Properties of Some Minimum Run Resolution IV Designs

by

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Abstract

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The regular resolution IV fractional factorial designs described in the literature, have many desirable properties; main effect estimates are uncorrelated and are unbiased by two-factor interactions, whilst estimates of the two-factor interaction effects are limited to the identification of a significant orthogonal aliased string. An augmenting experiment is required to identify which interactions in the string are significant and which are inert.

In addition to the regular class of fractional designs there exist a series of minimum run non-regular resolution IV designs. These non-regular designs can be further divided into orthogonal and non-orthogonal designs. To examine n factors these designs require $2n$ runs and are generated from minimum run resolution III designs via the Box and Wilson foldover thereom. These designs usually result in a saving of runs when compared with the corresponding regular fractional designs. Orthogonal designs provide uncorrelated main effect estimates. However, the two-factor interaction effects are usually confounded in an complex manner. Non-orthogonal designs provide correlated main effect estimates and also exhibit complex confounding in the two-factor interaction subspace.

This thesis examines a number of minimum run resolution IV non-regular

designs and determines whether it is possible to search for and estimate a small number of two-factor interactions without the need of augmenting trials. The orthogonal designs considered are the foldover designs generated from the 12,20 and 24 factor Plackett and Burman designs, whilst the non-orthogonal designs considered are the foldover designs generated from the Yang 6 and 14 factor designs and the Raghavarao 13 factor design. Unlike the regular factorial designs, some of these non-regular designs provide estimates of a small number of two-factor interactions without the addition of augmenting trials. In particular the 20 and 24 factor Plackett and Burman foldover designs are shown to be resolution V in every set of 5 factors and to allow the search and estimation of up to two two-factor interactions. The foldovers of the Yang and Raghavarao designs allow the search and estimation of up to one two-factor interaction.

As the number of interactions considered gets larger, the search and estimation cannot be done for some values of the interaction effects. In these cases a number of models are equally likely. In situations such as this augmenting trials are discussed, and a technique is devised to deal with the design of augmenting trials. This thesis is also concerned with the analysis of these non-regular designs and two methods are presented to analyse search designs which are illustrated through the examination of some simulated experiments.

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DEDICATION

This thesis is dedicated to my parents Brian and Margaret Simmons and to my wife Peta for their encouragement, patience and unyielding support.

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Chapter 1

INTRODUCTION

1.1 Factorial Design

Consider an arbitrary process with a measurable response and a number of potentially influential variables as inputs. Traditionally scientists would vary one variable whilst holding all others constant in an attempt to identify which of the variables affected the response and which did not.

Experimental design began as a statistical discipline in the 1920's with the work of R.A.Fisher. Fisher [20] introduced the idea of factorial design, where he showed it was more efficient to consider many factors simultaneously rather than one factor at a time.

One advantage of factorial design is that when the effect of one factor is dependent on the levels of one or more of the other factors it becomes possible to detect and estimate the magnitude of these interactions. A disadvantage of factorial designs is that as the number of variables to consider increases, the number of required observations becomes very large. In general a design involving n factors requires 2^n observations, when each factor is considered at two levels only.

1.2 *Fractional Factorial Design*

Finney [19], developed fractional factorial designs, which used the factorial design principle but required a fraction of the required observations. The direct consequence of this reduction in experimental observations is that certain effect estimates become indistinguishable from one another (the effect estimates are said to be *confounded* or *aliased*). The higher the fractionation of the design the more complex this confounding becomes. Box and Wilson [8] introduced the term *resolution* to classify these fractionated designs. The most commonly used fractional designs are of resolution III, IV and V.

A common assumption when applying these fractionated designs, is that all third order interactions and higher are inert. In cases when this assumption is feasible the problem becomes that of identifying significant main effects and two-factor interactions.

Resolution III designs require the least amount of experimental observations. However, one of the consequences of using such designs is that estimates of main effects are confounded with two-factor interaction estimates. For this reason they are normally called *main effect plans*. Whilst resolution III designs are the most economical to perform, if two-factor interactions are present the conclusions drawn will be inconclusive at best. For this reason resolution III designs are often used only to screen out the factors that are significant from the ones that are inert, before proceeding with further experimentation to resolve any ambiguities.

Resolution IV designs, although requiring more observations than resolution III designs, provide estimates of main effects confounded with three-factor interactions, whilst the two-factor interactions are normally aliased with a linear combination of one or more other two-factor interactions (called a *string of two-factor interactions*). Resolution IV designs provide estimation of the main effects unbiased by two-factor interaction estimates, however conclusions drawn about the significance of individual two-factor interactions will be limited only to the identification of which string of confounded interactions are significant.

Resolution V designs require considerably more experimental observations than resolution IV designs, but provide estimates of the main effects confounded with four-factor interactions, and two-factor interactions confounded with three-factor interactions. These designs are used when significant two-factor interaction effects are strongly suspected and individual estimates of each two-factor interaction are required.

1.3 Augmenting Fractional Designs

In many cases the cost of generating experimental observations is considerable. It is therefore desirable to run designs of low resolution initially, and to resolve any ambiguities via the addition of augmenting trials. For this reason there has been much interest in the design of augmenting trials.

An important augmenting technique was introduced by Box and Wilson [8] called the foldover. They defined the foldover of any run from a two-level design as that run with all factor levels reversed. The foldover of a design is defined similarly as the addition of all the foldover runs to the original design. For example if X is a 3 factor design defined as

$$X = \begin{pmatrix} + & + & + \\ + & - & - \\ - & + & - \\ - & - & + \end{pmatrix}$$

then the foldover of X would be defined as

$$X_f = \begin{pmatrix} + & + & + \\ + & - & - \\ - & + & - \\ - & - & + \\ - & - & - \\ - & + & + \\ + & - & + \\ + & + & - \end{pmatrix}$$

Box and Wilson showed that by folding over any design of resolution III in $(n-1)$ factors, a design of resolution IV in n factors is obtained.

Daniel [11] [12] developed a method to de-alias significant strings of two-factor interactions in resolution IV designs, via the addition of a small augmenting set. In general Daniel's method is concerned with the estimation of P and Q when $P+Q$ are an aliased string in a regular orthogonal resolution IV design. Daniel shows that the addition of trials to estimate $P-Q$ will "de-alias" P and Q thus allowing their independent estimation.

For example if a 2^{4-1} design was performed with design generator $I = ABCD$, and the two-factor interaction string $AB + CD$ proved to be significant, then it would be desirable to isolate the 2 aliased two-factor interactions. This can be achieved if a design can be found yielding an estimate of $AB - CD$.

Meyer, Steinberg and Box [27] presented a method for designing augmenting trials within the Bayesian framework. Their method is to choose runs that allow maximum discrimination among the models considered most likely from the initial Bayesian analysis.

1.4 Non-Regular Fractional Designs

The regular 2^{n-p} fractional designs described in the literature are derived from choosing a full factorial design in the first $(n-p)$ factors, then deliberately aliasing the further p main effects with interaction columns selected via the use of appropriate design generators. In addition to these regular fractional designs there exist a series of non-regular designs which are not derived in this manner. Most of these non-regular designs are saturated or resolution III designs. However, by utilizing the Box and Wilson foldover theorem, a series of non-regular minimum run resolution IV designs can be generated.

1.4.1 *Plackett and Burman Designs*

If $X = (x_{ij})$ is an $n \times n$ matrix with $x_{ij} = x_{1m}$ where

$$m = \begin{cases} j - i + 1 & \text{if } j \geq i \\ n - (j - i + 1) & \text{otherwise} \end{cases}$$

then X is called a circulant matrix.

For example, if X is a circulant matrix with first row defined as:

$$x_{1m} = \{- - + +\}$$

then X will be defined as:

$$X = \begin{pmatrix} - & - & + & + \\ + & - & - & + \\ + & + & - & - \\ - & + & + & - \end{pmatrix}$$

Plackett and Burman [30] presented a series of circulant matrices which formed a class of two-level, orthogonal, non-regular resolution III designs examining $N - 1$ factors in N runs, where N is a multiple of 4 and $N \leq 100$.

Any of the resolution III Plackett and Burman designs can be used to obtain a minimum run resolution IV design via Box and Wilson's foldover theorem. For example the Plackett and Burman 11 factor resolution III design in 12-runs becomes a 12-factor resolution IV design in 24 runs when folded over. Part of this thesis will be concerned with the foldover designs generated from the Plackett and Burman 12, 20 and 24 run designs which are defined as the union of one run with all factors at their low level and the circulant designs with first runs defined as

$$PB_{12}[1,] = (+ + - + + + - - - + -)$$
$$PB_{20}[1,] = (+ + - - + + + + - + - + - - - - + + -)$$
$$PB_{24}[1,] = (+ + + + + - + - + + - - + + - - + - + - - - -)$$

respectively, these designs provide 100% efficient main effect estimates but the aliasing pattern of the two-factor interaction estimates is quite complex, see Draper and Stoneman [18]. The full Plackett and Burman 12-run design is given in table 1.1.

Orthogonal Arrays

The Plackett and Burman designs are also examples of orthogonal arrays. Rao [32] first introduced orthogonal arrays. For convenience this thesis will use notation due to Owen [29]. An orthogonal array of strength t is a matrix of N rows and k columns with elements taken from a set of s symbols such that in any $N \times t$ matrix there are s^t distinct rows each occur λ times, where λ is called the index of the orthogonal array. Therefore any OA(N,k,s,t) yields λ copies of a complete factorial for any choice of t factors.

For example the Plackett and Burman 12-run design is an OA(12,11,2,2) as the rows $(--)$, $(-+)$, $(+-)$ and $(++)$ appear three times in every 12×2 sub-matrix. The design does not have strength 3 as the $2^3 = 8$ rows required cannot

A	B	C	D	E	F	G	H	J	K	L
+	+	−	+	+	+	−	−	−	+	−
−	+	+	−	+	+	+	−	−	−	+
+	−	+	+	−	+	+	+	−	−	−
−	+	−	+	+	−	+	+	+	−	−
−	−	+	−	+	+	−	+	+	+	−
−	−	−	+	−	+	+	−	+	+	+
+	−	−	−	+	−	+	+	−	+	+
+	+	−	−	−	+	−	+	+	−	+
+	+	+	−	−	−	+	−	+	+	−
−	+	+	+	−	−	−	+	−	+	+
+	−	+	+	+	−	−	−	+	−	+
−	−	−	−	−	−	−	−	−	−	−

Table 1.1: The Plackett and Burman 12-run, Resolution III Design

be observed an equal number of times in a total of 12-runs.

1.4.2 Non-Orthogonal Designs

In addition to the orthogonal non-regular fractional designs a number of non-orthogonal designs exist. Non-orthogonal designs partially correlate main effect estimates, it is therefore useful to discuss the efficiency of the main effect estimates of the design . Margolin [25] defined the *main effect trace efficiency* of a design X as

$$K/N \,\text{trace}\, (X'X)^{-1}$$

where K is the number of factors considered and N is the number of runs.

Many of these non-orthogonal designs provide a substantial saving in runs compared to corresponding regular designs. For example the 5 factor resolution III regular factorial design requires 8 runs whilst the corresponding Yang design to examine 5 factors in the presence of bias requires only 6 runs.

Weighing Designs

A number of the non-orthogonal, non-regular designs are examples of chemical balance weighing designs (Hotelling, [22]). These designs were developed to determine the weights of k objects in k weighings using a two-pan balance, with each row in the design matrix corresponding to an individual weighing and each column corresponding to one of the objects. Each row assigns either a $(-)$ or a $(+)$ to each of the individual objects with a $(-)$ meaning the object is placed in one of the pans and a $(+)$ meaning the object is placed in the other pan.

These designs were developed for a bias free weighing device. Unbiased estimates of the weights can only be obtained from a fair two-pan scale. If the design matrix has a column consisting of only +s, then unbiased estimates of the weights can be obtained even if the weighing device is biased. Mood [28] showed how to alter these bias free weighing designs to determine the weights of one less object at the cost of introducing bias.

Yang Designs

Yang [36] [37] presented some minimum-run, resolution III, weighing designs for $n = 2 \pmod 4$, with the exception of $n = 22$ and 34. The design matrix of these designs are in general

$$X = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}' & \mathcal{A}' \end{pmatrix}$$

where $\mathcal{A}$ and $\mathcal{B}$ are circulant matrices of order $n/2$ respectively. Any of the resolution III Yang designs for n factors in N runs in the absence of bias can be used to generate a resolution IV design for n factors in $2N$ runs in the presence of bias via the Box and Wilson [8] foldover thereom.

For example the 6 factor design due to Yang [36] has its first rows of $\mathcal{A}$ and $\mathcal{B}$ defined as

$$\mathcal{A} = \; + \; + \; +$$
$$\mathcal{B} = \; - \; + \; +$$

the design matrix is given as

$$X = \begin{pmatrix} + & + & + & - & + & + \\ + & + & + & + & - & + \\ + & + & + & + & + & - \\ + & - & - & + & + & + \\ - & + & - & + & + & + \\ - & - & + & + & + & + \end{pmatrix}$$

the foldover design is a resolution IV design for 6 factors in 12 runs given as

$$Y_{6:12} = \begin{pmatrix} X \\ -X \end{pmatrix}$$

Raghavarao Designs

Raghavarao [31] presented optimum weighing designs for $N = 5$, 13 and 25 factors. The 5 and 13 factor designs are circulant of order N. For example the resolution III design to examine 13 factors in 13 runs in the absence of bias is of circulant form with first row given by

$$- \; - \; + \; - \; + \; + \; + \; + \; + \; - \; + \; + \; +$$

These designs can be used to generate minimum-run resolution IV designs for N factors in the presence of bias using Box and Wilson's foldover theorem.

1.5 *Search Designs*

Whenever fractional designs are used there is the underlying assumption that only a fraction of the contrasts considered will be significant, Srivastava [33] conceptualized this in his theory for search designs. Srivastava divided factorial effects into 3 categories:

1. Effects that can be assumed negligible.

2. Effects which require estimation.

3. All remaining effects, some which are negligible and some of which will require estimation.

He termed designs which estimate all effects of type (2) and search the non-negligible effects of type (3) "Search Designs".

Srivastava considered the linear model

$$Y = A_1\xi_1 + A_2\xi_2 + e$$

$$E(e) = 0, \qquad V(e) = \sigma^2 I_N$$

where Y is a $N \times 1$ vector of observations, e is the $N \times 1$ error vector, A_i are the $N \times v_i$ design matrices, and ξ_i are $v_i \times 1$ vectors of fixed unknown parameters. If all the elements in ξ_2 are negligible except for possibly a set of at most r elements, where r is a known positive integer, we want A_1 and A_2 to be such that we can estimate all the elements of ξ_1 and the r non-negligible elements of ξ_2.

Srivastava's main theorem states:

Theorem 1 *[(Srivastava, [33])] Let T be a search design corresponding to the observations Y. If $e = 0$ then T is a search design of resolving power* $\{\xi_1; \xi_2, r\}$

iff for every sub-matrix A_{2k} $(N \times 2r)$ of A_2, we have

$$Rank\ (A_1 : A_{2r}) = v_1 + 2r$$

For example, in a four factor experiment with each factor at two levels, and supposing estimates are required of the mean and at most two main effects. Consider two sets of trials, the first $\{a, b, c, d, abcd\}$ has matrices A_1 and A_2 defined as:

$$A_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} A_2 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and the second set of trials $\{ad, abd, bc, d, c\}$ has matrices A_1 and A_2 defined as:

$$A_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} A_2 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$$

with $\xi_1 = \mu$ and $\xi_2 = (A\ B\ C\ D)'$ where at most two of the elements of ξ_2 are non-zero.

The rank of the $(A_1 : A_2)$ matrix for the first set of trials $\{a, b, c, d, abcd\}$ is of full rank, and therefore forms a search design for this situation. The second set of trials $\{ad, abd, bc, d, c\}$ has a rank deficient $(A_1 : A_2)$ matrix and therefore does not define a search design in this case.

In the context of this thesis A_1 will contain the design matrix of the mean and main effects, and A_2 will contain the design matrix of the two-factor interactions. When models can be resolved or separated only when parameters do not take particular values (Rank $(A_1 : A_2r) < v_1 + 2r)$, the design matrix is said to be a *weakly resolvable main-effect-plus r* search design. When for every A_2k submatrix of A_2 we have Rank $(A_1 : A_2r) = v_1 + 2r$, then design matrix is said to be a *strongly resolvable main-effect-plus r* search design.

1.6 Projection Properties

In many experiments it is often the case that of the many factors considered initially only a few affect the response to a significant degree. It is therefore useful when employing a design in k factors, to understand how the design behaves when projected onto any $p \leq k$ factors, or p-space.

The projection properties of regular fractional factorial designs were first discussed by Box and Hunter [4]. They showed that every resolution R design contained a full factorial design in every $R-1$ factors, and therefore are orthogonal arrays of strength $R-1$.

Box and Hunter's [4] result states any regular resolution III design contains a complete 2^2 factorial design in any 2 factors. However Lin and Draper [24] and Box and Bisgaard [2] when examining the projection properties of the Plackett and Burman 12-run design, which is also resolution III, showed when projected onto 3-space the design provides a 2^3 and a 2^{3-1} design. This is clearly an improvement on the regular fractional factorial projection.

Cheng [9] generalised this result showing the projection of any OA(N,k,2,t) with $k \geq t+1$ onto any $(t+1)$ columns results in three possible different designs:

1. $2^{-t}N$ copies of the half replicate of the 2^{t+1} factorial.

2. $2^{-(t+1)}N$ copies of the complete 2^{t+1} factorial.

3. Projection contains copies of both the complete 2^{t+1} factorial and the half replicate.

Type 1 projections are obviously the least desirable as they do not contain a complete 2^{t+1} factorial. Regular fractional designs always have a type 1 projection in at least one of the possible choices of $t+1$ columns.

Cheng further showed that if N is not a multiple of 2^{t+1} and $k \geq t+2$, then the projection must be of type 3. For example the Plackett and Burman 12-run design is OA(12,11,2,2) and therefore contains copies of both the complete 2^3 factorial and the 2^{3-1} half replicate in every choice of 3 columns.

Wang and Wu [34] when considering the projection properties of the Plackett and Burman designs observed that the complex aliasing pattern of the designs "allows some interactions to be entertained and estimated without making additional runs". They called this property the hidden projection property of the designs.

