Inequalities of Hermite-Hadamard's Type for Functions Whose Derivatives Absolute Values are Quasi-Convex

INEQUALITIES OF HERMITE-HADAMARD'S TYPE FOR FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

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Abstract. In this paper, some inequalities of Hermite-Hadamard type for functions whose detivatives absolute values are quasi-convex, are given. Some error estimates for the midpoint formula are also obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality, known as the *Hermite–Hadamard inequality* for convex functions, holds:

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In recent years many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [1] – [5] and [7] – [11].

In [2], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected with Hermite-Hadamard's inequality and they used the following lemma to prove it.

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)\,b)\,dt. \tag{1.2}$$

The main inequality in [2] is pointed out as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]. \tag{1.3}$$

In [10] Pearce and Pečarić using the same Lemma 1 proved the following theorem.

Key words and phrases. Convex function, Hermite-Hadamard inequality, Quasi-convex functions.

The financial support received from Universiti Kebangsaan Malaysia, Faculty of Science and Technology under the grant no. (UKM–GUP–TMK–07–02–107) is gratefully acknowledged.

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for some $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left[\frac{|f(a)|^q + |f(b)|^q}{2} \right]^{\frac{1}{q}}, \tag{1.4}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left[\frac{|f(a)|^q + |f(b)|^q}{2} \right]^{\frac{1}{q}}. \tag{1.5}$$

If $|f|^q$ is concave on $[a, b]$ for some $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right| \tag{1.6}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|. \tag{1.7}$$

In [7] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) = (b-a) \int_0^1 K(t)\, f'(ta + (1-t)b)\, dt \tag{1.8}$$

where,

$$K(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right], \\ t-1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

One more general result related to (1.7) was established in [8]. The main result in [7] is as follows:

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)| \right]. \tag{1.9}$$

Now, we recall that the notion of *quasi-convex functions* generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [6]).

Recently, D.A. Ion [6] established two inequalities for functions whose first derivatives in absolute value are quasi-convex. Namely, he obtained the following results

Theorem 4. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \max\{|f'(a)|, |f'(b)|\}. \tag{1.10}$$

Theorem 5. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\ &\qquad \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left(\max\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}. \end{aligned} \tag{1.11}$$

The main purpose of this paper is to establish inequalities related to the left hand side of Hermite-Hadamard's type for functions whose derivatives in absolute value are quasi-convex. The obtained results can be used to give estimates for the approximation error of the integral $\int_a^b f(x)\, dx$ by the use of the midpoint formula.

2. Hermite-Hadamard Type Inequalities

Let us start with an improvement and simplification of the constants in Theorem 5 and consolidate this result with Theorem 4.

Theorem 6. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}. \tag{2.1}$$

Proof. From Lemma 1, using the well-known power mean inequality, we have

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\ &= \left| \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b)\, dt \right| \\ &\leq \frac{b-a}{2} \int_0^1 |1-2t| \, |f'(ta + (1-t)b)|\, dt \\ &\leq \frac{b-a}{2} \left(\int_0^1 |1-2t|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2t| \, |f'(ta + (1-t)b)|^q\, dt \right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2} \left(\int_0^1 |1-2t|\, dt \right)^{1-\frac{1}{q}} \left(\max\left\{ |f'(a)|^q, |f'(b)|^q \right\} \int_0^1 |1-2t|\, dt \right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{4} \left(\max\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

□

Corollary 1. *Let f be as in Theorem 6. Additionally, if*

(1) $|f'|$ *is increasing, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left|f'(b)\right|. \tag{2.2}$$

(2) $|f'|$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left|f'(a)\right|. \tag{2.3}$$

Remark 1. *For $q=1$ this reduces to Theorem 4. For $q=p/(p-1)$ $(p>1)$ we have an improvement of the constants in Theorem 5, since $2^p>p+1$ if $p>1$ and accordingly*

$$\frac{1}{4}<\frac{1}{2(p+1)^{\frac{1}{p}}}.$$

Next, our main result(s) present new inequalities of midpoint type for quasi-convex functions.

