x,\,x-\varphi e\rangle\right)^{\frac{1}{2}}\right)^2\right)\cdot\left(\|y\|^2 - |\langle y,e\rangle|^2\right) \\ &\le \left(\frac{1}{2}|\Phi-\varphi|\cdot\|y\| - \left(\operatorname{Re}\langle \Phi e - x,\,x-\varphi e\rangle\right)^{\frac{1}{2}}\cdot|\langle y,\,e\rangle|\right)^2, \end{aligned}$$

giving the desired inequality (2.3). □

A similar version for Bessel's inequality is incorporated in the following theorem:

Theorem 6. *Let* $\{e_i\}_{i\in I}$ *, be a family of orthonormal vectors in* H*,* F *a finite part of* I*,* φ_i*,* $\Phi_i \in \mathbb{K}$*,* $i \in F$ *and and* x, y *are vectors in* H *such that either the condition*

$$\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - x,\, x - \sum_{i\in F}\varphi_i e_i \right\rangle \geq 0,$$

or equivalently,

$$\left\| x - \sum_{i\in F}\frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \varphi_i|^2\right)^{\frac{1}{2}}$$

holds. Then we have inequalities

$$\begin{aligned} &\left|\langle x, y\rangle - \sum_{i\in F}\langle x, e_i\rangle \, \langle e_i, y\rangle\right| \\ &\leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \varphi_i|^2\right)^{\frac{1}{2}} \sqrt{\left(\|y\|^2 - \sum_{i\in F}|\langle y,\, e_i\rangle|^2\right)} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\left|\langle x, y\rangle - \sum_{i\in F}\langle x, e_i\rangle \, \langle e_i, y\rangle\right| \\ &\leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \varphi_i|^2\right)^{\frac{1}{2}} \cdot \|y\| \\ &- \left(\sum_{i\in F}\left|\frac{\Phi_i + \varphi_i}{2} - \langle x,\, e_i\rangle\right|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F}|\langle y,\, e_i\rangle|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

Proof. It is obvious (see for example [3]) that :

$$\langle x,\, y\rangle - \sum_{i\in F}\langle x,\, e_i\rangle \langle e_i,\, y\rangle = \left\langle x - \sum_{i\in F}\langle x,\, e_i\rangle \, e_i,\, y - \sum_{i\in F}\langle y,\, e_i\rangle \, e_i \right\rangle.$$

Using Schwarz's inequality in inner product spaces, we have:

$$\begin{aligned} &\left|\left\langle x - \sum_{i\in F}\langle x,\, e_i\rangle \, e_i,\, y - \sum_{i\in F}\langle y,\, e_i\rangle \, e_i \right\rangle\right|^2 \\ &\leq \left\| x - \sum_{i\in F}\langle x,\, e_i\rangle \, e_i \right\|^2 \cdot \left\| x - \sum_{i\in F}\langle y,\, e_i\rangle \, e_i \right\|^2 \\ &= \left(\|x\|^2 - \sum_{i\in F}|\langle x,\, e_i\rangle|^2\right) \cdot \left(\|y\|^2 - \sum_{i\in F}|\langle y,\, e_i\rangle|^2\right). \end{aligned} \tag{2.11}$$

In a similar manner to the one in the proof of Theorem 5 we may conclude that (2.9) holds true.

Now, using (1.2) and (2.11) we also have:

$$\left|\left\langle x-\sum_{i\in F}\langle x,\,e_i\rangle\,e_i,\,y-\sum_{i\in F}\langle y,\,e_i\rangle\,e_i\right\rangle\right|^2$$
$$\leq\left(\frac{1}{2}\left(\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\right)^2-\left(\left(\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x,\,e_i\rangle\right|^2\right)^{\frac{1}{2}}\right)^2\right)$$
$$\times\left(\|y\|^2-\sum_{i\in F}|\langle y,\,e_i\rangle|^2\right)$$

Finally, utilizing the elementary inequality (2.8), we have

(2.12)
$$\left(\frac{1}{2}\left(\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\right)^2-\left(\left(\sum_{i\in F}\frac{\varphi_i+\Phi_i}{2}-\langle x,\,e_i\rangle^2\right)^{\frac{1}{2}}\right)^2\right)$$
$$\times\left(\|y\|^2-\sum_{i\in F}|\langle y,\,e_i\rangle|^2\right)$$
$$\leq\left(\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\cdot\|y\|^2-\left(\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x,\,e_i\rangle\right|^2\right)^{\frac{1}{2}}\cdot\sum_{i\in F}|\langle y,\,e_i\rangle|^2\right)^2$$

which gives the desired result (2.10). □

Another pre-Grüss type inequality associated to orthonormal families in inner product spaces is incorporated in the next theorem.