Diamond [15] when considering the Plackett and Burman 12-factor foldover design showed that as a consequence of this hidden projection property, estimates of all the main effects and two two-factor interactions could be attained without the addition of augmenting trials. The design is therefore a search design of resolving power 2 (Theorem 1) in the two-factor interaction subspace.

Box and Meyer [5] introduced the term *factor sparsity*. Their approach is to suppose that only a small number of factors in the design are responsible for most of what is happening, but these active factors may interact with each other. For example if a Plackett and Burman 12-run design was performed to consider 11 factors (A, $B \cdots K$) and subsequent analysis led one to expect only factors A, B and C are active, the design can now be collapsed into a 3 factor design in only the columns corresponding to factors A, B and C, allowing for the interaction estimates AB,AC,BC and ABC to now be formulated.

Hamada and Wu [21] outlined a technique to search for interactions in Plackett and Burman designs, exploiting the assumption of *effect heredity*, that is a two-factor interaction can only be real if one or both of the corresponding main effects are real. Their technique advocated a standard initial analysis using normal probability plots to identify the active main effects, then estimation of only the interactions containing at least one of the active main effects by means of a multiple regression.

Box and Meyer [7] introduced a Bayesian technique to identify active factors in highly fractionated designs. Their method consists of calculating posterior probabilities:

$$p\,(M_i \mid y)$$

of all possible models M_i given the response vector y. A marginal posterior probability:

$$P_j = \sum_{\mathrm{M_i:\, factor\ j\ active}} p\,(M_i \mid y)$$

can then be computed as a measure of determining whether a factor j is active. A large P_j value indicates that factor j is active, whilst small P_j values indicate factor j is inert and models involving factor j can be subsequently dropped from the analysis.

Box and Meyer [7] applied this technique to a 12-run Plackett and Burman experiment to study fatigue life of weld repaired castings (Hunter, Hodi and Eager [23]), finding after performing a Bayesian analysis that two factors were active and subsequently using the 2-space projection of the design to analyse it as a replicated 2^2 factorial.

Box and Meyer's technique is based on the principle of factor sparsity and offers the potential to greatly simplify the analysis of non-regular designs. In cases when the analysis leads to ambiguous conclusions a smaller set of trials or augmenting set is needed.

1.7 Thesis Objectives

Whilst there is no shortage of literature on regular full and fractional factorial experimental designs, with the possible exception of the Plackett Burman 12-run design, non-regular designs have been largely ignored.

This thesis is concerned with the examination of a number of the minimum-run, two-level, non-regular resolution IV designs, with reference to their projection properties, their resolvability as search designs and the design of appropriate augmenting trials when required.

Such an examination is desirable as in many cases these minimum-run designs result in a considerable saving of experimental observations, and as such non-regular designs form an important class within the broader experimental design framework.

Chapter 2 examines the foldover of the 13 factor design due to Raghavarao which is shown to be strongly resolvable main-effect-plus r search design when $r = 1$. When $r = 2$ the design is shown to be weakly resolvable and as such appropriate augmenting trials are given. The 3 and 4-space projections of the design are also presented.

Chapter 3 extends the work by Diamond [14] and Diamond and Simmons [16] on the Yang 6 factor foldover to consider the best augmenting trials in the presence of error. The chapter also looks at the Yang 14 factor foldover design, and the design is shown to be strongly resolvable when $k = 1$.

Diamond [15] showed the Plackett and Burman 12-factor foldover is a strongly resolvable main-effect-plus r search design when $r = 2$. Chapter 4 extends this work to consider the design when $r = 3$. The projection properties of the design in n-space are also presented.

In Chapter 5 it is shown that the 20 factor Plackett and Burman foldover design is resolution V in every set of five columns. The design is also shown to be a strongly resolvable main-effect-plus r search design when $r = 2$. The chapter then examines the 24 factor Plackett and Burman foldover in the same manner which is also shown to be resolution V in every choice of five columns, and to be a strongly resolvable main-effect-plus r search design when $r = 2$.

In Chapter 6 two methods are presented to analyse minimum-run resolution IV designs. The effectiveness of these techniques are illustrated through a comparison with the existing Bayesian analysis due to Box and Meyer [7] using two examples.

Finally, in Chapter 7 some overall conclusions on the use of these minimum-run, non-regular resolution IV designs will be made, and areas which require further consideration will be identified.

Chapter 2

THE RAGHAVARAO 13 FACTOR FOLDOVER DESIGN

2.1 Introduction

Raghavarao [31] presented a minimum run, non-orthogonal bias free weighing design, studying 13 factors in 13 runs, referred to as $2^{(13)}//13$ in the remainder of this thesis. This resolution III design is of circulant form with first row given by

$$- - + - + + + + + - + + +$$

and has main effect trace efficiency of 96.2%.

By applying the foldover technique this design can be moved from a bias free $2^{13}//13$ resolution III design, to a $2^{13}//26$ resolution IV design in the prensence of bias. The Raghavarao 13 factor foldover design is presented in Table 2.1[1] and is in fact one circulant matrix augmented by another circulant matrix, being the negative of the first, and is given by

$$R = \begin{pmatrix} X \\ -X \end{pmatrix}, \quad \text{where } R \text{ is a } 2n \times n \text{ matrix.}$$

Webb [35] has shown that the columns corresponding to the mean and main effects in any resolution IV design are orthogonal to the columns corresponding to the mean and two-factor interactions, and this applies to the Raghavarao 13 factor foldover design.

In this chapter it will be shown it is also possible for a small number of two-factor interactions to be searched for and estimated.

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[1] Throughout this chapter factors are labelled as numbers for convenience

Table 2.1: The Raghavarao 13 factor foldover design

Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	−	−	+	−	+	+	+	+	+	−	+	+	+
2	+	−	−	+	−	+	+	+	+	+	−	+	+
3	+	+	−	−	+	−	+	+	+	+	+	−	+
4	+	+	+	−	−	+	−	+	+	+	+	+	−
5	−	+	+	+	−	−	+	−	+	+	+	+	+
6	+	−	+	+	+	−	−	+	−	+	+	+	+
7	+	+	−	+	+	+	−	−	+	−	+	+	+
8	+	+	+	−	+	+	+	−	−	+	−	+	+
9	+	+	+	+	−	+	+	+	−	−	+	−	+
10	+	+	+	+	+	−	+	+	+	−	−	+	−
11	−	+	+	+	+	+	−	+	+	+	−	−	+
12	+	−	+	+	+	+	+	−	+	+	+	−	−
13	−	+	−	+	+	+	+	+	−	+	+	+	−
14	+	+	−	+	−	−	−	−	−	+	−	−	−
15	−	+	+	−	+	−	−	−	−	−	+	−	−
16	−	−	+	+	−	+	−	−	−	−	−	+	−
17	−	−	−	+	+	−	+	−	−	−	−	−	+
18	+	−	−	−	+	+	−	+	−	−	−	−	−
19	−	+	−	−	−	+	+	−	+	−	−	−	−
20	−	−	+	−	−	−	+	+	−	+	−	−	−
21	−	−	−	+	−	−	−	+	+	−	+	−	−
22	−	−	−	−	+	−	−	−	+	+	−	+	−
23	−	−	−	−	−	+	−	−	−	+	+	−	+
24	+	−	−	−	−	−	+	−	−	−	+	+	−
25	−	+	−	−	−	−	−	+	−	−	−	+	+
26	+	−	+	−	−	−	−	−	+	−	−	−	+

2.2 Projection Properties

In order to examine every possible design in p space one would normally consider $\binom{n}{p}$ different possible combinations. Due to the circulant nature of this design any choice of k columns $(x_1, x_2, \ldots, x_p)$ has 12 equivalent designs, given by $(x_{1+i}, x_{2+i}, \ldots, x_{k+i})$ for any $i = 1, \ldots, 12$, with reduction modulo 13 being performed whenever necessary. When considering the projection properties of the Raghavarao design in k space, therefore, only $\binom{13}{k}/13$ distinct choices of k columns need be considered, as all other choices are derived from this base set.

2.2.1 In 3-Space

When examining the design in 3-space $\binom{13}{3}/13 = 22$ combinations must be considered. Direct examination of all 22 potentially different possibilities yields, for all cases, a 2^3 full factorial design replicated three times, a run with all factors at their high level, and a run with all factors at their low level.

2.2.2 In 4-Space

There are $\binom{13}{4}/13 = 55$ potentially different choices of 4 columns which need to be considered when looking at the design in 4-space. Direct checking of each of the 55 cases yields 3 possible results:

(1) Three 2^{4-1} $(I = -1234)$ designs, a run with all factors at their high level, and a run with all factors at their low level. For example columns 1,2,4 and 10 form a projection of this type and are presented in Table 2.2.

(2) A 2^4, a 2^{4-1} $(I = 1234)$, a run with all factors at their high level, and a run with all factors at their low level. For example columns 1,2,3 and 6 form a projection of this type and are presented in Table 2.3.

(3) A 2^4, a 2^{4-1} ($I = -1234$), a run with all factors at their high level, and a run with all factors at their low level. For example columns 1,2,3 and 4 form a projection of this type and are presented in Table 2.4.

This immediately poses the problem, how to determine which of the three results is valid for a particular choice of 4 columns? Let $\alpha, \beta, \gamma, \delta$ be a unique choice of 4 columns in the foldover design and let X be the 26×78 matrix of two-factor interaction columns. If x_i is the column in X that corresponds to the $\alpha \times \beta$ interaction, and x_j is the column in X that corresponds to the $\gamma \times \delta$ interaction, the projection properties of the four columns corresponding to factors $\alpha, \beta, \gamma, \delta$ in the original foldover design is determined by examining the values of $x_i' x_j$ as follows:

$$x_i' x_j = \begin{cases} -22 & \text{Yields Result 1} \\ 10 & \text{Yields Result 2} \\ -6 & \text{Yields Result 3} \end{cases}$$

Determining the projection properties of the Raghavarao 13 factor foldover in 4-space is, therefore, as simple as looking up the relevant cell value for a given choice of four columns in the $X'X$ interaction matrix. For example if $x_i = 1 \times 2$ and $x_j = 4 \times 10$ then $x_i' x_j = -22$ and if $x_i = 1 \times 2$ and $x_j = 3 \times 4$ then $x_i' x_j = -6$.

2.3 Linear Dependencies

Highly saturated resolution III designs are most often used as screening designs. In such a design a large number of factors are considered under the assumption that either main effects are the only active effects, or that a few main effects are active and interactions are only considered between active main effects. This latter concept defines factor sparsity (Box and Meyer [5]).

Another concept, presented by Hamada and Wu [21] is effect heredity, which

Table 2.2: Columns 1,2,4 and 10 illustrating the first type of 4-space projection from the Raghavarao 13 factor foldover design

Run	1	2	4	10
20	−	−	−	+
18	+	−	−	−
15	−	+	−	−
3	+	+	−	+
16	−	−	+	−
2	+	−	+	+
5	−	+	+	+
7	+	+	+	−
22	−	−	−	+
24	+	−	−	−
19	−	+	−	−
4	+	+	−	+
17	−	−	+	−
6	+	−	+	+
11	−	+	+	+
9	+	+	+	−
23	−	−	−	+
26	+	−	−	−
25	−	+	−	−
8	+	+	−	+
21	−	−	+	−
12	+	−	+	+
13	−	+	+	+
10	+	+	+	−
1	−	−	−	−
14	+	+	+	+

Table 2.3: Columns 1,2,3 and 6 illustrating the second type of 4-space projection from the Raghavarao 13 factor foldover design

Run	1	2	3	6
22	−	−	−	−
18	+	−	−	−
19	−	+	−	−
3	+	+	−	−
1	−	−	+	−
26	+	−	+	−
15	−	+	+	−
4	+	+	+	−
17	−	−	−	+
2	+	−	−	+
13	−	+	−	+
7	+	+	−	+
16	−	−	+	+
6	+	−	+	+
5	−	+	+	+
9	+	+	+	+
21	−	−	−	+
24	+	−	−	−
25	−	+	−	−
14	+	+	−	+
20	−	−	+	−
12	+	−	+	+
11	−	+	+	+
8	+	+	+	−
23	−	−	−	−
10	+	+	+	+

Table 2.4: Columns 1,2,3 and 4 illustrating the third type of 4-space projection from the Raghavarao 13 factor foldover design

Run	1	2	3	4
21	−	−	−	−
24	+	−	−	−
25	−	+	−	−
3	+	+	−	−
20	−	−	+	−
6	+	−	+	−
5	−	+	+	−
10	+	+	+	−
23	−	−	−	+
2	+	−	−	+
13	−	+	−	+
7	+	+	−	+
1	−	−	+	+
12	+	−	+	+
11	−	+	+	+
4	+	+	+	+
22	−	−	−	−
18	+	−	−	+
19	−	+	−	+
14	+	+	−	−
16	−	−	+	+
26	+	−	+	−
15	−	+	+	−
8	+	+	+	+
17	−	−	−	−
9	+	+	+	+

is identical to factor sparsity except it also considers interactions between factors with one main effect active and the other main effect inert.

As stated in Chapter 1, Srivastava [33] developed the theory for search designs and he divided factorial effects into three categories as defined in section 1.5. The search design concept is in fact a superset of factor sparsity and effect heredity, assuming 3 factor interactions and higher are negligible. For example considering effect heredity, main effects would be classified as category 2, those k interactions involving at least one significant main effect as category 3, and three factor interactions and higher as category 1.

Srivastava showed that, in the error-free case, when estimating all the effects in category 2 and r effects in category 3, the design is a *strongly resolvable main-effect-plus r search design* iff every submatrix consisting of all the columns corresponding to category 2 and $2r$ of the columns corresponding to category 3 is of full rank.

2.3.1 *When* $r = 1$

The Raghavarao 13 factor foldover is a strongly resolvable main-effect-plus 1 search design iff none of the interaction columns are identical to each other. This would correspond to a value of 26 in an off-diagonal element of the $X'X$ interaction matrix. As off-diagonal elements can only take values of $2, -6$ and 10, the Raghavarao 13 factor foldover must be a strongly resolvable search design of resolving power 1.

2.3.2 *When* $r = 2$

Let X be a matrix of 4 interaction columns and consider any two columns of X say x_i and x_j; $i, j = 1, 2, 3, 4$. The vector product $x_i'x_j$ can take on one of five different values. If the interaction corresponding to x_i has no letters in common with the interaction x_j, $x_i'x_j$ can take one of 10,−6 or −22 as stated in section 2.2.2. Likewise if the interaction corresponding to x_i has one letter in common with interaction x_j, $x_i'x_j = 2$. Finally if $x_i = x_j$, $x_i'x_j = 26$. In summary :-

$$x_i'x_j = \begin{cases} 26 & \text{if i=j} \\ 2 & \text{if one letter in common} \\ \left.\begin{array}{c} 2 \pm 8 \\ -22 \end{array}\right\} & \text{if no letters in common} \end{cases}$$

To determine if the Raghavarao 13 factor foldover design is a strongly resolvable main-effect-plus 2 search design, every possible $X'X$ generated from the design must be of full rank, and each $X'X$ takes the following form:

$$X'X = \begin{pmatrix} 26 & x_{12} & x_{13} & x_{14} \\ x_{12} & 26 & x_{23} & x_{24} \\ x_{13} & x_{23} & 26 & x_{34} \\ x_{14} & x_{24} & x_{34} & 26 \end{pmatrix}$$

where $x_{ij} = 2, -6, 10, -22; i \neq j$.

Since x_{ij} can only take 4 possible values, there are $4^6 = 4096$ possible $X'X$ matrices to consider. Direct examination of these 4096 possibilities yields rank deficient matrices of 3 types. The upper triangles of these matrices are as follows:

$$\begin{pmatrix} 2 & 2 & -22 \\ & -22 & 2 \\ & & 2 \end{pmatrix} \begin{pmatrix} -6 & 10 & -22 \\ & -22 & 10 \\ & & -6 \end{pmatrix} \begin{pmatrix} -6 & 10 & 10 \\ & 10 & 10 \\ & & -6 \end{pmatrix}$$

Diamond [13] showed that every X can be represented by one of eleven graphs, given in Table 2.5, involving n vertices and 4 edges. Each vertex represents a factor whilst each edge represents a two-factor interaction. Note that for the Raghavarao 13-factor foldover design if two edges are co-incident at one of the vertices, the corresponding vector product $x_i'x_j$ must be 2. Since in each

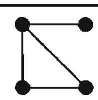
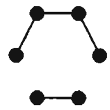
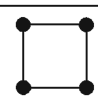
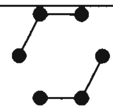
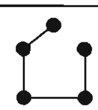
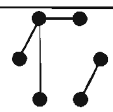
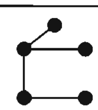
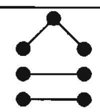
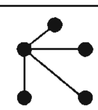
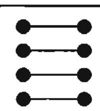
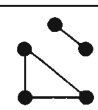
Case	Graph	Number of Isolated Vertices	Case	Graph	Number of Isolated Vertices
1		$(n-4)$	7		$(n-6)$
2		$(n-4)$	8		$(n-6)$
3		$(n-5)$	9		$(n-6)$
4		$(n-5)$	10		$(n-7)$
5		$(n-5)$	11		$(n-8)$
6		$(n-5)$			

Table 2.5: All graphs with n vertices and 4 edges

row of the first dependent matrix above there are two $x_i'x_j$ values equal to 2, every interaction in the dependent set must have two other interactions with one factor in common. This would mean that every vertex in Table 2.5 must have two edges co-incident with it, and can therefore be illustrated as linear graph 2 in Table 2.5. Any dependent sets of this type therefore involves two-factor interactions between 4 factors. The second and third dependent matrices have none of the $x_i'x_j$ equal to 2 and therefore involve two-factor interactions between 8 factors and correspond to Graph 11 in Table 2.5. For the Raghavarao 13-factor foldover design, therefore, only two possible linearly dependent cases need to be examined, when $r = 2$.

To consider the case consisting of four interactions with no letters in common (case 11) there are

$$\binom{13}{8} \times \frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 135,135$$

cases. Direct checking shows that these 135,135 possible cases generated from the Raghavarao 13-factor foldover are all linearly independent.

