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FEJÉR-TYPE INEQUALITIES (II)

K.-L. TSENG, SHIOW-RU HWANG, AND S.S. DRAGOMIR

ABSTRACT. In this paper, we establish some Fejér-type inequalities for convex functions. They complement the results from the previous recent paper [12].

1. INTRODUCTION

Throughout this paper, let $f : [a, b] \rightarrow \mathbb{R}$ be convex, $g : [a, b] \rightarrow [0, \infty)$ be integrable and symmetric to $\frac{a+b}{2}$ and define the following functions on $[0, 1]$:

$$G(t) = \frac{1}{2} \left[f \left(ta + (1-t) \frac{a+b}{2} \right) + f \left(tb + (1-t) \frac{a+b}{2} \right) \right];$$

$$H(t) = \frac{1}{b-a} \int_a^b f \left(tx + (1-t) \frac{a+b}{2} \right) dx;$$

$$H_g(t) = \int_a^b f \left(tx + (1-t) \frac{a+b}{2} \right) g(x) dx;$$

$$L(t) = \frac{1}{2(b-a)} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)] dx;$$

and

$$L_g(t) = \frac{1}{2} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)] g(x) dx.$$

If f is defined as above, then

$$(1.1) \quad f \left(\frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality [1].

For some results which generalize, improve, and extend this famous integral inequality see [2] – [17].

In [2], Dragomir established the following theorem which refines the first inequality of (1.1).

Theorem A. *Let f, H be defined as above. Then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$(1.2) \quad f \left(\frac{a+b}{2} \right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In [7], Dragomir, Milošević and Sándor established the following inequalities related to (1.1):

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Theorem B. *Let f, H be defined as above. Then:*

(1) *The following inequality holds*

$$(1.3) \quad \begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) dx \\ &\leq \int_0^1 H(t) dt \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f(x) dx \right]. \end{aligned}$$

(2) *If f is differentiable on $[a, b]$, then, for all $t \in [0, 1]$, we have the inequalities*

$$(1.4) \quad \begin{aligned} 0 &\leq \frac{1}{b-a} \int_a^b f(x) dx - H(t) \\ &\leq (1-t) \left[\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right] \end{aligned}$$

and

$$(1.5) \quad 0 \leq \frac{f(a) + f(b)}{2} - H(t) \leq \frac{(f'(b) - f'(a))(b-a)}{4}.$$

Theorem C. *Let f, H, G be defined as above. Then:*

(1) *G is convex and increasing on $[0, 1]$.*

(2) *We have*

$$\inf_{t \in [0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t \in [0,1]} G(t) = G(1) = \frac{f(a) + f(b)}{2}.$$

(3) *The following inequality holds for all $t \in [0, 1]$:*

$$(1.6) \quad H(t) \leq G(t).$$

(4) *The following inequality holds:*

$$(1.7) \quad \begin{aligned} \frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) dx &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \int_0^1 G(t) dt \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right]. \end{aligned}$$

(5) *If f is differentiable on $[a, b]$, then, for all $t \in [0, 1]$, we have the inequality*

$$(1.8) \quad 0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t).$$

Theorem D. *Let f, H, G, L be defined as above. Then:*

(1) *L is convex on $[0, 1]$.*

(2) *We have the inequality:*

$$(1.9) \quad G(t) \leq L(t) \leq \frac{1-t}{b-a} \int_a^b f(x) dx + t \cdot \frac{f(a) + f(b)}{2} \leq \frac{f(a) + f(b)}{2}$$

for all $t \in [0, 1]$ and

$$\sup_{t \in [0, 1]} L(t) = \frac{f(a) + f(b)}{2}.$$

(3) *For all $t \in [0, 1]$, we have the inequalities:*

$$H(1-t) \leq L(t) \quad \text{and} \quad \frac{H(t) + H(1-t)}{2} \leq L(t).$$

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem E. *Let f, g be defined as above. Then*

$$(1.10) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx$$

is known as *Fejér inequality*.

In [14], Yang and Tseng established the following theorem which refines the first inequality of (1.10) and generalizes Theorem A.

Theorem F. *Let f, g, H_g be defined as above. Then H_g is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$(1.11) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx = H_g(0) \leq H_g(t) \leq H_g(1) = \int_a^b f(x) g(x) dx.$$

In this paper, we establish some Fejér-type inequalities related to the functions G, H, H_g, L, L_g and generalize Theorems B – D. They complement the results from the recent paper [12].