Theorem 7. *Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\ \leq\frac{b-a}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(a)\right|\right\}\right].\end{aligned} \tag{2.4}$$

Proof. From Lemma 2, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\ &\leq(b-a)\left[\int_0^{\frac{1}{2}}t\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^1\left|1-t\right|\left|f'(ta+(1-t)b)\right|dt\right] \\ &\leq(b-a)\left[\int_0^{\frac{1}{2}}t\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right\}dt\right. \\ &\qquad\left.+\int_{\frac{1}{2}}^1(1-t)\max\left\{\left|f'(a)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}dt\right] \\ &\leq\frac{b-a}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(a)\right|\right\}\right].\end{aligned}$$

□

In the following, we deduce and improve some inequalities of Hermite-Hadamard type.

Corollary 2. *Let f be as in Theorem 7. Additionally, if*

(1) $|f'|$ *is increasing, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{8}\left[\left|f'(b)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{2.5}$$

(2) $|f'|$ *is decreasing, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{2.6}$$

(3) $f'\left(\frac{a+b}{2}\right) = 0$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]. \tag{2.7}$$

(4) $f'(a) = f'(b) = 0$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{2.8}$$

Proof. It follows directly by Theorem 7. □

Similar result(s) are embodied in the following theorem.

Theorem 8. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\qquad \left. + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right]. \end{aligned} \tag{2.9}$$

Proof. From Lemma 2, using well known Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
&\le (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 |1-t|\left|f'(ta+(1-t)b)\right|dt\right] \\
&\le (b-a)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad + (b-a)\left(\int_{\frac{1}{2}}^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\le (b-a)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&\qquad + (b-a)\left(\int_{\frac{1}{2}}^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, which completes the proof. □

Corollary 3. *Let f be as in Theorem 8. Additionally, if*

(1) $|f'|^{p/(p-1)}$ *is increasing, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f'(b)\right| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{2.10}$$

(2) $|f'|^{p/(p-1)}$ *is decreasing, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{2.11}$$

(3) $f'\left(\frac{a+b}{2}\right)=0$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{2.12}$$

(4) $f'(a)=f'(b)=0$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.13}$$

An improvement of the constants in Theorem 8 and consolidate this result with Theorem 7 is as follows:

Theorem 9. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}}\right]. \tag{2.14}$$

Proof. From Lemma 2, using the well-known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
&\leq (b-a)\int_0^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|\,dt + \int_{\frac{1}{2}}^1 (1-t)\,|f'(ta+(1-t)b)|\,dt \\
&\leq (b-a)\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \qquad (2.15)\\
&\quad + (b-a)\left(\int_{\frac{1}{2}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is quasi-convex we have

$$\int_0^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|^q\,dt \leq \frac{1}{8}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(b)|^q\right\}$$

and

$$\int_{\frac{1}{2}}^1 |1-t|\,|f'(ta+(1-t)b)|^q\,dt \leq \frac{1}{8}\max\left\{|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}.$$

Therefore, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{8}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Remark 2. *For $q = 1$ this reduces to Theorem 7. For $q = p/(p-1)$ $(p > 1)$ we have an improvement of the constants in Theorem 8, since $4^p > p+1$ if $p > 1$ and accordingly*

$$\frac{1}{8} < \frac{1}{4(p+1)^{\frac{1}{p}}}.$$

Improvements of the inequalities (2.5), (2.6), (2.7) and (2.8) are given in the following result:

Corollary 4. *Let f be as in Theorem 9. Additionally, if*

(1) $|f'|$ *is increasing, then (2.5) holds.*
(2) $|f'|$ *is decreasing, then (2.6) holds.*
(3) $f'\left(\frac{a+b}{2}\right) = 0$*, then (2.7) holds.*
(4) $f'(a) = f'(b) = 0$*, then (2.8) holds.*

Proof. Follows directly from Theorem 9. □

3. Applications to the Midpoint Formula

Let d be a division of the interval $[a,b]$, i.e., $d : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and consider the midpoint formula

$$M(f,d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right). \tag{3.1}$$

It is well known that if the mapping $f : [a,b] \to \mathbb{R}$, is differentiable such that $f''(x)$ exists on (a,b) and $K = \sup_{x\in(a,b)} |f''(x)| < \infty$, then

$$I = \int_a^b f(x)\,dx = M(f,d) + E(f,d), \tag{3.2}$$

where the approximation error $E(f,d)$ of the integral I by the midpoint formula $M(f,d)$ satisfies

$$|E(f,d)| \leq \frac{K}{24} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3. \tag{3.3}$$

It is clear that if the mapping f is not twice differentiable or the second derivative is not bounded on (a,b), then (3.3) cannot be applied.