To examine the case in which dependent models exist in four factors (case 2), the projection properties in 4-space can be utilised. Section 2.2.2 identified 3 distinct results for any choice of 4 columns, identified by $x_i'x_j = \{-22, 10, -6\}$; where x_i and x_j represent unique two-factor interactions in the design. Any choice of four factors in the design yielding $x_i'x_j = 10$ or -6, forms a full 2^4 factorial design, and thus provides unbiased estimation of all interactions between the four factors. Only choices of four columns yielding $x_i'x_j = -22$, therefore, require examination for linearly dependent models.

For any two-factor interaction $\alpha \times \beta$, corresponding to a column x_i in the $X'X$ interaction matrix, -22 will appear in only one of its 78 cells. If x_j is the row in $X'X$ corresponding to this cell, and x_j represents the two-factor interaction $\gamma \times \delta$, each linear dependency is of following form:

$$\alpha \times \beta + \gamma \times \delta = \alpha \times \gamma + \beta \times \delta = \alpha \times \delta + \beta \times \gamma \tag{2.1}$$

and by using the circulant properties of the design any one dependency of this form has $(k-1)$ equivalent dependencies found by:

$$\begin{aligned} \{(\alpha+k) \times (\beta+k)\} + \{(\gamma+k) \times (\delta+k)\} = \{(\alpha+k) \times (\gamma+k)\} + \\ \{(\beta+k) \times (\delta+k)\} = \{(\alpha+k) \times (\delta+k)\} + \{(\beta+k) \times (\gamma+k)\} \end{aligned} \tag{2.2}$$

where $k = 1, \ldots, 12$ and reduction modulo 13 is performed as necessary.

General Case	$\alpha \times \beta + \gamma \times \delta$	$\alpha \times \gamma + \beta \times \delta$	$\alpha \times \delta + \beta \times \gamma$
1	$1 \times 2 + 4 \times 10$	$1 \times 4 + 2 \times 10$	$1 \times 10 + 2 \times 4$
2	$1 \times 3 + 9 \times 13$	$1 \times 9 + 3 \times 13$	$1 \times 13 + 3 \times 9$
3	$1 \times 5 + 6 \times 8$	$1 \times 6 + 5 \times 8$	$1 \times 8 + 5 \times 6$
4	$1 \times 7 + 11 \times 12$	$1 \times 11 + 7 \times 12$	$1 \times 12 + 7 \times 11$
5	$2 \times 3 + 5 \times 11$	$2 \times 5 + 3 \times 11$	$2 \times 11 + 3 \times 5$
6	$2 \times 6 + 7 \times 9$	$2 \times 7 + 6 \times 9$	$2 \times 9 + 6 \times 7$
7	$2 \times 8 + 12 \times 13$	$2 \times 12 + 8 \times 13$	$2 \times 13 + 8 \times 12$
8	$3 \times 4 + 6 \times 12$	$3 \times 6 + 4 \times 12$	$3 \times 12 + 4 \times 6$
9	$3 \times 7 + 8 \times 10$	$3 \times 8 + 7 \times 10$	$3 \times 10 + 7 \times 8$
10	$4 \times 5 + 7 \times 13$	$4 \times 7 + 5 \times 13$	$4 \times 13 + 5 \times 7$
11	$4 \times 8 + 9 \times 11$	$4 \times 9 + 8 \times 11$	$4 \times 11 + 8 \times 9$
12	$5 \times 9 + 10 \times 12$	$5 \times 10 + 9 \times 12$	$5 \times 12 + 9 \times 10$
13	$6 \times 10 + 11 \times 13$	$6 \times 11 + 10 \times 13$	$6 \times 13 + 10 \times 11$

Table 2.6: Linear Dependencies in the Raghavarao 13-factor foldover design

Each dependency in the design is of the form given in equation 2.1. In total 13 linear dependencies exist and these form a closed set in which each of the 78 two-factor interactions of the design appear in one dependent model only. These 13 linear dependencies are listed in Table 2.6 for completeness, however by using equation 2.1 if any one linear dependency is known any other dependency can be derived.

For example if $(1 \times 2) + (4 \times 10) = (1 \times 4) + (2 \times 10) = (1 \times 10) + (2 \times 4)$ is a known linear dependency in the design, and it is desired to find the dependency which contains the 1×7 interaction. The interaction 1×7 can be expressed as

$(4 + 10) \times (10 + 10)$ and substituted into equation 2.2 as follows:

$$\{(1+10)\times(2+10)\} + \{(4+10)\times(10+10)\} = \{(1+10)\times(4+10)\} + \{(2+10)\times(10+10)\} = \{(1+10)\times(10+10)\} + \{(2+10)\times(10+10)\}$$

Evaluating this equation and taking modulo 13 as required the linear dependency $(1 \times 7) + (11 \times 12) = (1 \times 11) + (7 \times 12) = (1 \times 12) + (7 \times 1)$ is identified, which is the only linear dependency in which the 1×7 interaction appears.

From the above results it is apparent that the Raghavarao 13-factor foldover is not a strongly resolvable main-effect-plus 2 search design, and to estimate two-factor interactions in some cases requires the addition of augmenting trials.

2.4 Augmenting Design

If $(\alpha \times \beta + \gamma \times \delta)$ is the true model but is completely confused with the models $(\alpha \times \gamma + \beta \times \delta)$ and $(\alpha \times \delta + \beta \times \gamma)$ as described in section 2.3, then the addition of augmenting trials is required to identify the true model.

Let A be a 16×4 matrix with columns $\alpha, \beta, \gamma, \delta$ forming a full factorial design, and let $a^{[\alpha\beta]}$ correspond to the interaction column $\alpha \times \beta$ generated from A. Now let $C = (c^{[1]}, c^{[2]})$ be a 16×2 matrix with columns defined as follows:

$$c^{[1]} = a^{[\alpha\beta]} + a^{[\gamma\delta]} - (a^{[\alpha\gamma]} + a^{[\beta\delta]})$$
$$c^{[2]} = a^{[\alpha\beta]} + a^{[\gamma\delta]} - (a^{[\alpha\delta]} + a^{[\beta\gamma]})$$

Any $n \times 2$ submatrix of C corresponds to an augmenting set of trials in A that will separate the models iff each column of the submatrix in C satisfies the following:-

1. Neither column is equal to the null vector.

2. If a block term, that is, allowance for the augmenting trials to have a different mean than the original trials due to changes external to the experiment, is to be estimated neither column can be proportional to the unit vector.

3. No one column is proportional to any other column.

In order to separate the linear dependency $\alpha \times \beta + \gamma \times \delta = \alpha \times \gamma + \beta \times \delta = \alpha \times \delta + \beta \times \gamma$ the following A matrix is generated.

α	β	γ	δ
−	−	−	−
+	−	−	−
−	+	−	−
+	+	−	−
−	−	+	−
+	−	+	−
−	+	+	−
+	+	+	−
−	−	−	+
+	−	−	+
−	+	−	+
+	+	−	+
−	−	+	+
+	−	+	+
−	+	+	+
+	+	+	+

and the corresponding C matrix is as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 4 & 4 \\ 0 & 0 \\ -4 & 0 \\ 0 & -4 \\ 0 & 0 \\ 0 & 0 \\ 0 & -4 \\ -4 & 0 \\ 0 & 0 \\ 4 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A submatrix of C that satisfies the three criteria required to separate the dependency and to estimate the block effect is:

$$\begin{pmatrix} 4 & 4 \\ 0 & -4 \end{pmatrix}$$

which corresponds to a number of augmenting trials one of which is $(\alpha\beta\ ,\ \beta\gamma)$.

Thus the augmenting trials $(1, 2)$ and $(2, 4)$ are sufficient to separate the $(1 \times 2) + (4 \times 10) = (1 \times 4) + (2 \times 10) = (1 \times 10) + (2 \times 4)$ linear dependency.

2.5 Conclusion

In this chapter it has been shown that the Raghavarao 13-factor foldover design gives a 2^3, a run with all factors at their high level, and a run with all factors at their low level in every set of three factors. When examining the design in 4 factors three possible results can be obtained, given in section 2.2. The design has been shown to be a strongly resolvable main-effect-plus r search design when $r = 1$, but not when $r = 2$ since then in some cases a number of models fit the data equally well. Each linearly dependent case has been listed and a general result given to generate the augmenting trials required to separate the dependent models.

Chapter 3

THE YANG 6 AND 14 FACTOR DESIGNS

3.1 Introduction

Yang [36] presented some minimum run, resolution III, weighing designs for $n = 2 \pmod 4$, with the exception of $n = 22$ and 34. The design matrix of these designs are in general

$$X = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}' & \mathcal{A}' \end{pmatrix}$$

where $\mathcal{A}$ and $\mathcal{B}$ are circulant matrices of order $n/2$ respectively.

Using the foldover theorem any of the bias-free resolution III designs stated above will move to a resolution IV design with bias. Margolin [25] when examining the Yang foldover designs showed that their effect variances are $2\sigma^2/(n-1)$ and the main effect trace efficiencies are $(n-1)/n$ for all values of n. This Chapter will examine the foldovers of the Yang 6 and 14-factor designs, which have main effect trace efficiencies of 83.3% and 92.9%, respectively.

3.2 The Yang 6-factor Foldover

The 6-factor design due to Yang [36] has its first rows of $\mathcal{A}$ and $\mathcal{B}$ defined as

$$\mathcal{A} = \; + \; + \; +$$
$$\mathcal{B} = \; - \; + \; +$$

The foldover design generated from this design was examined by Diamond [14], who showed the foldover is a strongly resolvable main-effect-plus r search design when $r = 1$ but only weakly resolvable when $r = 2$. In the next section the linearly dependent models as identified by Diamond [14] are summarised and finally a technique for resolving the dependent models is presented in section 3.2.2.

3.2.1 Linear Dependencies

The Yang 6-factor foldover design can be partitioned as follows:

$$\left[\begin{array}{r|r} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}' & \mathcal{A}' \\ \hline -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & -\mathcal{A}' \end{array}\right]$$

The first partition defined as the columns $\mathcal{A}=\{A,B,C\}$ will be labelled S_1 and the second partition defined as the columns $\mathcal{B}=\{D,E,F\}$ will be labelled S_2.

Diamond [14] identified the ten essentially different sets of linearly dependent models which are presented in a general form as follows:

Case 1: $\Delta\alpha\beta - \Delta\alpha\gamma = -\Delta\beta\theta + \Delta\gamma\theta = -\Delta\beta\rho + \Delta\gamma\rho = -\Delta\beta\phi + \Delta\gamma\phi$

Case 2: $\Delta\theta\rho - \Delta\theta\phi = -\Delta\alpha\phi + \Delta\alpha\rho = -\Delta\beta\phi + \Delta\beta\rho = -\Delta\gamma\phi + \Delta\gamma\rho$

Case 3: $\Delta\alpha\beta + \Delta\alpha\theta = \Delta\beta\gamma + \Delta\gamma\theta = -\Delta\beta\theta - 2\Delta\rho\phi$

Case 4: $\Delta\theta\rho - \Delta\alpha\theta = \Delta\rho\phi - \Delta\alpha\phi = \Delta\alpha 1\rho - 2\Delta\beta\gamma$

Case 5: $\Delta\alpha\theta - \Delta\alpha\rho = \Delta\beta\theta - \Delta\gamma\rho = \Delta\gamma\theta - \Delta\gamma\rho$

Case 6: $\Delta\alpha\beta - \Delta\gamma\theta = -\Delta\alpha\theta + \Delta\beta\gamma = -\Delta\beta\theta + \Delta\alpha\gamma$

Case 7: $2\Delta\theta\rho + \Delta\alpha\beta = -\Delta\alpha\phi - \Delta\beta\phi$

Case 8: $2\Delta\alpha\beta + \Delta\theta\rho = \Delta\gamma\theta + \Delta\gamma\rho$

Case 9: $\Delta\theta\rho + \Delta\alpha\phi = \Delta\alpha\theta + \Delta\rho\phi = \Delta\alpha\rho + \Delta\theta\phi$

Case 10: $\Delta\alpha\theta + \Delta\beta\rho = \Delta\alpha\rho - \Delta\beta\theta$

where α, β and γ correspond to factors in S_1 and θ, ρ and ϕ correspond to factors in S_2 and Δ represents the magnitude of an effect.

For example, from case 1 the model where the interaction between factors A (α) and B (β) takes the value Δ and the interaction between factors A (α) and C (γ) takes the value of $-\Delta$ is confounded with 3 other models.

The next section considers the design of augmenting trials for each of the above cases.

3.2.2 Augmenting When Error is Present

Diamond [14] presented a method for augmenting designs in the presence of error for a series of modified one-factor-at-a-time foldover designs. This section will summarise the technique and present augmenting trials to resolve the linearly dependent models described in section 3.2.1.

Consider two models M_1 and M_2, where M_1 and M_2 involve the mean, all main effects and k_i two-factor interactions. The models can be rewritten as:

$$M_1 : y = X_0\beta_0 + X_1\beta_1 + \epsilon_1$$

$$M_2 : y = X_0\beta_0 + X_2\beta_2 + \epsilon_2$$

where X_0 is the matrix corresponding to the mean and main effects, X_1 and X_2 consist of k_i two-factor interaction columns and $\epsilon_1 \sim N(0, \sigma^2 I)$ assuming that M_1 is the true model whereas $\epsilon_2 \sim N(X_1\beta_1 - X_2\beta_2, \sigma^2 I)$.

Both models can be rewritten as:

$$M_1 : y_0 = X_{1\cdot 0}\beta_1 + \epsilon_{1\cdot 0}$$

$$M_2 : y_0 = X_{2\cdot 0}\beta_2 + \epsilon_{2\cdot 0}$$

where y_0,$X_{1\cdot 0}$,$X_{2\cdot 0}$,$\epsilon_{1\cdot 0}$ and $\epsilon_{2\cdot 0}$ are the matrices of residuals of y,X_1,X_2,ϵ_1 and ϵ_2 regressed on X_0 respectively.

Diamond [14] showed that the standardised residual sum of squares for M_2 follows a noncentral χ^2 distribution with a noncentrality parameter of

$$\lambda = \beta_1' X_{1\cdot 0}'(I - X_{2\cdot 0}(X_{2\cdot 0}' X_{2\cdot 0})^{-1} X_{2\cdot 0}') X_{1\cdot 0}\beta_1$$

If M_1 and M_2 are two models that are linearly dependent then $X_{1\cdot 0}\beta_1 = X_{2\cdot 0}\beta_2$ and $\lambda = 0$.

In Chapter 2 a method was presented to calculate augmenting trials to separate linearly dependent models in the error free case. This method can be generalised to design augmenting trials in the presence of error. Let A be an $n \times p$ matrix of candidate runs, and $M_1, M_2 \cdots M_j$ be the j linearly dependent models among which we wish to discriminate. Let C be the $n \times j$ matrix with columns defined as $C[i-1] = M_1 - M_i$ where $i = 2 \cdots j$. Any $n \times (j-1)$ submatrix of C corresponds to an augmenting set of trials in A that will separate the models in the error free case iff each column of the submatrix in C satisfies the following:-

1. Each column is not equal to the null vector.

2. Each column is not proportional to the unit vector (if a block term is required).

3. No one column is proportional to any other column.

Consider fitting the model M_2 when the true model is in fact M_1. The residual sum of squares of the false model M_2 can be defined as

$$RSS_f = RSS_e + RSS_n$$

where RSS_e is the error component and RSS_n is the compnent of M_1 not accounted for when fitting M_2, which will be called the noiseless component. Diamond [14] showed that for any true and false model combination the noiseless component is proportional to the second eigenvalue λ_2 of the quadratic form

$$\beta_1' X_{1\cdot 0}'(I - X_{2\cdot 0}(X_{2\cdot 0}' X_{2\cdot 0})^{-1} X_{2\cdot 0}') X_{1\cdot 0}\beta_1$$

One way to examine the performance of each set of augmenting trials fulfilling the above criteria in the presence of error is to estimate λ_2 for each of the $j(j-1)$ combinations of true and false models selected from the M_j models in the linear dependency. If each of the M_j linearly dependent models is equally likely to be true then the best set of augmenting trials would maximise the minimum value of λ_2 across the $j(j-1)$ λ_2's.

This method was applied to each of the 10 sets of essentially different linearly dependent models in the Yang 6-factor foldover design when $k = 2$ and the results are summarised below:

Case 1

Dependent models: $\{AB = \Delta, AC = -\Delta\}, \{BD = -\Delta, CD = \Delta\}, \{BE = -\Delta, CE = \Delta\}, \{BF = -\Delta, CF = \Delta\}$ One possible maximum minimum eigenvalue producing case is (ab, bf, be).

True Model	Maximum Minimum λ_2
$\{AB = \Delta, AC = -\Delta\}$	3.396648
$\{BD = -\Delta, CD = \Delta\}$	2.502793
$\{BE = -\Delta, CE = \Delta\}$	3.322404
$\{BF = -\Delta, CF = \Delta\}$	3.322404

For example if A is the 2^6 full factorial design then from the corresponding C matrix every possible set of augmenting trials can be selected. For each possible set of augmenting trials λ_2 is calculated for the following twelve cases:

1. True Model : $AB = \Delta, AC = -\Delta$; False Model : $BD = -\Delta, CD = \Delta$
2. True Model : $AB = \Delta, AC = -\Delta$; False Model : $BE = -\Delta, CE = \Delta$
3. True Model : $AB = \Delta, AC = -\Delta$; False Model : $BF = -\Delta, CF = \Delta$
4. True Model : $CD = \Delta, BD = -\Delta$; False Model : $AC = -\Delta, AB = \Delta$
5. True Model : $CD = \Delta, BD = -\Delta$; False Model : $BE = -\Delta, CE = \Delta$
6. True Model : $CD = \Delta, BD = -\Delta$; False Model : $BF = -\Delta, CF = \Delta$
7. True Model : $CE = \Delta, BE = -\Delta$; False Model : $AC = -\Delta, AB = \Delta$
8. True Model : $CE = \Delta, BE = -\Delta$; False Model : $BD = -\Delta, CD = \Delta$
9. True Model : $CE = \Delta, BE = -\Delta$; False Model : $BF = -\Delta, CF = \Delta$
10. True Model : $CF = \Delta, BF = -\Delta$; False Model : $AC = -\Delta, AB = \Delta$
11. True Model : $CF = \Delta, BF = -\Delta$; False Model : $BD = -\Delta, CD = \Delta$
12. True Model : $CF = \Delta, BF = -\Delta$; False Model : $BE = -\Delta, CE = \Delta$

The set of augmenting trials (ab, bf, be) yielded the maximum of the minimum values of λ_2 over the above twelve true/false model combinations. This method assumes that each of the dependent models is equally likely to be the true model, and therefore maximises the minimum value of the noncentrality parameter over all true/false model combinations. If there was cause to suspect that one of the dependent models, say M_1, was most likely to be the true model, then a set of augmenting trials that would maximise the minimum value of λ_2 across the true/false model combinations where M_1 is the true model could be considered.