2. MAIN RESULTS

In order to prove our main results, we need the following lemma:

Lemma 1 (see [9]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and let $a \leq A \leq C \leq D \leq B \leq b$ with $A + B = C + D$, then*

$$f(C) + f(D) \leq f(A) + f(B).$$

Now, we are ready to state and prove our results.

Theorem 2. *Let f, g, H_g be defined as above. Then we have the following Fejér-type inequalities:*

(1) *The following inequality holds:*

$$\begin{aligned}
 (2.1) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq 2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) g\left(2x - \frac{a+b}{2}\right) dx \\
 &\leq \int_0^1 H_g(t) dt \\
 &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx + \int_a^b f(x) g(x) dx \right].
 \end{aligned}$$

(2) *If f is differentiable on $[a, b]$ and g is bounded on $[a, b]$, then, for all $t \in [0, 1]$, we have the inequality*

$$\begin{aligned}
 (2.2) \quad 0 &\leq \int_a^b f(x) g(x) dx - H_g(t) \\
 &\leq (1-t) \left[\frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(x) dx \right] \|g\|_\infty,
 \end{aligned}$$

where $\|g\|_\infty = \sup_{x \in [a, b]} |g(x)|$.

(3) *If f is differentiable on $[a, b]$, then, for all $t \in [0, 1]$, we have the inequality*

$$\begin{aligned}
 (2.3) \quad 0 &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - H_g(t) \\
 &\leq \frac{(f'(b) - f'(a))(b-a)}{4} \int_a^b g(x) dx.
 \end{aligned}$$

Proof. (1) Using simple techniques of integration and the hypothesis of g , we have the following identities:

$$(2.4) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx = 4 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right) g(x) dt dx;$$

$$\begin{aligned}
 (2.5) \quad 2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) g\left(2x - \frac{a+b}{2}\right) dx \\
 = 2 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \right] g(x) dt dx;
 \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad &\int_0^1 H_g(t) dt \\
 &= \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t \frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t) \frac{a+b}{2}\right) \right] g(x) dt dx \\
 &\quad + \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t(a+b-x) + (1-t) \frac{a+b}{2}\right) \right. \\
 &\quad \left. + f\left(t \frac{a+b}{2} + (1-t)(a+b-x)\right) \right] g(x) dt dx;
 \end{aligned}$$

and

$$\begin{aligned}
 (2.7) \quad & \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx + \int_a^b f(x) g(x) dx \right] \\
 &= \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f(x) + f\left(\frac{a+b}{2}\right) \right] g(x) dt dx \\
 &\quad + \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{a+b}{2}\right) + f(a+b-x) \right] g(x) dt dx.
 \end{aligned}$$

By Lemma 1, the following inequalities hold for all $t \in [0, \frac{1}{2}]$ and $x \in [a, \frac{a+b}{2}]$.

$$(2.8) \quad 4f\left(\frac{a+b}{2}\right) \leq 2 \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \right]$$

holds when $A = \frac{x}{2} + \frac{a+b}{4}$, $C = D = \frac{a+b}{2}$ and $B = \frac{3(a+b)}{4} - \frac{x}{2}$ in Lemma 1.

$$(2.9) \quad 2f\left(\frac{x}{2} + \frac{a+b}{4}\right) \leq f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right)$$

holds when $A = t\frac{a+b}{2} + (1-t)x$, $C = D = \frac{x}{2} + \frac{a+b}{4}$ and $B = tx + (1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}
 (2.10) \quad & 2f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \\
 & \leq f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right)
 \end{aligned}$$

holds when $A = t(a+b-x) + (1-t)\frac{a+b}{2}$, $C = D = \frac{3(a+b)}{4} - \frac{x}{2}$ and $B = t\frac{a+b}{2} + (1-t)(a+b-x)$ in Lemma 1.

$$(2.11) \quad f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \leq f(x) + f\left(\frac{a+b}{2}\right)$$

holds when $A = x$, $C = t\frac{a+b}{2} + (1-t)x$, $D = tx + (1-t)\frac{a+b}{2}$ and $B = \frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}
 (2.12) \quad & f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right) \\
 & \leq f\left(\frac{a+b}{2}\right) + f(a+b-x)
 \end{aligned}$$

holds for $A = \frac{a+b}{2}$, $C = t(a+b-x) + (1-t)\frac{a+b}{2}$, $D = t\frac{a+b}{2} + (1-t)(a+b-x)$ and $B = a+b-x$ in Lemma 1. Multiplying the inequalities (2.8) – (2.12) by $g(x)$ and integrating them over t on $[0, \frac{1}{2}]$, over x on $[a, \frac{a+b}{2}]$ and using identities (2.4) – (2.7), we derive (2.1).