Theorem 7. *Let $\{e_i\}$, $i\in I$ be a family of orthonormal vectors in H, F a finite part of I, φ_i, $\Phi_i\in\mathbb{K}$, $i\in F$ and x,y vectors in H such that either the condition*

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-x,\,x-\sum_{i\in F}\varphi_i e_i\right\rangle\geq 0,$$

or equivalently,

$$\left\|x-\sum_{i\in F}\frac{\Phi_i+\varphi_i}{2}e_i\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}},$$

holds. Then we have inequalities

$$\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\,\langle e_i,y\rangle\right|$$
$$\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\cdot\|y\|-\sum_{i\in F}\left|\frac{\Phi_i+\varphi_i}{2}-\langle x,\,e_i\rangle\right|\,|\langle y,\,e_i\rangle|.$$

Proof. Using Schwarz's inequality (2.11) with the reverse of Bessel's inequality (1.2) we have:

$$\begin{aligned}
&\left|\left\langle x-\sum_{i\in F}\langle x, e_i\rangle e_i,\, y-\sum_{i\in F}\langle y, e_i\rangle e_i\right\rangle\right|^2\\
\leq\ &\left(\|x\|^2-\sum_{i\in F}|\langle x, e_i\rangle|^2\right)\cdot\left(\|y\|^2-\sum_{i\in F}|\langle y, e_i\rangle|^2\right)\\
\leq\ &\left(\frac{1}{4}\sum_{i\in F}|\Phi_i-\varphi_i|^2-\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x, e_i\rangle\right|^2\right)\cdot\left(\|y\|^2-\sum_{i\in F}|\langle y, e_i\rangle|^2\right).
\end{aligned}$$

Further, on utilizing Aczél's inequality [4, p. 117] for two sequences of real numbers $\mathbf{a}=(a_1,\dots,a_n)$ and $\mathbf{b}=(b_1,\dots,b_n)$ such that $a_1^2-a_2^2-\cdots-a_n^2>0$ or $b_1^2-b_2^2-\cdots-b_n^2>0$, that is

$$\left(a_1^2-a_2^2-\cdots-a_n^2\right)\left(b_1^2-b_2^2-\cdots-b_n^2\right)\leq\left(a_1b_1-a_2b_2-\cdots-a_nb_n\right)^2,$$

we have

$$\begin{aligned}
&\left(\frac{1}{4}\sum_{i\in F}|\Phi_i-\varphi_i|^2-\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x, e_i\rangle\right|^2\right)\cdot\left(\|y\|^2-\sum_{i\in F}|\langle y, e_i\rangle|^2\right)\\
\leq\ &\left(\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\|y\|-\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x, e_i\rangle\right||\langle y, e_i\rangle|\right)^2.
\end{aligned}$$

This completes the proof. □

3. Applications for Integrals

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L^2(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and 2–integrable on Ω, i. e.

$$\int_\Omega |f(s)|^2\,d\mu(s)<\infty.$$

The following proposition holds.

Proposition 1. *If $f, g, h\in L^2(\Omega,\mathbb{K})$ and $\varphi, \Phi\in\mathbb{K}$, are such that $\int_\Omega |h(s)|^2\,d\mu(s)=1$ and, either*

$$\int_\Omega \operatorname{Re}\left(\left(\Phi h(s)-f(s)\right)\left(\bar{f}(s)-\overline{\varphi}\bar{h}(s)\right)\right)d\mu(s)\geq 0, \tag{3.1}$$

or, equivalently,

$$\left(\int_\Omega\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}|\Phi-\varphi|,$$