Case 2

Dependent models: $\{DE = \Delta, DF = -\Delta\}, \{AF = -\Delta, AE = \Delta\}, \{BF = -\Delta, BE = \Delta\}, \{CF = -\Delta, CE = \Delta\}$ One possible maximum minimum eigenvalue producing case is (f, cdf, bdf).

True Model	Maximum Minimum λ_2
$\{DE = \Delta, DF = -\Delta\}$	3.396648
$\{AF = -\Delta, AE = \Delta\}$	2.502793
$\{BF = -\Delta, BE = \Delta\}$	3.322404
$\{CF = -\Delta, CE = \Delta\}$	3.322404

Case 3

Dependent models: $\{AB = \Delta, AD = \Delta\}, \{BC = \Delta, CD = \Delta\}, \{BD = -\Delta, EF = -2\Delta\}$ One possible maximum minimum eigenvalue producing case is (bdf, bcd).

True Model	Maximum Minimum λ_2
$\{AB = \Delta, AD = \Delta\}$	2.275181
$\{BC = \Delta, CD = \Delta\}$	2.857143
$\{BD = -\Delta, EF = -2\Delta\}$	1.1167395

Case 4

Dependent models: $\{DE = \Delta, AD = -\Delta\}, \{EF = \Delta, AF = -\Delta\}, \{AE = \Delta, BC = -2\Delta\}$ One possible maximum minimum eigenvalue producing case is $(bde, bcdef)$.

True Model	Maximum Minimum λ_2
$\{DE = \Delta, AD = -\Delta\}$	2.275181
$\{EF = \Delta, AF = -\Delta\}$	2.857143
$\{AE = \Delta, BC = -2\Delta\}$	1.1167395

Case 5

Dependent models: $\{AD = \Delta, AE = -\Delta\}, \{BD = \Delta, BE = -\Delta\}, \{CD = \Delta, CE = -\Delta\}$ One possible maximum minimum eigenvalue producing case is (ce, be).

True Model	Maximum Minimum λ_2
$\{AD = \Delta, AE = -\Delta\}$	2.000000
$\{BD = \Delta, BE = -\Delta\}$	2.666667
$\{CD = \Delta, CE = -\Delta\}$	2.666667

Case 6

Dependent models: $\{AB = \Delta, CD = -\Delta\}, \{AD = -\Delta, BC = \Delta\}, \{BD = -\Delta, AC = \Delta\}$ One possible maximum minimum eigenvalue producing case is (ce, be).

True Model	Maximum Minimum λ_2
$\{AB = \Delta, CD = -\Delta\}$	2.857143
$\{AD = -\Delta, BC = \Delta\}$	2.857143
$\{BD = -\Delta, AC = \Delta\}$	1.529744

Case 7

Dependent models: $\{DE = 2\Delta, AB = \Delta\}, \{AF = -\Delta, BF = -\Delta\}$ One possible maximum minimum eigenvalue producing case is $((1), be)$.

True Model	Maximum Minimum λ_2
$\{DE = 2\Delta, AB = \Delta\}$	4.004456
$\{AF = -\Delta, BF = -\Delta\}$	6.023956

Case 8

Dependent models: $\{AB = 2\Delta, DE = \Delta\}, \{CD = \Delta, CE = \Delta\}$ One possible maximum minimum eigenvalue producing case is (de, be).

True Model	Maximum Minimum λ_2
$\{AB = 2\Delta, DE = \Delta\}$	4.004456
$\{CD = \Delta, CE = \Delta\}$	6.023956

Case 9

Dependent models: $\{DE = \Delta, AF = \Delta\}, \{AD = \Delta, EF = \Delta\}, \{AE = \Delta, DF = \Delta\}$ One possible maximum minimum eigenvalue producing case is (cdf, acd).

True Model	Maximum Minimum λ_2
$\{DE = \Delta, AF = \Delta\}$	1.529744
$\{AD = \Delta, EF = \Delta\}$	2.857143
$\{AE = \Delta, DF = \Delta\}$	2.857143

Case 10

Dependent models: $\{AD = \Delta, BE = \Delta\}, \{AE = \Delta, BD = \Delta\}$ One possible maximum minimum eigenvalue producing case is (bd, be).

True Model	Maximum Minimum λ_2
$\{AD = \Delta, BE = \Delta\}$	8.000000
$\{AE = \Delta, BD = \Delta\}$	8.000000

3.3 The Yang 14-factor Foldover

3.3.1 Linear Dependencies

As for the Yang 6-factor foldover design the Yang 14-factor foldover design can be partioned as follows:

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}' & \mathcal{A}' \\ \hline -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & -\mathcal{A}' \end{array}\right]$$

Where $\mathcal{A}$ and $\mathcal{B}$ are circulant matrices defined as:

$$\begin{array}{ccccccc} - & + & + & + & + & + & + \\ - & - & + & - & + & + & + \end{array}$$

The first partition defined as the columns $\mathcal{A}= \{A, B, C, D, E, F, G\}$ will be labelled S_1 and the second partition defined as the columns $\mathcal{B}= \{H, J, K, L, M, N, O\}$ will be labelled S_2. The set of interactions involving only factors from S_1 will be labelled $S_{1\cdot1}$, interactions involving only factors from S_2 will be labelled $S_{2\cdot2}$ and the remaining interactions involving one factor in S_1 and one from S_2 will be labelled $S_{1\cdot2}$.

When r=1

For the design to be a strongly resolvable main-effect-plus 1 search design, no two interaction columns may be identical. Direct checking reveals each of the 91 interaction columns in the design is distinct, and therefore the Yang 14 foldover enables the search and estimation of at least one non zero interaction in the error free case.

When r=2

For the design to be strongly resolvable when main-effect-plus 2 search design, every one of the $\binom{91}{4} = 2672670$ possible choices of four two-factor interaction columns must be of full rank.

It is desired to divide the 2672670 possible choices of four two-factor interaction columns into more manageable pieces for the purpose of direct checking. Each of the 2672670 possible choices of four two-factor interaction columns can be allocated into one of the 11 different sets, as defined in Table 2.5. Then each of the 11 different sets can be recursively searched for rank deficient possibilities.

Direct checking of each of the 11 possible sets establishes that all but the set corresponding to Case 2, Table 2.5, to be linearly independent. Of the $\binom{14}{4} \times 3 = 3003$ possible choices of 4 two-factor interaction columns corresponding to Case 2, 231 are rank deficient. Each of the 231 linear dependencies take one of the two forms summarised below:

1. $\Delta X_{\alpha \cdot \beta} + \Delta X_{\alpha \cdot \gamma} = -\Delta X_{\beta \cdot \delta} - \Delta X_{\gamma \cdot \delta}$

2. $\Delta X_{\alpha \cdot \beta} - \Delta X_{\alpha \cdot \gamma} = \Delta X_{\beta \cdot \delta} - \Delta X_{\gamma \cdot \delta}$

where $X_{\alpha \cdot \beta}$ is a two-factor interaction in $S_{1 \cdot 1}$, $X_{\gamma \cdot \delta}$ is a two factor interaction in $S_{2 \cdot 2}$ and both of $X_{\alpha \cdot \gamma}$ and $X_{\beta \cdot \delta}$ are two-factor interactions in $S_{1 \cdot 2}$

As S_1 and S_2 are circulant matrices of order 7, any choice of n columns in the design has a further 6 equivalent designs formed using circulant permutation in S_1 and S_2. The same rationale can be used for each linear dependency, meaning from any one linear dependency a further 6 dependencies can be derived using circulant permutations in S_1 and S_2.

Using the above result the full set of 231 linear dependendencies can be derived from $231/7 = 33$ dependencies, circulantly permuting to obtain the remainder. Another point to note is that all dependencies involve at least 2 two-factor interactions from $S_{1\cdot 2}$.

As the linear dependencies present in the design are all in 4-space it would be worthwhile to check projections of the Yang 14-factor foldover onto 4-space. In order to examine every possible choice of 4 columns from the design one needs to consider $\binom{14}{4} = 1001$ different designs. However using the circulant nature of the design, any choice of 4 columns has 6 other equivalent designs found by circulant permutation. We therefore need to check $1001/7 = 143$ different designs. Direct checking of the 143 different designs yields the following:

1. If the four columns belong to S_1 a 2^4, a 2^{4-1} and $2 \times 2^{4-3}$ are obtained.

2. If the four columns belong to S_2 a 2^4, a 2^{4-1} and $2 \times 2^{4-3}$ are obtained.

3. The remainder formed from columns in S_1 and S_2 yield a projection which is not of the regular fractional factorial type, or any known non-regular design.

As cases 1 and 2 both contain full factorial designs in 4-space any linear dependencies in the design must come from the third case. This result is useful in cases when effect sparsity is expected and illustrates why every linear dependency must have at least 2 two-factor interactions from $S_{1\cdot 2}$.

3.4 Augmenting Runs

The method outlined in section 2.4 can be used to determine appropriate augmenting trials in the absence of error case for the two general forms of dependencies identified for the Yang 14-factor foldover in section 3.3.1. As both forms of dependencies within the design involve 4 factors the following A matrix is generated.

α	β	γ	δ
$-$	$-$	$-$	$-$
$+$	$-$	$-$	$-$
$-$	$+$	$-$	$-$
$+$	$+$	$-$	$-$
$-$	$-$	$+$	$-$
$+$	$-$	$+$	$-$
$-$	$+$	$+$	$-$
$+$	$+$	$+$	$-$
$-$	$-$	$-$	$+$
$+$	$-$	$-$	$+$
$-$	$+$	$-$	$+$
$+$	$+$	$-$	$+$
$-$	$-$	$+$	$+$
$+$	$-$	$+$	$+$
$-$	$+$	$+$	$+$
$+$	$+$	$+$	$+$

And the C_1 matrix that corresponds to the first type of dependency is defined as $\Delta X_{\alpha \cdot \beta} + \Delta X_{\alpha \cdot \gamma} - (-\Delta X_{\beta \cdot \delta} - \Delta X_{\gamma \cdot \delta})$ and is as follows:

$$\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4 \\ 0 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

A submatrix of C_1 that satisfies the three criteria required to separate the dependency and to estimate the block effect is:

$$\begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

which corresponds to a number of different augmenting trials two of which being $\{(1)\ ,\ \beta\gamma\}$ or $\{(\alpha\delta\ ,\ \alpha\beta\gamma\delta)\}$.

The C_2 matrix that corresponds to the second type of dependency is defined as $\Delta X_{\alpha\cdot\beta} - \Delta X_{\alpha\cdot\gamma} - (\Delta X_{\beta\cdot\delta} - \Delta X_{\gamma\cdot\delta})$ and is as follows:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A submatrix of C_2 that satisfies the three criteria required to separate the dependency and to estimate the block effect is:

$$\begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

which corresponds to a number of different augmenting trials two of which being $\{\alpha\beta\ ,\ \alpha\gamma\}$ or $\{(\beta\delta\ ,\ \gamma\delta)\}$.

3.5 Conclusion

In this chapter a method for deriving minimum run augmenting designs has been presented for the Yang 6-factor foldover design for the case when the number of non-zero interactions is at most two in the presence of error. The augmenting trials will be useful when the interaction effects take one of the 10 general forms presented in section 3.2.1.

This chapter also showed that the Yang 14-factor foldover design is a strongly resolvable main-effect-plus 1 search design, and hence no augmenting runs will be required when the number of non-zero interactions is one. The Yang 14-factor foldover is not a strongly resolvable main-effect-plus 2 search design, augmenting runs may be required when the number of non-zero interactions is two and take one of the general forms presented in section 3.3.1. Augmenting trials were presented that are sufficient to separate each dependency and to estimate a block effect. The performance of these augmenting trials were not examined when error is present.

Chapter 4

PLACKETT AND BURMAN 12 FACTOR FOLDOVER DESIGN

4.1 Introduction

The class of two-level, non-regular, orthogonal, resolution III designs due to Plackett and Burman [30] are the most commonly discussed non-regular designs in the literature. These designs are useful as they provide orthogonal and 100% efficient estimates of the main effects. The alias structure is extremely complex and each main effect is aliased with a long linear combination of two-factor interactions. For this reason the designs have been traditionally used as screening designs only. Whilst this complex aliasing has been traditionally thought of as a disadvantage, recently a number of authors have shown, when coupled with the assumption of effect sparsity, that the estimation of a number of two-factor interactions is sometimes possible without the addition of augmenting trials, see Box and Meyer [5], Hamada and Wu [21], and Wang and Wu [34].

Plackett and Burman [30] presented a minimum run, $2^{11}//12$ resoution III, orthogonal array of strength 2. By applying the foldover technique this design can be moved to a $2^{12}//24$ resolution IV, orthogonal array of strength 3. This chapter will examine the projection properties of the foldover design and whether it is possible to search for and estimate up to three two-factor interactions.

4.2 *Projection properties of the Plackett and Burman 12-factor foldover design*

In an orthogonal array of strength t all possible 2^t rows occur an equal number of times in every set of t columns. Therefore the Plackett and Burman 12-factor foldover design must have only one projection onto 3-space being a 2^3 replicated 3 times.

Cheng [9] showed that the projection of an OA$(N, k, 2, t)$ with $k \geq t + 1$ onto $(t + 1)$-space results in 3 possible different designs summarised as follows:

1. $2^{-t}N$ copies of the half replicate of 2^{t+1}.

2. $2^{-(t+1)}N$ copies of the complete 2^{t+1} factorial.

3. Projection contains copies of both the complete 2^{t+1} factorial and the half replicate.

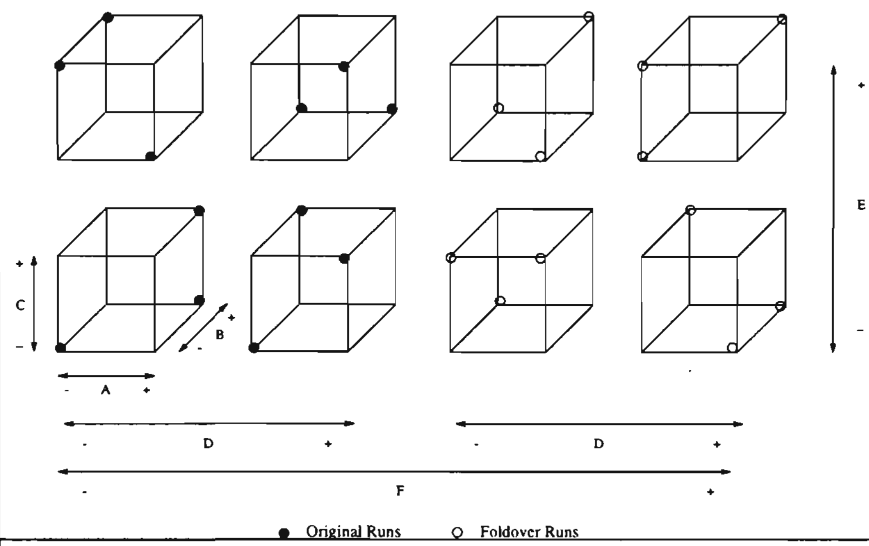
He further showed that if N is not a multiple of 2^{t+1}, then the projection must be of type 3, and therefore contains at least one copy of the complete 2^{t+1} factorial.

This result can be used to determine the projection properties of the Plackett and Burman 12-factor foldover design onto 4-space, which must therefore be of type 3, as $N = 24$ is not a multiple of 2^4. The only valid projection of type 3 in 24 runs onto 4-space is a full 2^4 factorial plus a 2^{4-1} fractional design. There is therefore only one projection onto 4-space.

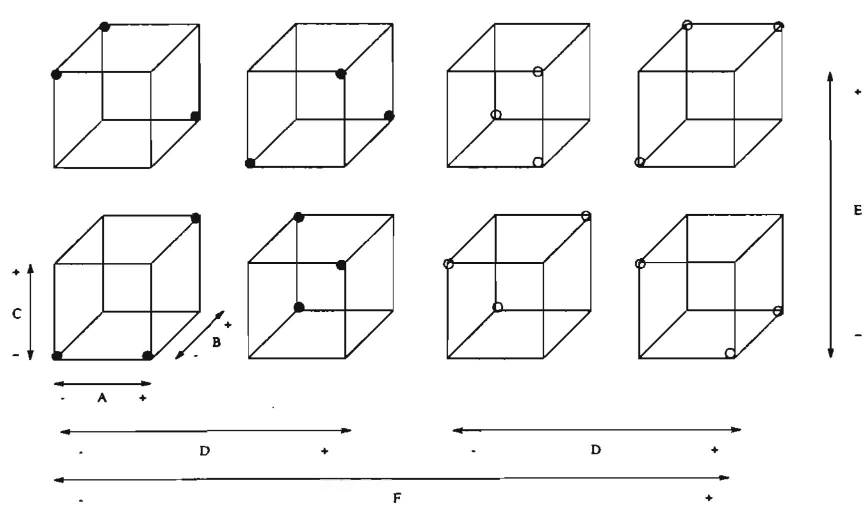
Diamond [15] showed any choice of 5 columns from the Plackett and Burman 12-factor foldover design yields only one projection which is of resolution V.