(2) By integration by parts, we have

$$\begin{aligned}
 (2.13) \quad & \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right) [f'(a+b-x) - f'(x)] dx \\
 &= \int_a^b \left(x - \frac{a+b}{2} \right) f'(x) dx \\
 &= \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(x) dx.
 \end{aligned}$$

Using substitution rules for integration and the hypothesis of g , we have the following identities

$$(2.14) \quad \int_a^b f(x) g(x) dx = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] g(x) dx$$

and

$$\begin{aligned}
 (2.15) \quad H_g(t) &= \int_a^{\frac{a+b}{2}} \left[f \left(tx + (1-t) \frac{a+b}{2} \right) \right. \\
 &\quad \left. + f \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right] g(x) dx.
 \end{aligned}$$

Now, using the convexity of f and the hypothesis of g , the inequality

$$\begin{aligned}
 & \left[f(x) - f \left(tx + (1-t) \frac{a+b}{2} \right) \right] g(x) \\
 & \quad + \left[f(a+b-x) - f \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right] g(x) \\
 & \leq (1-t) \left(x - \frac{a+b}{2} \right) f'(x) g(x) \\
 & \quad + (1-t) \left(\frac{a+b}{2} - x \right) f'(a+b-x) g(x) \\
 & = (1-t) \left(\frac{a+b}{2} - x \right) [f'(a+b-x) - f'(x)] g(x) \\
 & \leq (1-t) \left(\frac{a+b}{2} - x \right) [f'(a+b-x) - f'(x)] \|g\|_\infty
 \end{aligned}$$

holds for all $t \in [0, 1]$ and $x \in [a, \frac{a+b}{2}]$. Integrating the above inequalities over x on $[a, \frac{a+b}{2}]$ and using (2.13) – (2.15) and (1.11), we derive (2.2).

(3) Using the convexity of f , we have

$$\frac{f(a) - f(\frac{a+b}{2})}{2} \leq \frac{1}{2} \left(a - \frac{a+b}{2} \right) f'(a) = \frac{a-b}{4} f'(a)$$

and

$$\frac{f(b) - f(\frac{a+b}{2})}{2} \leq \frac{1}{2} \left(b - \frac{a+b}{2} \right) f'(b) = \frac{b-a}{4} f'(b)$$

and taking their sum we obtain

$$\frac{f(a) + f(b)}{2} - f \left(\frac{a+b}{2} \right) \leq \frac{(f'(b) - f'(a))(b-a)}{4}.$$

Thus,

$$(2.16) \quad \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \\ \leq \frac{(f'(b) - f'(a))(b-a)}{4} \int_a^b g(x) dx.$$

Finally, (2.3) follows from (1.10), (1.11) and (2.16). This completes the proof. ■

Remark 3. Let $g(x) = \frac{1}{b-a}$ ($x \in [a, b]$) in Theorem 2. Then $H_g(t) = H(t)$ ($t \in [0, 1]$) and Theorem 2 reduces to Theorem B.

In the following theorems, we point out some inequalities for the functions H, H_g, G, L_g, Q considered above:

Theorem 4. Let f, g, G, H_g be defined as above. Then we have the following Fejér-type inequalities:

(1) The following inequality holds for all $t \in [0, 1]$:

$$(2.17) \quad H_g(t) \leq G(t) \int_a^b g(x) dx.$$

(2) The following inequality holds:

$$(2.18) \quad 2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) g\left(2x - \frac{a+b}{2}\right) dx \leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \int_a^b g(x) dx \\ \leq (b-a) \int_0^1 G(t) g((1-t)a + tb) dt \\ \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] \int_a^b g(x) dx.$$

(3) If f is differentiable on $[a, b]$ and g is bounded on $[a, b]$, then, for all $t \in [0, 1]$, we have the inequality

$$(2.19) \quad 0 \leq H_g(t) - f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq (b-a) [G(t) - H(t)] \|g\|_\infty$$

where $\|g\|_\infty = \sup_{x \in [a, b]} |g(x)|$.

Proof. (1) Using simple techniques of integration and the hypothesis of g , we have that the following identity holds on $[0, 1]$:

$$(2.20) \quad G(t) \int_a^b g(x) dx = \int_a^{\frac{a+b}{2}} \left[f\left(ta + (1-t)\frac{a+b}{2}\right) \right. \\ \left. + f\left(tb + (1-t)\frac{a+b}{2}\right) \right] g(x) dx.$$

By Lemma 1, the following inequality holds for all $x \in [a, \frac{a+b}{2}]$:

$$(2.21) \quad f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \\ \leq f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right).$$

It holds when

$$\begin{aligned} A &= ta + (1-t) \frac{a+b}{2}, & C &= tx + (1-t) \frac{a+b}{2}, \\ D &= t(a+b-x) + (1-t) \frac{a+b}{2} \quad \text{and} \quad B = tb + (1-t) \frac{a+b}{2} \end{aligned}$$

in Lemma 1. Multiplying the inequality (2.21) by $g(x)$, integrating both sides over x on $[a, \frac{a+b}{2}]$ and using identities (2.15) and (2.20), we derive (2.17).