In the following, we propose some new estimates for the remainder term $E(f,d)$ in terms of the first derivative which are better than the estimations of [7, 8] and [10].

Proposition 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$, then in (3.2), for every division d of $[a,b]$, the following holds:*

$$|E(f,d)| \leq \frac{1}{8} \sum_{i=1}^{n-1} (x_{i+1} - x_i) \left[\max\left\{\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\} + \max\left\{\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right]. \tag{3.4}$$

Proof. Applying Theorem 6 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n-1)$ of the division d, we get

$$\left|(x_{i+1}-x_i)\, f\left(\frac{x_i+x_{i+1}}{2}\right)-\int_{x_i}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq (x_{i+1}-x_i)\left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right.$$
$$\left.+\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right].$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|$ is quasi-convex, we deduce that

$$\left|M(f,d)-\int_a^b f(x)\,dx\right| \leq \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right.$$
$$\left.+\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right],$$

which completes the proof. □

Corollary 5. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. Given that $|f'|$ is quasi-convex on $[a, b]$, then in (3.2), for every division d of $[a, b]$,*

(1) *if $|f'|$ is increasing, then we have*

$$|E(f,d)| \leq \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+|f'(x_{i+1})|\right). \tag{3.5}$$

(2) *if $|f'|$ is decreasing, then we have*

$$|E(f,d)| \leq \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+|f'(x_i)|\right). \tag{3.6}$$

(3) *if $f'\left(\frac{x_i+x_{i+1}}{2}\right) = 0$, then we have*

$$|E(f,d)| \leq \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(|f'(x_i)|+|f'(x_{i+1})|\right). \tag{3.7}$$

(4) *if $f'(x_i) = f'(x_{i+1}) = 0$, then we have*

$$|E(f,d)| \leq \frac{1}{4}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|. \tag{3.8}$$

Proof. The proof is similar to that of Proposition 1, using Corollary 2. □

Proposition 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, $p > 1$, then in (3.2), for every* *division d of $[a,b]$, the following holds:*

$$|E(f,d)| \le \frac{1}{4(p+1)^{\frac{1}{p}}} \sum_{i=1}^{n-1} (x_{i+1}-x_i) \left[\left(\max\left\{ \left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^{\frac{p}{p-1}}, |f'(x_{i+1})|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} + \left(\max\left\{ \left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^{\frac{p}{p-1}}, |f'(x_i)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \right]. \tag{3.9}$$

Proof. The proof is similar to that of Proposition 1, using Theorem 8. □

Corollary 6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. Given that $|f'|$ is quasi-convex on $[a,b]$, then in (3.2), for every division d of $[a,b]$,*

(1) *if $|f'|$ is increasing, then we have*

$$|E(f,d)| \le \frac{1}{4(p+1)^{\frac{1}{p}}} \sum_{i=1}^{n-1} (x_{i+1}-x_i) \left(\left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right| + |f'(x_{i+1})| \right).$$

(2) *if $|f'|$ is decreasing, then we have*

$$|E(f,d)| \le \frac{1}{4(p+1)^{\frac{1}{p}}} \sum_{i=1}^{n-1} (x_{i+1}-x_i) \left(\left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right| + |f'(x_i)| \right).$$

Proof. The proof is similar to that of Proposition 1, using Corollary 3. □

Proposition 3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \ge 1$, then in (3.2), for every division d of $[a,b]$, the following holds:*

$$|E(f,d)| \le \frac{1}{8} \sum_{i=1}^{n-1} (x_{i+1}-x_i) \left[\left(\max\left\{ \left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^{q}, |f'(x_{i+1})|^{q} \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ \left| f'\left(\frac{x_i+x_{i+1}}{2} \right) \right|^{\frac{1}{q}}, |f'(x_i)|^{q} \right\} \right)^{\frac{1}{q}} \right]. \tag{3.10}$$

Proof. The proof is similar to that of Proposition 1, using Theorem 9. □

Corollary 7. *Let f as in Proposition 3, if in addition*

(1) *$|f'|$ is increasing, then (3.5) holds.*
(2) *$|f'|$ is decreasing, then (3.6) holds.*

Proof. The proof is similar to that of Proposition 3, using Corollary 4. □

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