Table 4.1: The 4 possible different 6-space projections from the PB12 Foldover Design

1



2



Continued on next page

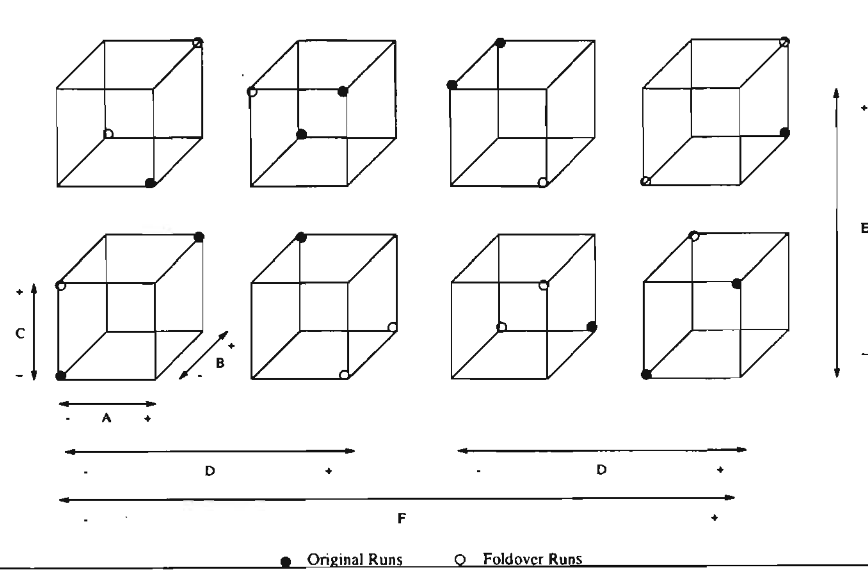
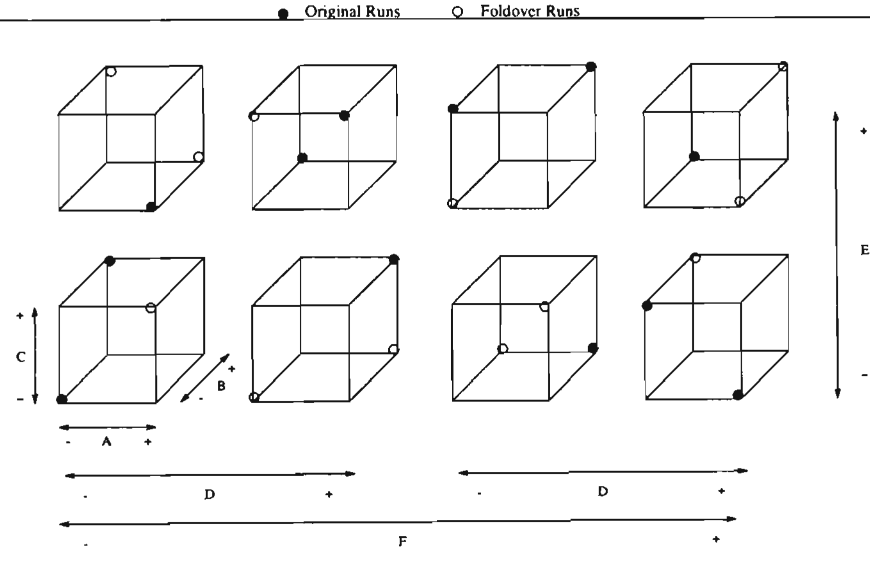
<table>
<tr><td colspan="2">Continued from previous page</td></tr>
<tr><th>Case</th><th>Cube Diagram</th></tr>
<tr><td>3</td><td></td></tr>
<tr><td>4</td><td></td></tr>
</table>

Table 4.1: The 4 possible different 6-space projections from the PB12 Foldover Design

Draper and Lin [17] investigated the projection properties of the Plackett and Burman 11 factor design and listed the different possible projections of the design in 5 and 6 space. They showed the design had 2 essentially different designs in 5-space and 2 essentially different designs in 6-space, formed as the complement of the 2 different 5-space designs. This result can be used to investigate the projection of the foldover onto 6-space by considering the following fact:

For any design design X the projection of the foldover design $\tilde{X} = \begin{pmatrix} X & -1 \\ -X & 1 \end{pmatrix}$ onto n space can be determined by considering:

- each projection of X onto $n-1$ space in $\tilde{X}$ adding the foldover column as the nth factor.

- each projection of X onto n space in $\tilde{X}$

There are therefore 4 possible different designs that must be considered, which are represented as cube diagrams in Table 4.1. Each of the 4 designs contain no repeat runs and 12 mirror image points. Each of the four designs can be shown to be equivalent as follows:

- Case 1 can be obtained from Case 2 by changing the signs of A and C and setting $D = -F$, $E = D$ and $F = E$.

- Case 1 can be obtained from Case 3 by changing the sign of F and setting $D = -E$ and $E = -D$.

- Case 1 can be obtained from Case 4 by changing the sign of D and setting $E = F$ and $F = E$.

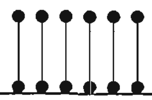
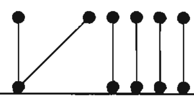
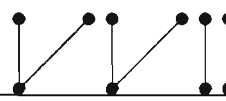
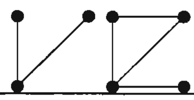
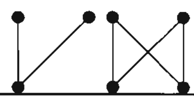
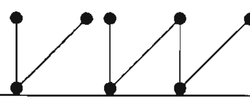
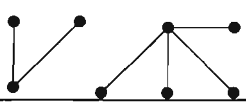
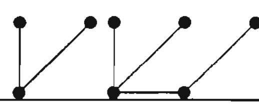
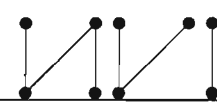
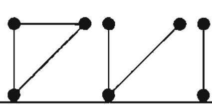
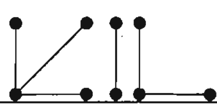
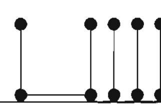
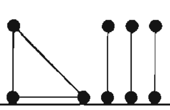
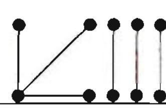
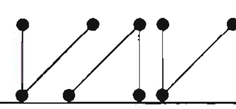
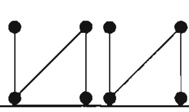
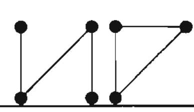
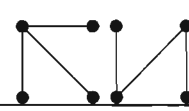
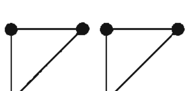
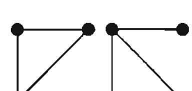
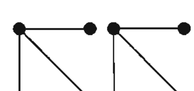
There is therefore only one projection of the Plackett and Burman 12-factor foldover design onto 6-space.

Projections of the foldover design in 7,8,9,10, and 11-space are formed by taking the complement of the designs in 6,5,4,3, and 2-space respectively, and therefore there is only one projection of the Plackett and Burman 12-factor foldover onto n-space.

4.3 *Searching for interactions in the Plackett and Burman 12-factor foldover design*

Diamond [15] showed that the Plackett and Burman 12-factor foldover is strongly resolvable main-effect-plus 2 search design. To establish whether a design is a strongly resolvable main-effect-plus 3 search design every 6 columns selected from the two-factor interaction matrix must be of full rank. To check this directly one would need to consider the ranks of $\binom{66}{6} = 90858768$ matrices. Obviously a simpler method is desirable.

Table 4.2: All graphs with n vertices and 6 edges

Case	Graph	Case	Graph	Case	Graph
1		2		3	
4		5		6	
7		8		9	
10		11		12	
13		14		15	
16		17		18	
19		20		21	

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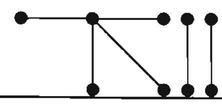
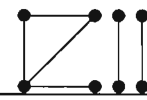
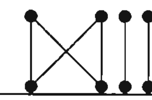
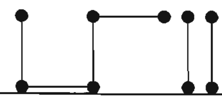
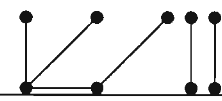
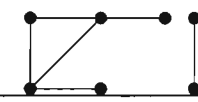
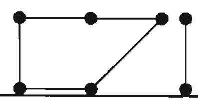
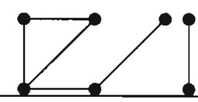
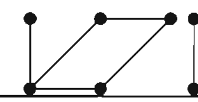
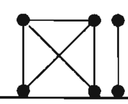
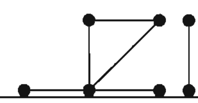
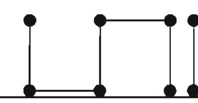
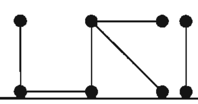
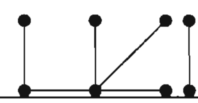
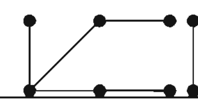
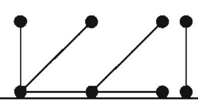
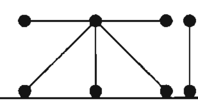
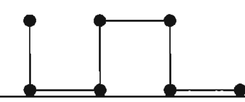
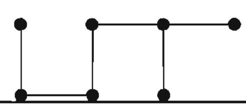
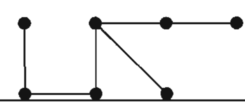
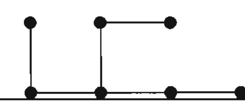
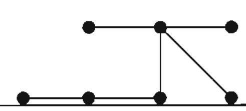
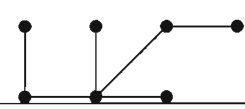
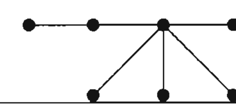
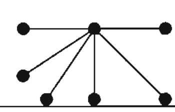
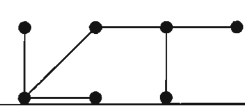
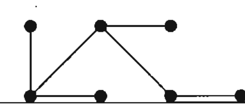
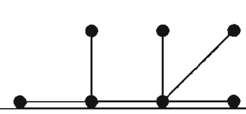
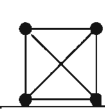
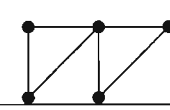
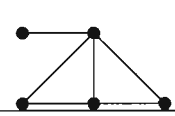
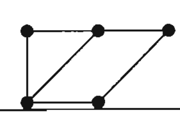
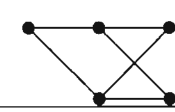
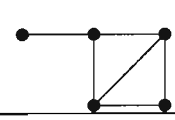
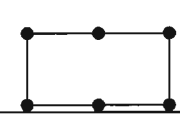
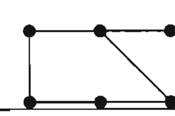
22		23		24	
25		26		27	
28		29		30	
31		32		33	
34		35		36	
37		38		39	
40		41		42	
43		44		45	
46		47		48	
49		50		51	
52		53		54	
55		56		57	

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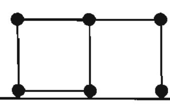
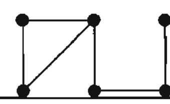
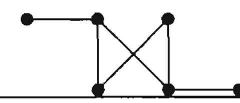
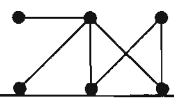
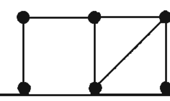
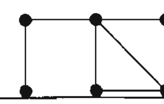
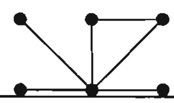
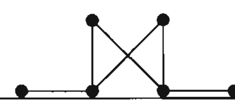
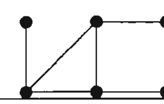
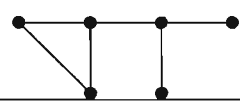
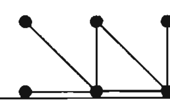
58		59		60	
61		62		63	
64		65		66	
67		68			

Table 4.2: All graphs with n vertices and 6 edges

Another method would be to consider all the possible graphs with n vertices and 6 edges. There are 68 possible linear graphs of this type which are displayed in Table 4.2. For example one possible choice of 6 interactions from case 56 would be $A \times B$, $A \times C$, $B \times D$, $C \times E$, $D \times F$ and $E \times F$, the rank of these six interaction columns would then checked and if rank deficient would denote a linear dependency. To check every possible choice for case 56 in Table 4.2, the ranks of

$$\binom{12}{6}\binom{6}{3} \times 6 \times 3 = 332640$$

two-factor interaction matrices must be checked, this can be reduced further to

$$\binom{6}{3} \times 6 \times 3 = 360$$

cases as there is only one projection in 6-space.

Therefore in order to determine if the design is a strongly resolvable main-effect-plus 3 search design, it is sufficient to examine the ranks of every two-factor interaction matrix of the form defined in each of the 68 different linear graphs in Table 4.2 selected from n factors where n is the number of vertices in each graph. Examination of each of the 68 possible cases shows that 65 of the linear graphs produce linearly independent cases of full rank, and 3 graphs produce linearly dependent rank deficient cases. These rank deficient cases are summarised below:

1. Table 4.2, Case 1

   Direct checking of the

$$\frac{\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6!} = 10395$$

   possible cases yields 2475 linear dependent cases. Each of the 2475 linear dependencies are of the form:

$$\Delta X_{ij} + \Delta X_{kl} + \Delta X_{mn} - \Delta X_{op} - \Delta X_{qr} - \Delta X_{st} = 0$$

   which is summarised in Table 4.3, Case 1. For example if the "true model" consists of the interactions AB, CD, and EF all taking the value Δ then the model consisting of the interactions GK, HM and JL all taking the value Δ fits the data equally as well.

2. Table 4.2, Case 24

   Direct checking of the

$$\frac{\binom{8}{4} \times 3 \times \binom{4}{2}\binom{2}{2}}{2!} = 630$$

   cases reveals 6 linear dependent cases. Therefore $\binom{12}{8} \times 6 = 2970$ of the possible $\binom{12}{8} \times 630 = 311850$ cases are linearly dependent. The general form of this dependency is presented in Table 4.3, cases 2 and 3.

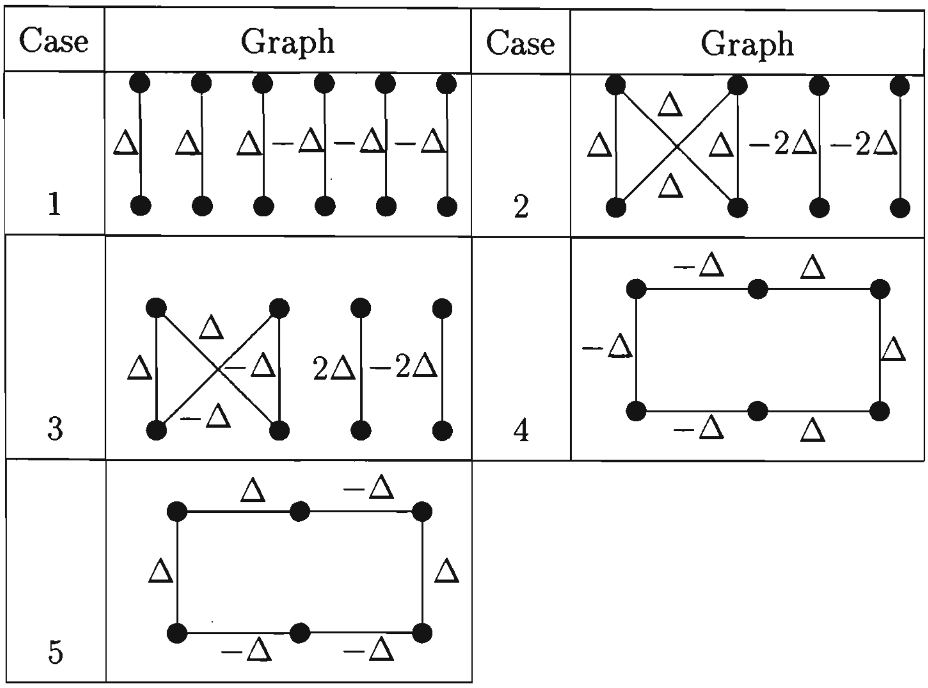


Table 4.3: The 5 linear dependent forms present in the Plackett and Burman 12-factor foldover design when $k = 3$

3. Table 4.2, Case 56

   Direct checking of the $\binom{6}{3} \times 6 \times 3 = 360$ cases reveals 10 linear dependent cases. Therefore $\binom{12}{6} \times 10 = 9240$ of the possible $\binom{12}{6} \times 360 = 332640$ cases are linearly dependent. The general form of this dependency is presented in Table 4.3, cases 4 and 5.

4.4 Conclusion

In this chapter the Plackett and Burman 12-factor foldover design was shown to have only one type of projection onto n-space, where $n = 1, \ldots, 12$. Using this result a method was developed to investigate whether the design was a strongly resolvable main-effect-plus 3 search design. It consisted of considering the 68 different cases formed from all possible linear graphs with n vertices and 6 edges, and checking all the possible arrangements in each case for rank deficiencies. The design was shown to be only weakly resolvable when $r = 3$ and hence augmenting trials may be required if the number of non-zero interactions is 3 and the interactions take one of the general forms presented in Table 4.3.

Chapter 5

THE PLACKETT AND BURMAN 20 AND 24 FACTOR FOLDOVER DESIGNS

5.1 Introduction

This chapter will examine the resolution IV, foldover designs generated from the Plackett and Burman 20 and 24 run designs [30]. A simple method will be derived to determine the projection properties of each design, which are then used to determine each design's resolvability as a search design.

5.2 The Plackett and Burman 20-factor foldover Design

Plackett and Burman [30] presented a $2^{19}//20$, resolution III design, which is defined as the union of one run with all factors at their low level and the circulant design generated from the following row:

$$++--++++-+-+----++-$$

The foldover thereom can be used to generate a $2^{20}//40$ resolution IV design from the $2^{19}//20$ resolution III design. The $2^{20}//40$ foldover design is an example of an orthogonal array (OA$(40, 20, 2, 3)$). In this design main effects become orthogonal to the two-factor interactions and unlike the regular fractional designs the two-factor interactions are not confounded in orthogonal strings. As a consequence the estimation of some two-factor interactions may become possible under certain conditions, without the need for augmenting trials. This section will examine some properties of this foldover design, including its projections and resolvability as a search design.

5.2.1 Searching for Interactions in the Plackett and Burman 20-factor foldover Design

To determine if the Plackett and Burman 20-factor foldover design is a strongly resolvable main-effect-plus 2 search design, every possible 4×4 matrix generated from $X'X$, where X is the interaction matrix, must be of full rank.

Cheng [9] showed that the projection of an OA$(N, k, 2, t)$ with $k \geq t+1$ onto $t+1$ space results in 3 possible different types of design summarized as follows:

1. $2^{-t}N$ copies of the half replicate of 2^{t+1}.

2. $2^{-(t+1)}N$ copies of the complete 2^{t+1} factorial.

3. Projection contains copies of both the complete 2^{t+1} factorial and the half replicate.

He further showed that if N is not a multiple of 2^{t+1}, then the projection must be of type 3, and therefore contains at least one copy of the complete 2^{t+1} factorial.

Since the Plackett and Burman 20-factor foldover design is a OA$(40, 20, 2, 3)$, the above result can be used to determine the projections onto 4 space, which therefore must be of type 3. Now there are only two valid projections of type 3 in 40 runs, summarized as follows:

1. 2^4 plus a 2^{4-1} replicated three times (Figure 5.1).