(2) As for (1), we have the following identities:

$$\begin{aligned} (2.22) \quad & 2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) g\left(2x - \frac{a+b}{2}\right) dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \right] g(x) dx; \end{aligned}$$

$$\begin{aligned} (2.23) \quad & \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \int_a^b g(x) dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] g(x) dx; \end{aligned}$$

$$\begin{aligned} & (b-a) \int_0^1 G(t) g((1-t)a + tb) dt \\ &= \frac{b-a}{2} \left[\int_{\frac{1}{2}}^1 f\left(ta + (1-t) \frac{a+b}{2}\right) g(ta + (1-t)b) dt \right. \\ &\quad + \int_0^{\frac{1}{2}} f\left(ta + (1-t) \frac{a+b}{2}\right) g((1-t)a + tb) dt \\ &\quad + \int_0^{\frac{1}{2}} f\left(tb + (1-t) \frac{a+b}{2}\right) g((1-t)a + tb) dt \\ &\quad \left. + \int_{\frac{1}{2}}^1 f\left(tb + (1-t) \frac{a+b}{2}\right) g(ta + (1-t)b) dt \right] \\ &= \int_a^{\frac{a+b}{2}} \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{2a+b-x}{2}\right) \right. \\ (2.24) \quad & \left. + f\left(\frac{b+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right] g(x) dx; \end{aligned}$$

and

$$\begin{aligned} (2.25) \quad & \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \int_a^b g(x) dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] g(x) dx. \end{aligned}$$

By Lemma 1, the following inequalities hold for all $x \in [a, \frac{a+b}{2}]$.

$$(2.26) \quad f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \leq f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)$$

holds when $A = \frac{3a+b}{4}$, $C = \frac{x}{2} + \frac{a+b}{4}$, $D = \frac{3(a+b)}{4} - \frac{x}{2}$ and $B = \frac{a+3b}{4}$ in Lemma 1.

$$(2.27) \quad f\left(\frac{3a+b}{4}\right) \leq \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{2a+b-x}{2}\right) \right]$$

holds when $A = \frac{x+a}{2}$, $C = D = \frac{3a+b}{4}$ and $B = \frac{2a+b-x}{2}$ in Lemma 1.

$$(2.28) \quad f\left(\frac{a+3b}{4}\right) \leq \frac{1}{2} \left[f\left(\frac{b+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right]$$

holds when $A = \frac{b+x}{2}$, $C = D = \frac{a+3b}{4}$ and $B = \frac{a+2b-x}{2}$ in Lemma 1.

$$(2.29) \quad f\left(\frac{x+a}{2}\right) + f\left(\frac{2a+b-x}{2}\right) \leq f(a) + f\left(\frac{a+b}{2}\right)$$

holds when $A = a$, $C = \frac{x+a}{2}$, $D = \frac{2a+b-x}{2}$ and $B = \frac{a+b}{2}$ in Lemma 1.

$$(2.30) \quad f\left(\frac{b+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \leq f\left(\frac{a+b}{2}\right) + f(b)$$

holds when $A = \frac{a+b}{2}$, $C = \frac{b+x}{2}$, $D = \frac{a+2b-x}{2}$ and $B = b$ in Lemma 1. Multiplying the inequalities (2.26) – (2.30) by $g(x)$, integrating both sides over x on $[a, \frac{a+b}{2}]$ and using identities (2.22) – (2.25), we derive (2.18).