holds, then we have the inequalities

$$\left|\int_\Omega f(s)\bar{g}(s)\,d\mu(s) - \int_\Omega f(s)\bar{h}(s)\,d\mu(s)\int_\Omega h(s)\bar{g}(s)\,d\mu(s)\right| \le \frac{1}{2}|\Phi-\varphi|\cdot\sqrt{\left(\int_\Omega |g(s)|^2\,d\mu(s) - \left|\int_\Omega h(s)\bar{g}(s)\,d\mu(s)\right|^2\right)}$$

and

$$\left|\int_\Omega f(s)\bar{g}(s)\,d\mu(s) - \int_\Omega f(s)\bar{h}(s)\,d\mu(s)\int_\Omega h(s)\bar{g}(s)\,d\mu(s)\right|$$
$$\le \frac{1}{2}|\Phi-\varphi|\cdot\left(\int_\Omega |g(s)|^2\,d\mu(s)\right)^{1/2}$$
$$-\left(\int_\Omega \operatorname{Re}\left((\Phi h(s)-f(s))\left(h(s)\bar{f}(s)-\varphi h(s)\right)\right)d\mu(s)\right)^{\frac{1}{2}}$$
$$\times\left|\int_\Omega h(s)\bar{g}(s)\,d\mu(s)\right|.$$

Proof. The proof follows by Theorem 5 on choosing $H = L^2(\Omega, K)$ with the inner product

$$\langle f,g\rangle = \int_\Omega f(s)\bar{g}(s)\,d\mu(s).$$

□

Remark 1. *We observe that, a sufficient condition for the condition (3.1) to hold, is that*

$$\operatorname{Re}(\Phi h(s)-f(s))\left(\bar{f}(s)-\overline{\varphi}\bar{h}(s)\right)\ge 0, \tag{3.2}$$

for $\mu-$a.e. $s\in\Omega$.

If the functions are real-valued, then, for Φ and φ real numbers, a sufficient condition for (3.2) to hold is

$$\Phi h(s)\ge f(s)\ge \varphi h(s)$$

for $\mu-$a.e. $s\in\Omega$.

In this way we can see the close connection that exists between the classical Grüss' inequality and the results we obtained above.

Now, consider the family $\{f_i\}_{i\in I}$ of functions in $L^2(\Omega,\mathbb{K})$ with the properties that

$$\int_\Omega f_i(s)\overline{f_j}(s)\,d\mu(s)=\delta_{ij},\quad i,j\in I,$$

where δ_{ij} is 0 if $i\neq j$ and $\delta_{ij}=1$ if $i=j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2(\Omega,\mathbb{K})$.

The following proposition holds.

Proposition 2. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2(\Omega,\mathbb{K})$, F a finite subset of I, $\phi_i,\Phi_i\in\mathbb{K}$ $(i\in F)$ and $f\in L^2(\Omega,\mathbb{K})$, so that either*

$$\int_\Omega \operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s)-f(s)\right)\left(\overline{f}(s)-\sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right]d\mu(s)\ge 0 \tag{3.3}$$

or, equivalently,

$$\int_{\Omega}\left|f(s)-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2 d\mu(s)\le\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

holds. Then we have the inequalities

$$\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)-\sum_{i\in F}\int_{\Omega}f(s)\overline{f_i}(s)\,d\mu(s)\int_{\Omega}f_i(s)\overline{g(s)}d\mu(s)\right|$$
$$\le\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{1/2}\left(\int_{\Omega}|g(s)|^2\,d\mu(s)-\sum_{i\in F}\left|\int_{\Omega}g(s)\overline{f_i(s)}d\mu(s)\right|^2\right)^{1/2}$$

and

$$\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)-\sum_{i\in F}\int_{\Omega}f(s)\overline{f_i}(s)\,d\mu(s)\int_{\Omega}f_i(s)\overline{g(s)}d\mu(s)\right|$$
$$\le\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{1/2}\left(\int_{\Omega}|g(s)|^2\,d\mu(s)\right)^{1/2}$$
$$-\left(\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-\int_{\Omega}f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\right)^{1/2}\left(\sum_{i\in F}\left|\int_{\Omega}f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\right)^{1/2}.$$

The proof is obvious by Theorem 7 and we omit the details.

Remark 2. *In the real case, we observe that a sufficient condition for (3.3) to hold, is that*

$$\sum_{i\in F}\Phi_i f_i(s)\ge f(s)\ge\sum_{i\in F}\varphi_i f_i(s)$$

for $\mu-a.e.\ s\in\Omega$.

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