2. 2^4 replicated twice plus a single 2^{4-1} (Figure 5.2).

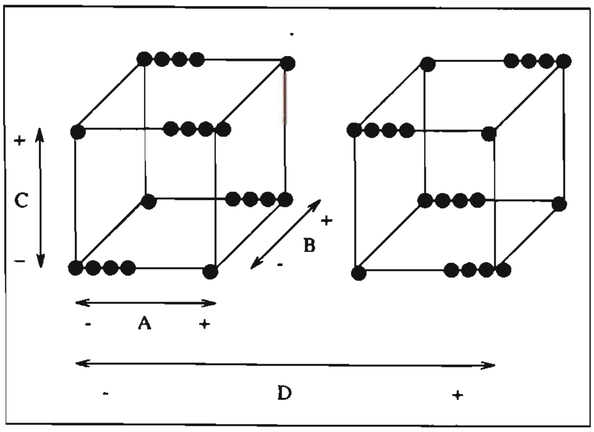


Figure 5.1: A 2^4 plus $3 \times 2^{4-1}$ designs

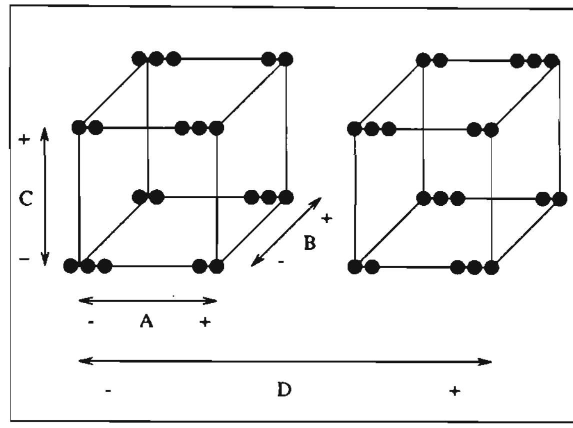


Figure 5.2: A 2^{4-1} plus 2×2^4 designs

Figure 5.3: The two different possible 4-space projections for an OA(40, 20, 2, 3)

Note that in the 2^{4-1} design interactions with one letter in common are in different orthogonal strings whilst interactions with no letters in common are in the same string since the defining relation is $I = \pm ABCD$.

Using this result if x_i and x_j are two columns selected from X, where X is the two-factor interaction matrix generated from the Plackett and Burman 20-factor foldover, then the vector product $x_i'x_j$ can take one of six values summarized as follows :-

$$x_i'x_j = \begin{cases} 40 & \text{if } i = j \\ 0 & \text{if one letter in common} \\ \left.\begin{array}{c} \pm 8 \\ \pm 24 \end{array}\right\} & \text{if no letters in common} \end{cases}$$

Using the above result each 4×4 matrix generated from $X'X$ must be of the following form:

$$X'X = \begin{pmatrix} 40 & x_{12} & x_{13} & x_{14} \\ x_{12} & 40 & x_{23} & x_{24} \\ x_{13} & x_{23} & 40 & x_{34} \\ x_{14} & x_{24} & x_{34} & 40 \end{pmatrix}$$

where $x_{ij} = 0, \pm 8, \pm 24; i \neq j$.

Since $x_{ij}; i \neq j$ can take only 5 different values, there are $5^6 = 15625$ possible matrices to consider. Direct examination of the 15625 matrices yields 5 essentially different rank deficient matrices whose upper triangles are as follows:

$$\begin{pmatrix} 0 & -8 & -8 \\ & 24 & 24 \\ & & -8 \end{pmatrix} \begin{pmatrix} 0 & -8 & 24 \\ & 24 & -8 \\ & & 24 \end{pmatrix} \begin{pmatrix} 8 & -24 & 8 \\ & 8 & -24 \\ & & 8 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 24 & 24 \\ & 24 & 24 \\ & & 8 \end{pmatrix} \begin{pmatrix} 8 & -24 & -8 \\ & -24 & 8 \\ & & -24 \end{pmatrix}$$

Diamond [13] showed that every 4×4 sub-matrix of X can be represented by one of eleven graphs involving n vertices and 4 edges, these graphs are displayed in Table 2.5. Each vertex represents a factor whilst each edge represents a two-factor interaction. Note that for any orthogonal design if two edges are co-incident at one of the vertices, the corresponding vector product $x_i' x_j$ must be 0 since there is one letter in common.

The first two dependent matrices above involves two-factor interactions between 7 factors, and can therefore be illustrated as Graph 10 in Table 2.5, whilst the remaining dependent matrices involve two-factor interactions between 8 factors and correspond to Graph 11 in Table 2.5. Therefore to determine if the Plackett and Burman 20-factor foldover design is a strongly resolvable main-effect-plus 2 search design when, only two possible linearly dependent cases need to be examined.

When selecting the 7 columns in the Plackett and Burman 20-factor foldover design corresponding to Graph 10 there exist

$$\binom{20}{7} \times \frac{\binom{7}{2} \times \binom{5}{2} \times 3}{2!} = 24,418,800$$

possible choices to consider. When selecting the 8 columns which correspond to Graph 11 there are

$$\binom{20}{8} \times \frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 13,226,850$$

possible choices to consider. Since direct checking of this many matrices is not a realistic option, a method is required that will reduce the number of choices in n space to a more manageable level.

Draper and Lin [17] when considering the projection properties of the Plackett and Burman arrays listed the n-space projections for the Plackett and Burman 20 run design. They showed that the design had 17 different projections in 6 space, 9 different projections in 7 space and 5 different projections in 8 space.This result can be used to generate the projections of the foldover design in 7 and 8 space respectively by considering the following fact:

For any design design X the projection of the foldover design $\tilde{X} = \begin{pmatrix} X & -1 \\ -X & 1 \end{pmatrix}$ onto n space can be determined by considering:

- each projection of X onto $n-1$ space in $\tilde{X}$ adding the foldover column as the nth factor.
- each projection of X onto n space in $\tilde{X}$

To consider the foldover in 7 space, therefore, one must consider the 9 different projections of the original design in 7 space, plus the 17 different projections of the design in 6 space with the foldover column added to make up the 7th factor. The possible different projections of the foldover in 8 space is derived in the same manner by considering the 5 different projections of the original design in 8 space plus the 9 different projections of the design in 7 space with the foldover column added as the 8th factor.

Examination of the 26 possible arrangements in 7 columns reveals that only 4 essentially different designs exist. Likewise the 14 possible different designs in 8 space reduces to 3 essentially different designs.

To consider the rank of every possible 4×4 submatrix corresponding to Graph 10 it is sufficient to consider the

$$\frac{\binom{7}{2} \times \binom{5}{2} \times 3}{2!} = 315$$

arrangements of the 4 different possible projections of the foldover in 7 space as defined above. Likewise the case corresponding to Graph 11 can be considered by checking the ranks of every 4×4 sub-matrix formed from

$$\frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 105$$

arrangements of the 3 different possible projections of the foldover in 8 space.

Direct checking of all these possibilities yielded no rank deficient matrices therefore the Plackett and Burman 20-factor factor foldover is a strongly resolvable main-effect-plus 2 search design.

5.2.2 Resolution of the Plackett and Burman 20-factor foldover

In the previous section it was shown the Plackett and Burman 20-factor foldover design contained a full 2^4 factorial design in every choice of 4 columns. In this

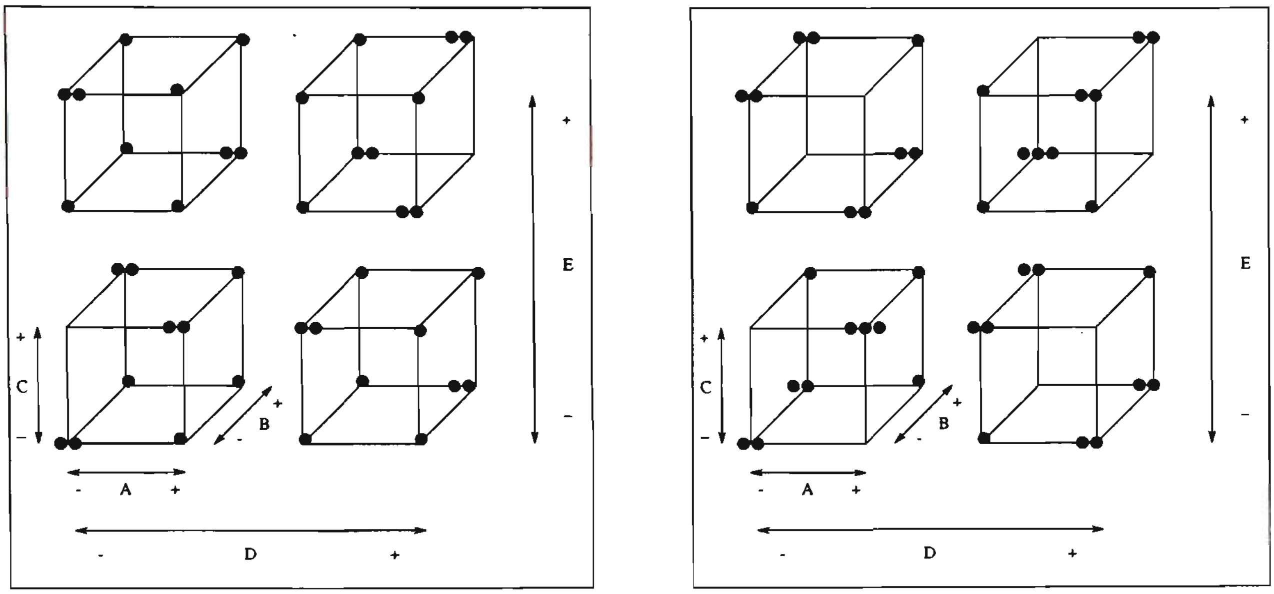


Figure 5.4: The two different projections of the PB20 foldover onto 5 space

section it will be shown the Plackett and Burman 20-factor foldover is resolution V in every 5 factors.

Draper and Lin [17] showed the Plackett and Burman 20 run design has 3 different projections onto 4 space and 9 different projections onto 5 space. Using this result the projection properties of the foldover onto 5 space can be determined using the method described in section 5.2.1.

Examination of the 12 possible different designs reveals there are only 2 essentially different projections of the foldover design onto 5 space (Figure 5.4). To show that the foldover design is resolution V in 5 factors each $X'X$, where X is the interaction matrix formed from 5 columns in the foldover design, must be of full rank. The first projection has a $X'X$ matrix given as follows:

$$X'X = \begin{pmatrix} 40 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & -8 \\ 0 & 40 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 8 \\ 0 & 0 & 40 & 0 & 8 & 0 & -8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 40 & 8 & -8 & 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 8 & 40 & 0 & 0 & 0 & 0 & 8 \\ 0 & 8 & 0 & -8 & 0 & 40 & 0 & 0 & 8 & 0 \\ 0 & 8 & -8 & 0 & 0 & 0 & 40 & 8 & 0 & 0 \\ 8 & 0 & 0 & 8 & 0 & 0 & 8 & 40 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 & 8 & 0 & 0 & 40 & 0 \\ -8 & 8 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 40 \end{pmatrix}$$

which is of full rank, whilst the second projection has a $X'X$ given as:

$$X'X = \begin{pmatrix} 40 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & -8 \\ 0 & 40 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 24 \\ 0 & 0 & 40 & 0 & 8 & 0 & -8 & 0 & 24 & 0 \\ 0 & 0 & 0 & 40 & 8 & -8 & 0 & 24 & 0 & 0 \\ 0 & 0 & 8 & 8 & 40 & 0 & 0 & 0 & 0 & -8 \\ 0 & 8 & 0 & -8 & 0 & 40 & 0 & 0 & -8 & 0 \\ 0 & 8 & -8 & 0 & 0 & 0 & 40 & -8 & 0 & 0 \\ 8 & 0 & 0 & 24 & 0 & 0 & -8 & 40 & 0 & 0 \\ 8 & 0 & 24 & 0 & 0 & -8 & 0 & 0 & 40 & 0 \\ -8 & 24 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 40 \end{pmatrix}$$

which is also of full rank.

This result shows the Plackett and Burman 20-factor foldover to be resolution V in any choice of 5 columns.

5.3 *The Plackett and Burman 24-factor foldover Design*

Plackett and Burman [30] presented a $2^{23}//24$ resolution III design, defined as the union of one run with all factors at their low level and the circulant design permuted from the first row:

$$+++++-+-++--++--+-+----$$

This design can be used to generate a $2^{24}//48$ resolution IV design using the foldover thereom. This section will examine some properties of this foldover design, including its projections and resolvability as a search design.

5.3.1 *Searching for interactions in the Plackett and Burman 24-factor foldover design*

To determine if the Plackett and Burman 24-factor foldover design is a strongly resolvable main-effect-plus 2 search a method similair to that used for the 20-factor foldover is used. Every possible 4×4 matrix generated from $X'X$, where X is the interaction matrix, must be of full rank.

The Plackett and Burman 24-factor foldover is an OA$(48, 24, 2, 3)$, Cheng [9] showed the projection must be of three types, and in 48 runs there are 4 possible designs that suit this criteria which are summarised as follows:

1. 2^{4-1} replicated six times.

2. 2^4 and a 2^{4-1} replicated four times.

3. 2^4 replicated twice, and a 2^{4-1} replicated twice.

4. 2^4 replicated three times.

Draper and Lin [17] identified the different projections of the Plackett and Burman 24 run design onto 3 and 4 space, which are 2 and 3 respectively. To

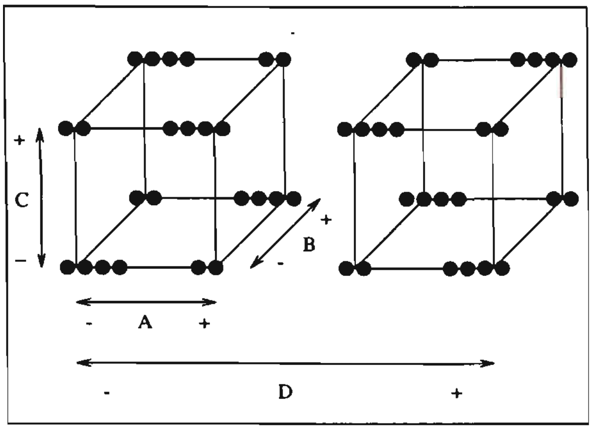


Figure 5.5: 2^4 replicated twice, and a 2^{4-1} replicated twice

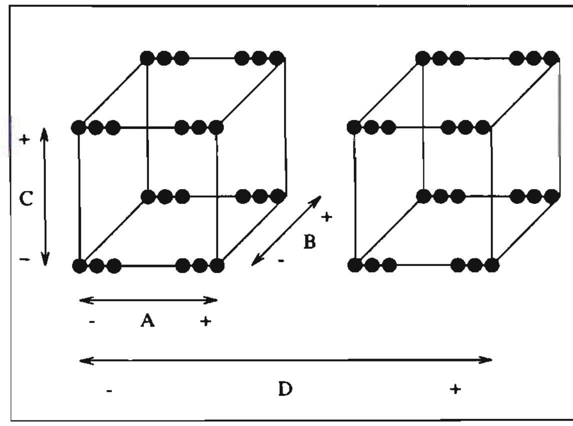


Figure 5.6: 2^4 replicated three times

determine the projection of the foldover design onto 4 space the 2 different projections of the original design onto 3 space are applied to the foldover with the addition of the 24th (foldover) column as the fourth factor, plus the 3 different projections of the original design onto 4 space applied to the foldover, must be considered. Examination of these 5 potentially different designs reveals two essentially different projections onto 4 space, a full 2^4 factorial replicated three times (Figure 5.6), or a full 2^4 factorial replicated twice and a 2^{4-1} replicated twice (Figure 5.5). Note also that these two designs form a subset of the four possible designs as outlined above.

Using the above result if x_i and x_j are two columns selected from X then the vector product $x_i'x_j$ can take one of four values summarised as follows :-

$$x_i'x_j = \begin{cases} 48 & \text{if } i = j \\ 0 & \text{if one letter in common} \\ \begin{pmatrix} 0 \\ \pm 16 \end{pmatrix} & \text{if no letters in common} \end{cases}$$

Each 4×4 matrix generated from $X'X$ must therefore be of the following form:

$$X'X = \begin{pmatrix} 48 & x_{12} & x_{13} & x_{14} \\ x_{12} & 48 & x_{23} & x_{24} \\ x_{13} & x_{23} & 48 & x_{34} \\ x_{14} & x_{24} & x_{34} & 48 \end{pmatrix}$$

where $x_{ij} = 0, \pm 16; i \neq j$.

Since $x_{ij}; i \neq j$ can take only 3 different values, there are $3^6 = 729$ possible matrices to consider. Direct examination of these 729 matrices yields 1 essentially different matrix whose upper diagonal is as follows:

$$\begin{pmatrix} 48 & -16 & -16 & -16 \\ & 48 & -16 & -16 \\ & & 48 & -16 \\ & & & 48 \end{pmatrix}$$

which involves two-factor interactions between 8 factors and corresponds to linear Graph 11 in Table 1. There are

$$\binom{24}{8} \times \frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 77,224,455$$

combinations of 4 interaction in 8 columns which need to be considered. Once again we need to reduce the possibilities to a more manageable level.

Draper and Lin [17] listed the projections of the Plackett and Burman 24 run design in 7 and 8 space which are 12 and 5 different designs respectively. Therefore using the technique described in the previous sections we need to examine 17 possible different designs. Direct examination of these 17 designs reveals there are only 3 essentially different projections onto 8 space.