(3) By integration by parts, we have

$$\begin{aligned} & t \int_a^{\frac{a+b}{2}} \left[\left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) \right. \\ & \quad \left. + \left(\frac{a+b}{2} - x \right) f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right] dx \\ & = t \int_a^b \left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) dx \\ (2.31) \quad & = (b-a) [G(t) - H(t)]. \end{aligned}$$

Now, using the convexity of f and the hypothesis of g , the inequality

$$\begin{aligned} & \left[f \left(tx + (1-t) \frac{a+b}{2} \right) - f \left(\frac{a+b}{2} \right) \right] g(x) \\ & \quad + \left[f \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) - f \left(\frac{a+b}{2} \right) \right] g(x) \\ & \leq t \left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) g(x) \\ & \quad + t \left(\frac{a+b}{2} - x \right) f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) g(x) \end{aligned}$$

$$\begin{aligned}
&= t \left(\frac{a+b}{2} - x \right) \left[f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right. \\
&\quad \left. - f' \left(tx + (1-t) \frac{a+b}{2} \right) \right] g(x) \\
&\leq t \left(\frac{a+b}{2} - x \right) \left[f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right. \\
&\quad \left. - f' \left(tx + (1-t) \frac{a+b}{2} \right) \right] \|g\|_{\infty}
\end{aligned}$$

holds for all $t \in [0, 1]$ and $x \in [a, \frac{a+b}{2}]$. Integrating the above inequality over x on $[a, \frac{a+b}{2}]$ and using (2.31) and (1.11), we derive (2.17). This completes the proof. ■

Remark 5. Let $g(x) = \frac{1}{b-a}$ ($x \in [a, b]$) in Theorem 4. Then $H_g(t) = H(t)$ ($t \in [0, 1]$) and Theorem 4 reduces to Theorem C.

Theorem 6. Let f, g, G, H_g, L_g be defined as above. Then we have the following results:

- (1) L_g is convex on $[0, 1]$.
- (2) The following inequalities hold for all $t \in [0, 1]$:

$$\begin{aligned}
(2.32) \quad G(t) \int_a^b g(x) dx &\leq L_g(t) \\
&\leq (1-t) \int_a^b f(x) g(x) dx + t \cdot \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \\
&\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx;
\end{aligned}$$

and

$$(2.33) \quad H_g(1-t) \leq L_g(t);$$

$$(2.34) \quad \frac{H_g(t) + H_g(1-t)}{2} \leq L_g(t).$$

- (3) The following bound is true:

$$(2.35) \quad \sup_{t \in [0, 1]} L_g(t) = \frac{f(a) + f(b)}{2} \int_a^b g(x) dx.$$

Proof. (1) It is easily observed from the convexity of f that L_g is convex on $[0, 1]$.

(2) As for (1) in Theorem 4, we have that the following identity holds on $[0, 1]$:

$$\begin{aligned}
(2.36) \quad L_g(t) &= \frac{1}{2} \int_a^{\frac{a+b}{2}} [f(ta + (1-t)x) + f(ta + (1-t)(a+b-x)) \\
&\quad + f(tb + (1-t)x) + f(tb + (1-t)(a+b-x))] g(x) dx.
\end{aligned}$$

By Lemma 1, the following inequalities hold for all $x \in [a, \frac{a+b}{2}]$.

$$(2.37) \quad 2f\left(ta + (1-t) \frac{a+b}{2}\right) \leq f(ta + (1-t)x) + f(ta + (1-t)(a+b-x))$$

holds when $A = ta + (1 - t)x$, $C = D = ta + (1 - t)\frac{a+b}{2}$ and $B = ta + (1 - t)(a + b - x)$ in Lemma 1.

$$(2.38) \quad 2f\left(tb + (1 - t)\frac{a+b}{2}\right) \leq f(tb + (1 - t)x) + f(tb + (1 - t)(a + b - x))$$

holds when $A = tb + (1 - t)x$, $C = D = tb + (1 - t)\frac{a+b}{2}$ and $B = tb + (1 - t)(a + b - x)$ in Lemma 1. Multiplying the inequalities (2.37) – (2.38) by $g(x)$, integrating them over x on $[a, \frac{a+b}{2}]$ and using identities (2.20) and (2.36), we derive the first inequality of (2.32). Using the convexity of f and the inequality (1.10), the last part of (2.32) holds. Again from the convexity of f , we get

$$(2.39) \quad \begin{aligned} H_g(1 - t) &= \int_a^b f\left((1 - t)x + t\frac{a+b}{2}\right) g(x) dx \\ &= \int_a^b f\left(\frac{ta + (1 - t)x}{2} + \frac{tb + (1 - t)x}{2}\right) g(x) dx \\ &\leq L_g(t) \end{aligned}$$

and (2.33) is proved. From (2.17), (2.32) and (2.33), we get (2.34).

(3) Using (2.32), the inequality (2.35) holds. This completes the proof. ■

Remark 7. Let $g(x) = \frac{1}{b-a}$ ($x \in [a, b]$) in Theorem 6. Then $H_g(t) = H(t)$ ($t \in [0, 1]$), $L_g(t) = L(t)$ ($t \in [0, 1]$) and Theorem 6 reduces to Theorem D.

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