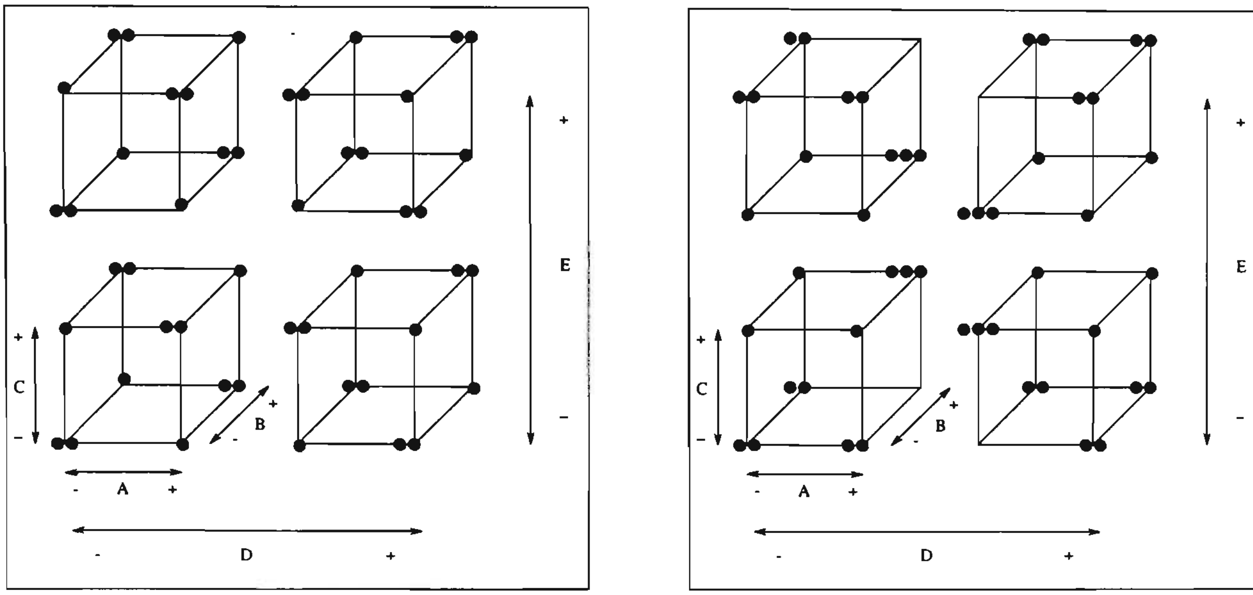


Figure 5.7: The two different projections of the PB24 foldover onto 5 space

Checking the ranks of the

$$\frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 105$$

arrangements of the 3 possible foldover projections revealed no linear dependencies and therefore the design is a strongly resolvable main-effect-plus 2 search design.

5.3.2 Resolution of the Plackett and Burman 24-factor foldover

The previous section showed that the Plackett and Burman 24-factor foldover design yields at least a full 2^4 in every choice of four columns. In this section it will be shown that the Plackett and Burman 24-factor foldover is resolution V in every choice of five columns.

Draper and Lin [17] listed the different possible projections for the Plackett Burman 24 run design in 4 and 5 space, which are 3 and 9 respectively. Using the technique described in previous sections 12 potentially different designs are formed and from these designs the projection properties of the foldover design in 5 space were determined. Examination of these 12 potentially different designs revealed the Plackett and Burman 24-factor foldover has two essentially different projections onto 5 space (Figure 5.7).

Given this result one needs only to investigate the rank of the 2 $X'X$ matrices, where X is the interaction matrix formed from the two different choices of five columns in the design. The first projection has a $X'X$ as follows:

$$X'X = \begin{pmatrix} 48 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 48 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 48 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 48 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 48 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 \end{pmatrix}$$

which is of full rank, whilst the second projection has a $X'X$ defined as:

$$X'X = \begin{pmatrix} 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 \\ 0 & 48 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 16 \\ 0 & 0 & 48 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 48 & -16 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & -16 & 48 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & 16 & 0 \\ 0 & -16 & 0 & 0 & 0 & 0 & 48 & 16 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 16 & 48 & 0 & 0 \\ -16 & 0 & 16 & 0 & 0 & 16 & 0 & 0 & 48 & 0 \\ 0 & 16 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 48 \end{pmatrix}$$

which is also of full rank.

This result shows the Plackett and Burman 24-factor foldover design to be resolution V in every set of 5 columns.

5.4 Conclusion

In this chapter the Plackett and Burman 20 and 24-factor foldover designs were shown to be strongly resolvable main-effect-plus 2 search designs. A method was developed, based on Cheng's [9] results on orthogonal arrays, to determine the 4-space projection of each design and thus the structure of the two-factor interaction $X'X$ matrix. Every possible 4×4 submatrix of the $X'X$ was then examined for rank deficiencies. Each possible rank deficient submatrix identified was then checked against the design to determine whether it was present or not. The projection properties of the design were utilised to greatly simplify this task.

The 5 space projection of each of the foldover designs were determined and shown to be of resolution V. This result is potentially useful in situations where factor sparsity is assumed as estimates of all two-factor interactions in every set of five factors can be attained.

Chapter 6

ANALYSING MINIMUM RUN RESOLUTION IV DESIGNS

6.1 Introduction

The problem of analysing non-regular or fractionated designs with complex aliasing structures has been addressed by a number of authors.

Box and Meyer [7] presented a Bayesian method of identifying active factors. Their technique is summarised in Section 1.6 and consists of calculating posterior probabilities of all possible models and then calculating marginal posterior probabilities that each factor is active. This technique works well when the data exhibit factor sparsity as the design can be collapsed down into active factors and analysed exploiting the individual factors projection properties. Hamada and Wu [21] introduced a technique that consisted of analysing the design by identifying significant main effects through the use of normal plots then entertaining interactions comprising of at least one significant main effect.

This chapter presents two methods for analysing minimum run resolution IV designs. One method is approximate and is designed specifically for minimum run resolution IV designs and uses existing code MBCQPI5 developed by Box and Meyer [7], which is freely available on Statlib. The other method directly adapts the Box and Meyer approach for a search design situation.

6.2 The Box and Meyer Procedure

For a $N \times k$ design Box and Meyer consider every possible model M_i in f_i factors where $0 \leq f_i \leq k$. There are 2^k models M_i to consider starting from $i = 0$ (no active factors) to $i = 2^k - 1$ (k active factors). The posterior probability of M_i can be written as:

$$p(M_i \mid y) = C\left(\frac{\pi}{1-\pi}\right)^{f_i} \gamma^{-t_i} \times \frac{\mid X_0'X_0 \mid^{\frac{1}{2}}}{\mid \Gamma_i + X_i'X_i \mid^{\frac{1}{2}}} \left(\frac{S(\hat{\beta}_i) + \hat{\beta}_i\Gamma_i\hat{\beta}_i}{S(\hat{\beta}_0)}\right)^{(n-1)/2}$$

where

$$\Gamma_i = \frac{1}{\gamma^2}\begin{bmatrix} 0 & 0 \\ 0 & I_i \end{bmatrix}$$

$$\hat{\beta}_i = (\Gamma_i + X_i'X_i)^{-1} X_i'y$$

$$S(\hat{\beta}_i) = (y - X_i\hat{\beta}_i)'(y - X_i\hat{\beta}_i)$$

and C is a normalising constant, X_i is the design matrix corresponding to the t_i effects in model M_i, β_i is the vector of regression effects under M_i, π is the prior probability that any one factor is active, and γ is the magnitude of an effect relative to noise.

The probabilities $p(M_i \mid y)$ can be summed to compute the marginal posterior probability that a factor j is active by:

$$P_j = \sum_{\text{M}_\text{i}\text{: factor j active}} p(M_i \mid y)$$

A large value for P_j would therefore indicate factor j is active and similarly if P_j is close to zero the factor would be assumed inert.

6.3 Two Proposed Methods

6.3.1 An Approximate Method

The first method uses the fact that the main effects and two-factor interactions are orthogonal in a foldover design. Stage 1 involves the identification of active main effects. A foldover design can be written as:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & U & V \\ 1 & -U & V \end{pmatrix} \begin{pmatrix} \mu \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

where 1 is the vector for the mean, y_1 and y_2 are the response vectors of the original and foldover trials respectively, U and $-U$ are the design matrices corresponding to the main effects and V is the design matrix for the two-factor interactions.

Estimates of the main effects can be achieved by examining the difference between foldover pairs, which gives the model

$$(y_1 - y_2) = 2U\beta_1 + (\varepsilon_1 - \varepsilon_2)$$

Note that this model does not have an intercept term. The Box-Meyer program does not allow this explicitly but an equivalent model can be obtained by analysing the original responses including a block term for each foldover pair.

Stage 2 involves the identification of the active two-factor interactions. This is achieved by examining the sum of the foldover pairs

$$(y_1 + y_2) = \begin{pmatrix} 1 & V \end{pmatrix} \begin{pmatrix} 2\mu \\ 2\beta_2 \end{pmatrix} + (\varepsilon_1 + \varepsilon_2)$$

In Stage 1 the maximum order of interactions is set to 1 since only main effects are of concern for the difference of foldover pairs. In Stage 2 the columns of V are used as the input factors, the maximum order of interactions is again set to 1, and the maximum order of the model is set equal to the maximum number of interactions thought possible.

The method is only approximate since information on the variance σ^2 generated by Stage 1 is not used in Stage 2 and vice-versa. It turns out that, at least for the examples considered here, qualitative conclusions are similar to that obtained by using a more direct method. A more serious concern is that the method is not easily adapted to the case when augmenting trials are required and this was the reason the more direct method was implemented.

6.3.2 A Direct Method

Let MM_i denote the model where f_i of the k main effects are active and MI_j denote the model where g_j of the t two-factor interactions are active with $g_j \leq r$ and $t = k(k-1)/2$. Altogether there are 2^k different main effect models with there being $\binom{k}{f_i}$ different MM_i models. There are $\binom{t}{g_j}$ different MI_j models so that there are $1 + \binom{t}{1} + \cdots + \binom{t}{r}$ different two-factor interaction models with at most r two-factor interactions. Each combined model M_{ij} is the direct sum of a single MM_i and a single MI_j and its prior probability is given by

$$\pi_e^{f_i+g_j}(1-\pi_e)^{k-f_i+tg_j}$$

where π_e is the prior probability of an effect (or interaction) being active. If desired, the prior probability of a main effect being active could be set higher than the prior probability of a two-factor interaction being active although in the examples below this has not been done. Similarly effect heredity (and also factor sparsity if three-factor interactions and higher can be ignored) can be induced by setting to zero the prior probabilities of models not satisfaying effect heredity (or factor sparsity).

The posterior probabilities of all the possible models M_{ij} are computed using the method given by Box and Meyer. The probability of model MM_i is found by

$$PMM_i = \Sigma_j p(M_{ij}|y).$$

and then the probability of main effect k being active is given by

$$PM_k = \sum_{\text{MM}_i\text{: factor k active}} PMM_i$$

Similarly, the probability of model MI_j is found by

$$PMI_j = \Sigma_i p(M_{ij}|y)$$

and then the probability of two-factor interaction l being active is given by

$$PI_l = \sum_{\text{MI}_j\text{: 2fi } l \text{ active}} PMI_j$$

6.4 Example 1

Table 6.1 gives the results of the twelve runs generated for the Yang 6 factor foldover design using the model

$$y = 2A + 1.5B - 3C + BD + CD + \epsilon$$

where $\epsilon \sim N(0, 0.5)$. This model satisfies effect heredity but not factor sparsity since there are interactions involving D but the main effect of D is absent.

6.4.1 Results from Box and Meyer Technique

The Box and Meyer [7] technique of identifying active factors was applied to this design. Results were generated with $\pi = 0.25$ and $\gamma = 2.0$, and the maximum order of interaction was set to 2.

Table 6.2 summarises the 10 models that fit the data the best. Table 6.3 gives the posterior probabilities that each factor is active. Factors A, B and C are clearly identified as being active. However, this technique fails to identify factor D and therefore the subsequent analysis would fail to identify the two two-factor interactions that should be included in the model.

6.4.2 Results from the Approximate and Direct Methods

The analysis of main effects using the approximate method was generated with $\pi_e = 0.25$ and $\gamma = 2.0$, six blocks corresponding to each foldover pair and the

A	B	C	D	E	F	response
+	+	+	−	+	+	−0.5470
+	+	+	+	−	+	0.0554
+	+	+	+	+	−	−0.1297
+	−	−	+	+	+	2.1668
−	+	−	+	+	+	0.8004
−	−	+	+	+	+	−2.5952
−	−	−	+	−	−	0.5087
−	−	−	−	+	−	0.3274
−	−	−	−	−	+	0.1328
−	+	+	−	−	−	−2.1648
+	−	+	−	−	−	−5.4021
+	+	−	−	−	−	7.1028

Table 6.1: Results for Example 1

Factors	Post. Prob.
$A\ B\ C$	.946
None	.014
C	.011
$A\ B\ C\ D$	.007
$A\ B\ C\ E$	.006
$A\ B\ C\ F$	.005
$B\ C$	.003
$A\ C$	.001
B	.001
A	.001

Table 6.2: Box and Meyer Analysis: Best 10 fitting Models for Example 1

Factor	Post.Prob
None	.014
A	.966
B	.970
C	.982
D	.009
E	.007
F	.007

Table 6.3: Box and Meyer Analysis: Posterior Probabilities for Example 1

Factor	post.prob (Approx.)	post.prob (Direct)
None	.001	.004
A	.919	.951
B	.992	.986
C	.998	.994
D	.054	.058
E	.046	.046
F	.046	.046

Table 6.4: Proposed Analysis (Main effects): Posterior Probabilities for Example 1

maximum order of interaction set to 1. For the direct method the maximum order of interaction was set equal to two, π_e was set to 0.25 and γ was set to 2.0. The posterior probabilities for the main efects were calculated by summing over all possible interaction models and then summing over those models that include the particular main effect.

Table 6.4 displays the posterior probability that each main effect is active, factors A, B and C are identified as the most likely main effects.

Table 6.5 presents the two-factor interaction design matrix for the design in Table 6.1. This design was also analysed with $\pi_e = 0.25$ and $\gamma = 2.0$, maximum order of interaction set to 1 and the maximum number of active factors set to 2.

The upper portion of Figure 6.1 shows the posterior probabilities of all possible models involving two two-factor interactions using the direct method. A similar diagram was also obtained using the approximate method. Table

AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF	Response
+	+	−	+	+	+	−	+	+	−	+	+	−	−	+	−0.0382
+	+	+	−	+	+	+	−	+	+	−	+	−	+	−	0.3828
+	+	+	+	−	+	+	+	−	+	+	−	+	−	−	0.0032
−	−	+	+	+	+	−	−	−	−	−	−	+	+	+	0.0039
−	+	−	−	−	−	+	+	+	−	−	−	+	+	+	−4.6016
+	−	−	−	−	−	−	−	−	+	+	+	+	+	+	4.5075

Table 6.5: Example 1: Two factor Interaction design Matrix (with foldover pairs added)

6.6 presents the posterior probability that each two-factor interaction is active using the approximate and direct method. The diagram, in particular, identifies four models that fit the data equally as well, $M_1 = (AB, AC)$, $M_2 = (BE, CE)$,$M_3 = (BD, CD)$, and $M_4 = (BF, CF)$. This result corresponds to the first linear dependency presented in Chapter 3, and therefore the addition of augmenting trials will be required to estimate which of the two-factor interaction models is the true model.

6.5 Augmenting for Example 1

The maximum-minimum eigenvalue case given in Chapter 3 is (ab, bf, be). The generated reults for these runs were $(7.2789, 6.4040, 5.0932)$. The lower portion of Figure 6.1 shows the posterior probabilities of all possible models involving two two-factor interactions using the direct method. The right column of Table 6.6 presents the posterior probability that each two-factor interaction is active using the approximate and direct method. Both the diagram and the table shows that of the four models that fitted the data equally as well after the

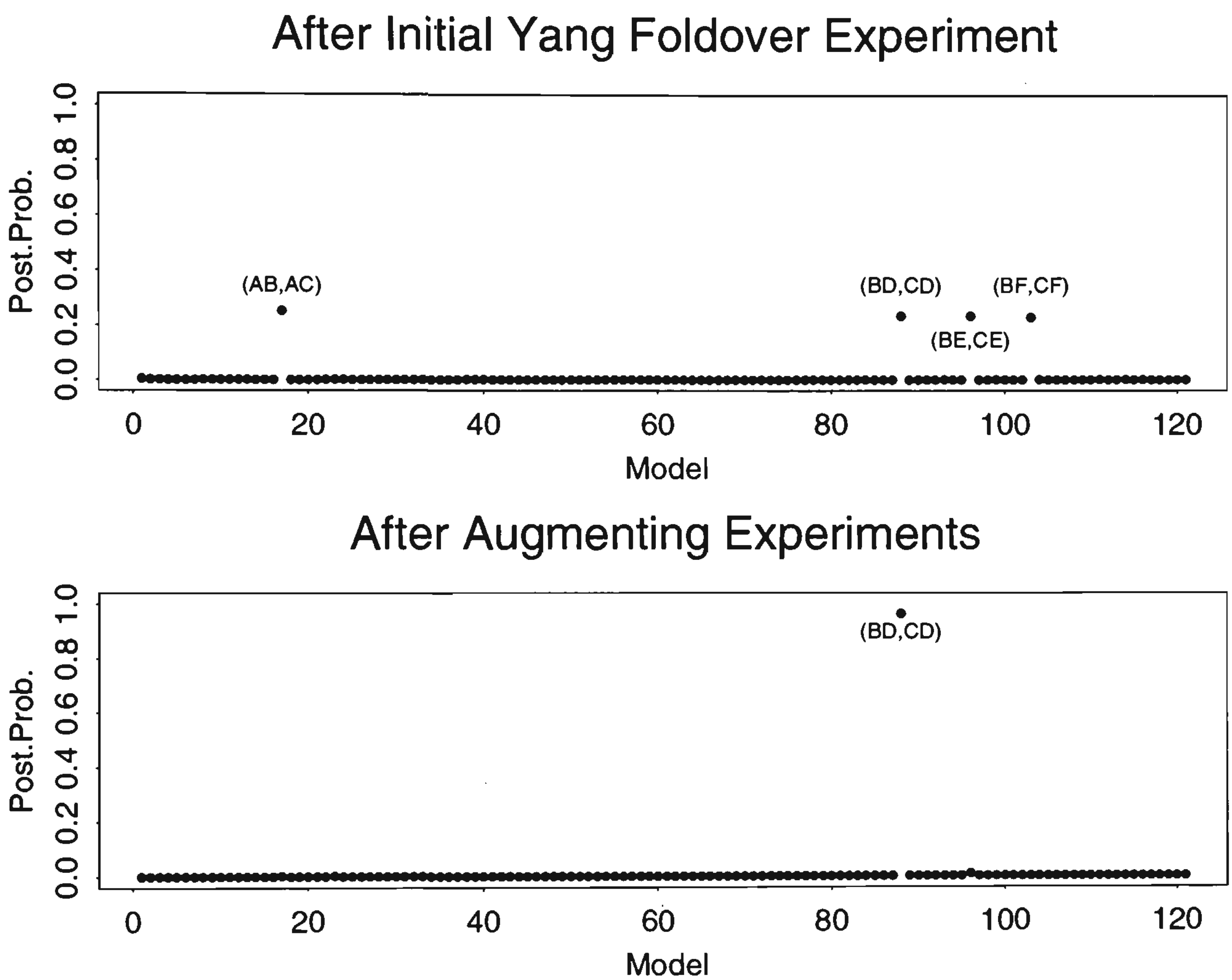


Figure 6.1: Posterior Probabilities of models involving two two-factor interaction models before the addition of augmenting trials (top) and after addition of augmenting trials (bottom)

factor	post.Prob. (Approx.)	Post.Prob. (Direct)	Post.prob. (Augmented)
none	.038	.003	.000
AB	.239	.262	.007
AC	.235	.260	.007
AD	.008	.002	.001
AE	.008	.002	.001
AF	.008	.002	.001
BC	.008	.002	.001
BD	.216	.240	.967
BE	.221	.242	.014
BF	.213	.237	.002
CD	.220	.241	.969
CE	.215	.239	.012
CF	.216	.238	.003
DE	.013	.004	.001
DF	.013	.004	.002
EF	.013	.004	.002

Table 6.6: Posterior Probabilities of Two-Factor Interactions using the Approximate method and Direct method, and after the addition of augmenting trials

A	B	C	D	E	F	response
+	+	+	−	+	+	3.4530
+	+	+	+	−	+	7.0554
+	+	+	+	+	−	2.8703
+	−	−	+	+	+	0.1688
−	+	−	+	+	+	1.8004
−	−	+	+	+	+	1.4048
−	−	−	+	−	−	−3.4913
−	−	−	−	+	−	−6.6726
−	−	−	−	−	+	−10.8672
−	+	+	−	−	−	−0.1648
+	−	+	−	−	−	−2.4021
+	+	−	−	−	−	7.1028

Table 6.7: Results for Example 2

initial design now only model $M_3 = (BD, CD)$ is supported by the data.

6.6 Example 2

Table 6.7 gives the results of the twelve runs generated for the Yang 6 factor foldover design using the model

$$y = 2A + 3B + 1.5D - 2AE - 2BD + \epsilon$$

where $\epsilon \sim N(0, 0.5)$. Again this model satisfies effect heredity but not factor sparsity since there is an interaction involving E but the main effect E is absent.

Factors	Post. Prob.
B	.452
None	.176
A	.090
$A\ B\ D\ E$	.072
$A\ B\ D\ F$	.037
$B\ D$	.036
$A\ B$	.024
C	.021
D	.015
$B\ C$	.013

Table 6.8: Box and Meyer Analysis: best 10 fitting Models for Example 2

6.6.1 Results from Box and Meyer Technique

The Box and Meyer [7] technique of identifying active factors was applied to this design. Results were generated with $\pi = 0.25$ and $\gamma = 2.0$, and the maximum order of interaction was set to 2.

Table 6.8 summarises the 10 models that fit the data the best. Table 6.9 gives the posterior probabilities that each factor is active. While factor B appears to be active the method does not identify factors A, D and E.

6.6.2 Results from the Approximate and Direct Methods

The analysis of main effects using the approximate method was generated with $\pi_e = 0.25$ and $\gamma = 2.0$, six blocks corresponding to each foldover pair and the maximum order of interaction set to 1. For the direct method the maximum order of interaction was set equal to two, π_e was set to 0.25 and γ was set to 2.0.

Factor	Post.Prob
None	.176
A	.255
B	.672
C	.057
D	.188
E	.103
F	.062

Table 6.9: Box and Meyer Analysis: Posterior Probabilities for Example 2

The posterior probabilities for the main efects were calculated by summing over all possible interaction models and then summing over those models that include the particular main effect.

Table 6.10 displays the posterior probability that each main effect is active. Factors A, B and D are identified as the most likely main effects.

Table 6.11 presents the two-factor interaction design matrix for the design in Table 6.7. This design was also analysed with $\pi_e = 0.25$ and $\gamma = 2.0$, maximum order of interaction set to 1 and the maximum number of active factors set to 2.

The upper portion of Figure 6.2 shows the posterior probabilities of all possible models involving two two-factor interactions using the direct method. A similar diagram was also obtained using the approximate method. Table 6.12 presents the posterior probability that each two-factor interaction is active using the approximate and direct method. The diagram, in particular, identifies two models that fit the data equally as well, $M_1 = (AD, BE)$ and $M_2 = (AE, BD)$. This result corresponds to the tenth linear dependency pre-

Factor	post.prob (Approx.)	post.prob (Direct)
none	.000	.003
A	.981	.977
B	.998	.994
C	.108	.172
D	.959	.966
E	.049	.049
F	.049	.049

Table 6.10: Proposed Analysis (Main effects): Posterior Probabilities for Example 1

AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF	Response
+	+	−	+	+	+	−	+	+	−	+	+	−	−	+	−0.03382
+	+	+	−	+	+	+	−	+	+	−	+	−	+	−	0.3828
+	+	+	+	−	+	+	+	−	+	+	−	+	−	−	−7.9968
−	−	+	+	+	+	−	−	−	−	−	−	+	+	+	0.0039
−	+	−	−	−	−	+	+	+	−	−	−	+	+	+	0.6016
+	−	−	−	−	−	−	−	−	+	+	+	+	+	+	8.5075

Table 6.11: Example 2: Two factor Interaction design Matrix (with foldover pairs added)

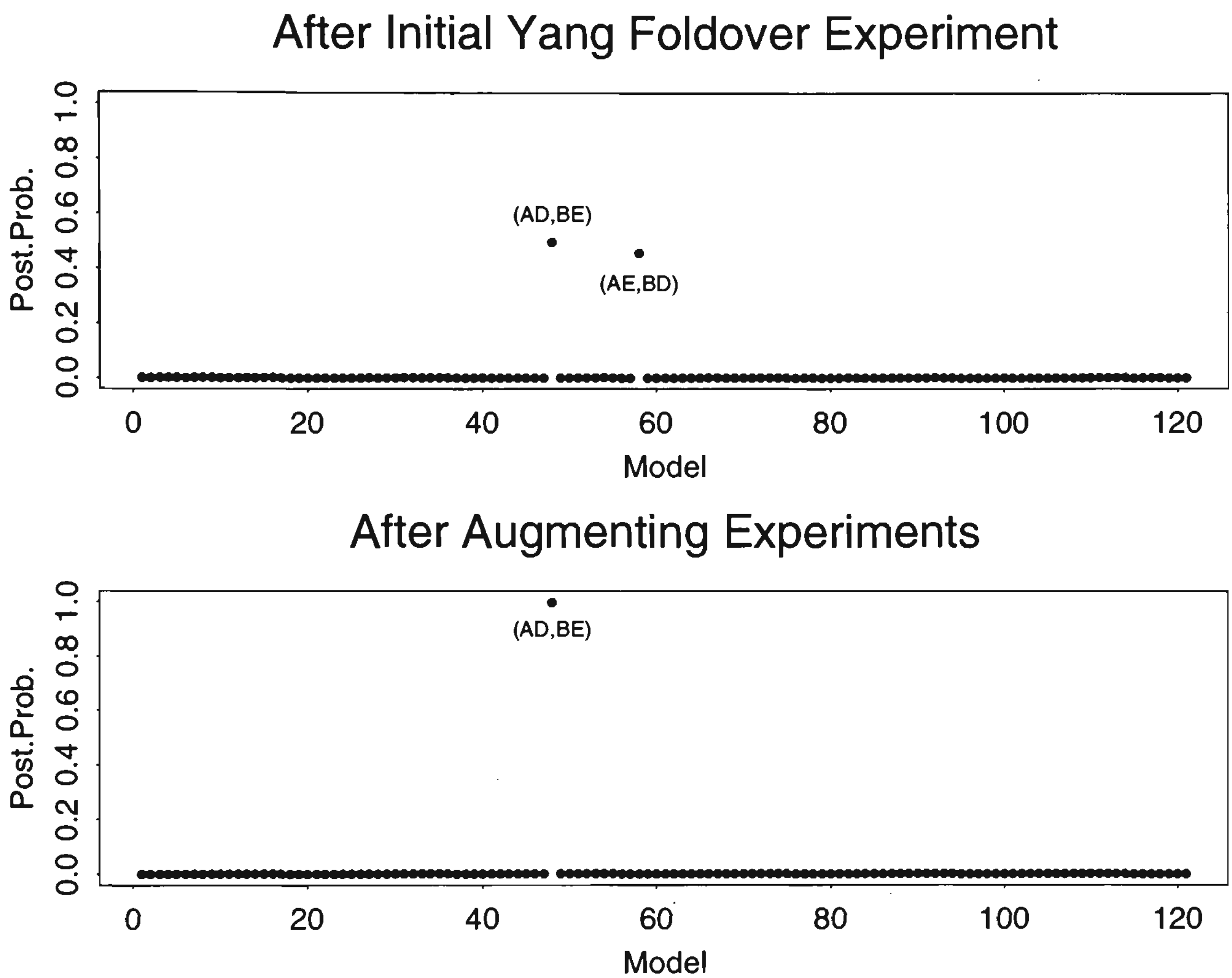


Figure 6.2: Posterior Probabilities of models involving two two-factor interaction models before the addition of augmenting trials (top) and after addition of augmenting trials (bottom)

factor	post.Prob. (Approx.)	Post.Prob. (Direct)	Post.prob. (Augmented)
none	.064	.002	.000
AB	.019	.003	.000
AC	.053	.009	.000
AD	.352	.501	.997
AE	.333	.460	.000
AF	.018	.003	.000
BC	.044	.007	.000
BD	.334	.460	.000
BE	.361	.503	.998
BF	.019	.003	.000
CD	.019	.003	.000
CE	.020	.003	.000
CF	.057	.010	.000
DE	.020	.003	.000
DF	.049	.008	.000
EF	.049	.007	.001

Table 6.12: Posterior Probabilities of Two-Factor Interactions using the Approximate method and Exact method, and after the addition of augmenting trials.

sented in Chapter 3, and therefore the addition of augmenting trials will be required to estimate which of the two-factor interaction models is the true model.

6.7 Augmenting for Example 2

The maximum-minimum eigenvalue case given in Chapter 3 is (bd, be). The generated reults for these runs were $(6.2789, -3.5960)$. The lower portion of Figure 6.2 shows the posterior probabilities of all possible models involving two two-factor interactions using the direct method. The right column of Table 6.12 presents the posterior probability that each two-factor interaction is active using the approximate and direct method. Both the diagram and the table shows that of the two models that fitted the data equally as well after the initial design now only model $M_2 = (AE, BD)$ is supported by the data.

6.8 Conclusion

In this chapter two methods for analysing search designs were presented. The first method is approximate only and utilises existing freely available software. This method is useful as an exploratory tool but is not easily adapted to the case when augmenting trials are required. For this reason a direct Bayesian technique was implemented which can accomodate augmenting trials. For the two examples presented which satisfy effect heredity but not factor sparsity both the approximate and direct methods performed considerably better than the standard Box and Meyer technique of identifying active factors, which failed to identify significant effects. If the models had exhibited factor Sparsity it would be expected that the Box and Meyer technique and the two methods presented in this chapter would give similiar results. Furthermore the direct method was used to analyse both examples after the addition of the augmenting trials suggested in Chapter 3, and for each example the "true" model was correctly identified.

Chapter 7

DISCUSSION

7.1 Introduction

Previous chapters have detailed some properties of a series of non-regular, minimum-run resolution IV designs. These designs are termed non-regular as they are not members of the well known class of two-level fractional designs as derived by Finney [19].

Regular fractional factorial designs have been successfully used in all areas of quality improvement. The designs have many nice properties such as minimum variance, orthogonality of main effects and simple aliasing structures that enable them to be easily employed and analysed. Normal probability plots (Daniel [10]) and Bayes plots (Box and Meyer [5]) enable one to easily study the data and to determine any real effects. In addition methods for the design of augmenting trials have been derived including methods due to Daniel [12] and Diamond [14] to resolve ambiguities in the data as the need arises.

Regular fractional resolution IV factorial designs are used when it is important to estimate the main effects unbiased by the two-factor interactions, this is often the case when a small number of significant two-factor interaction effects are suspected.

It can sometimes be the case that the number of experimental observations required are larger than can be afforded, in these cases non-regular, minimum run resolution IV designs can be employed. These non-regular designs can be orthogonal or non-orthogonal and usually exhibit a complex aliasing structure between the two-factor interactions. Consequently, they have been considered to be too difficult to interpret. However results due to Diamond [14] and in this thesis show this complex aliasing can in fact be quite advantageous as it sometimes allows for the identification of a small number of two-factor interactions without the need for augmenting trials. Two techniques where developed specifically for the analysis of these non-regular minimum run designs.

In some cases the addition of augmenting trials may still be required. This thesis has identified these cases when considering models with one, two or three interactions for a number of non-regular minimum run designs. Augmenting trials which separate these dependencies have been derived for a number of these designs also.

7.2 Comments On The Non-regular Designs

7.2.1 The Raghavarao Foldover Designs

Raghavarao [31] presented two-level weighing designs for 5,13 and 25 factors that are the most efficient of their class. The 5 factor foldover design is in fact the modified one factor at a time foldover design studied by Diamond [13]. The 13 factor foldover design has been shown to be a strongly resolvable main-effect-plus r search design when $r = 1$ and weakly resolvable when $r = 2$. Augmenting runs have been derived to separate the dependency when $r = 2$.

7.2.2 The Yang Foldover Designs

The foldover of the $n = 2 \pmod 4$ factor designs due to Yang [36] are also very efficient designs. As a class, the Yang foldovers provide designs with a convenient number of runs. For example, the 6 factor foldover involves 12 runs and falls halfway between the 8 and 16 run regular orthogonal resolution IV designs, the 10 factor foldover in 20 runs fits between the 16 and 24 run regular fractional replicates and the 14 factor foldover in 28 runs fits between the regular orthogonal resolution IV designs in 24 and 32 runs.

Diamond [14] showed that the Yang 6 factor foldover was a strongly resolvable main-effect-plus r search design when $r = 1$ but only weakly resolvable when $r = 2$. Results in this thesis present augmenting trials in the presence of error for the dependent cases when $r = 2$. Diamond also showed the 10 factor foldover design is not resolvable when $r = 1$. Results in this thesis show the 14 factor foldover design to be a strongly resolvable main-effect-plus r search design when $r = 1$ but only weakly resolvable when $r = 2$. Augmenting trials were presented for the general case when $r = 2$.

7.2.3 Plackett and Burman Foldover Designs

The Plackett and Burman designs give 100% efficient estimates of the main effects and unlike the designs due to Raghavarao and Yang, main effects are also orthogonal. Diamond showed the 12 factor foldover design to be strongly resolvable main-effect-plus r search design when $r = 2$. He also showed the design was resolution V in every choice of 5 columns. Results in this thesis show that the 12 factor foldover has only one projection in any N factors (where $N \leq 12$). It was also shown that the design is weakly resolvable when $r = 3$.

The foldovers generated from the 20 and 24 factor Plackett and Burman designs also have nice projection properties yielding a resolution V design in every choice of 5 columns. Each of the designs were also shown to be strongly resolvable when $r = 2$.

7.3 Limitations

7.3.1 The Assumption of No Three-Factor Interactions

This thesis has been concerned wholely with the identification of a small number of real two-factor interactions in minimum-run resolution IV designs. The analysis of each design has been contingent upon the assumption that higher order interactions are negligible. If, in fact, three-factor interactions are real they will bias main effect estimates but not the two-factor interactions. If four-factor interactions are real then the two-factor interaction estimates will be biased. Hence if there is cause to suspect that higher order interactions are real, conclusions drawn from an analysis of these minumum-run resolution IV designs will be affected and this should be considered when interpreting the experimental results.

7.3.2 Appropriate Metric

Finding an appropriate transformation for the experimental data is an issue that has not been addressed in this thesis, nor has it in any of the search design literature. Box and Cox [3] published a paper which generated considerable interest in this area for regular fractional factorials. They presented an example where the analysis of a replicated resolution III design in one metric identified two main effects as being real, whilst for other metrics the corresponding two-factor interaction begins to appear.

Box and Cox employed the family of transformations called the power transformations, where the response y is transformed according to

$$y^{(\lambda)} = \frac{y^{(\lambda)} - 1}{\lambda \dot{y}^{(\lambda-1)}}, (y^0 = \dot{y} \ln y)$$

where $\dot{y}$ is the geometric mean of the response. Box and Cox employed a lamdba plot which is a plot of various test statistics versus the transformation parameter λ to determine an appropriate transformation for the data. The λ plot is discussed in detail by Box [1].

One technique for determining the appropriate transformation of a search design was suggested by Diamond [14] who suggested producing a lamdba plot of the main effects to determine an appropriate metric. This technique, however, is yet to be examined.

7.4 Some Further Extensions

7.4.1 Augmenting with and without error

In Chapter 2 a method was presented for devising augmenting runs from a candidate set to separate linearly dependent models for the error free case. This method was extended in chapter 3 to include the with error case. Augmenting trials in the presence of error need to be developed for the Yang 10 and 14-factor foldover designs and also the Plackett and Burman 12 factor foldover when $k = 3$.

7.4.2 The effect of outliers

Daniel [10] showed how by using a normal probability plot it is often possible to identify outliers in a regular fractional design. The minimum-run resolution IV designs require a method for the identification of outliers to be developed.

7.4.3 Extensions to the Designs

The technique described in chapter 4 could be used to investigate the Plackett and Burman 20 and 24 factor foldovers when $k = 3$ and also for larger Plackett and Burman designs. It would also be worthwhile to examine the Plackett and

Burman resolution III designs as search designs although the problem would be considerably more difficult as the main effects would need to be taken into account.

7.4.4 Generalising the Results

The emphasis in this thesis has been on obtaining results using a combination of mathematical and computing power. It would be desirable to prove many of the results mathematically. This would give further insight into the properties of the designs and suggest additional research. For example, the result obtained in this thesis that the foldovers of the PB20 and PB24 designs are, like the foldover of the PB12 design, resolution V in every set of 5 factors may generalise to a larger class of Plackett and Burman designs.

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