VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Refinements of Fejér's Inequality for Convex Functions

This is the Published version of the following publication

Tseng, Kuei-Lin, Hwang, Shiow-Ru and Dragomir, Sever S (2009) Refinements of Fejér's Inequality for Convex Functions. Research report collection, 12 (Supp).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository https://vuir.vu.edu.au/17956/

REFINEMENTS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS

K.-L. TSENG, SHIOW-RU HWANG, AND S.S. DRAGOMIR

Abstract. In this paper, we establish some new refinements for the celebrated Fejér's and Hermite-Hadamard's integral inequalities for convex functions.

1. Introduction

One of the most important integral inequalities with various applications for generalised means, information measures, quadrature rules, etc., is the well known *Hermite-Hadamard inequality* [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function on the interval $[a,b]$.

In order to refine and generalize this classical result for weighted integrals, we define the following functions on $[0,1]$, namely

$$G(t) = \frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right];$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx;$$

$$H_g(t) = \int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)g(x)\,dx;$$

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy;$$

$$\begin{aligned}K(t) = \int_a^b\int_a^b \frac{1}{4}\Bigg[&f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)\\&+f\left(t\frac{x+a}{2}+(1-t)\frac{y+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{y+b}{2}\right)\Bigg]g(x)g(y)\,dxdy;\end{aligned}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hermite-Hadamard inequality, Fejér inequality, Convex function.

This research was partially supported by grant NSC 98-2115-M-156-004.

$$L(t)=\frac{1}{2(b-a)}\int_a^b [f(ta+(1-t)x)+f(tb+(1-t)x)]\,dx;$$

$$L_g(t)=\frac{1}{2}\int_a^b [f(ta+(1-t)x)+f(tb+(1-t)x)]\,g(x)\,dx;$$

$$\begin{aligned}S_g(t)=\frac{1}{4}\int_a^b \bigg[&f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(ta+(1-t)\frac{x+b}{2}\right)\\&+f\left(tb+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\bigg]g(x)\,dx\end{aligned}$$

and

$$N(t)=\int_a^b \frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx.$$

where $f:[a,b]\to\mathbb{R}$ is convex, $g:[a,b]\to[0,\infty)$ is integrable and symmetric to $\frac{a+b}{2}$.

Remark 1. *We note that $H=H_g=I$, $F=K$ and $L=L_g=S_g$ on $[0,1]$ as $g(x)=\frac{1}{b-a}\,(x\in[a,b])$.*

For some results which generalize, improve, and extend the famous Hermite-Hadamard integral inequality see [2] – [20].

In [8], Fejér established the following weighted generalization of (1.1).

Theorem A. *Let f,g be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

In [11], Tseng et al. established the following Fejér-type inequalities.

Theorem B. *Let f,g be defined as above. Then we have*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx&\le\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx\\&\le\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\\&\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx\\&\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.\end{aligned}\tag{1.3}$$

In [2], Dragomir improved the first part of the Hermite-Hadamard inequality by considering the functions H,F as follows:

Theorem C. *Let f,H be defined as above. Then H is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have*

$$f\left(\frac{a+b}{2}\right)=H(0)\le H(t)\le H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.4}$$

Theorem D. *Let f,F be defined as above. Then*

(1) *F is convex on $[0,1]$, symmetric about $\frac{1}{2}$, F is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$, and for all $t\in[0,1]$,*

$$\sup_{t\in[0,1]} F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t)=F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.$$

(2) *We have:*

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right);\qquad H(t)\le F(t),\qquad t\in[0,1]. \tag{1.5}$$

In [11], Tseng et al. established the following Fejér-type inequality related to the functions I,N, which is also the weighted generalization of Theorem C.

Theorem E. *Let f,g,I,N be defined as above. Then I,N are convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0)\le I(t)\quad\le I(1)\\
&=\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\\
&=N(0)\le N(t)\le N(1)\\
&=\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \qquad (1.6)
\end{aligned}$$

In [7], Dragomir et al. established the following Hermite-Hadamard-type inequality related to the functions H,G,L.

Theorem F. *Let f,H,G,L be defined as above. Then G is convex, increasing on $[0,1]$, L is convex on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned}
H(t)\le G(t)\le L(t) &\le \frac{1-t}{b-a}\int_a^b f(x)\,dx+t\cdot\frac{f(a)+f(b)}{2}\\
&\le\frac{f(a)+f(b)}{2}. \qquad (1.7)
\end{aligned}$$

In [12] – [13], Tseng et al. established the following theorem related to Fejér-type inequalities concerning the functions G,H_g,L_g,I,S_g and which provides a weighted generalizations of the inequality (1.7).

Theorem G ([12])**.** *Let f,g,G,H_g,L_g be defined as above. Then L_g is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned}
H_g(t)\le G(t)&\int_a^b g(x)\,dx\le L_g(t)\\
&\le(1-t)\int_a^b f(x)\,g(x)\,dx+t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\\
&\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \qquad (1.8)
\end{aligned}$$

Theorem H ([13])**.** *Let f, g, G, I, S_g be defined as above. Then S_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} I(t) &\le G(t) \int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t) \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \\ &\qquad + t \cdot \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx. \end{aligned} \tag{1.9}$$

Finally, we notice that in [5], Dragomir established the following Hermite-Hadamard-type inequalities related to the functions H, F, L.

Theorem I. *Let F, H, L be defined as above. Then we have the inequality*

$$0 \le F(t) - H(t) \le L(1-t) - F(t) \tag{1.10}$$

for all $t \in [0,1]$.

In this paper, we establish some Fejér-type and Hermite-Hadamard-type inequalities related to the functions $H, F, L, H_g, L_g, I, S_g, K$ defined above. As an important consequence we also obtain the weighted generalizations of Theorems D and I.

2. Main Results

The following lemma plays a key role in proving the new results:

Lemma 2 (*see* [9])**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

We can state now the following result:

Theorem 3. *Let f, g, I, K be defined as above. Then:*

1. *K is convex on $[0,1]$ and symmetric about $\frac{1}{2}$.*
2. *K is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$\begin{aligned} \sup_{t\in[0,1]} K(t) &= K(0) = K(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \cdot \int_a^b g(x)\,dx \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \inf_{t\in[0,1]} K(t) &= K\left(\frac{1}{2}\right) \\ &= \int_a^b \int_a^b \frac{1}{4}\left[f\left(\frac{x+y+2a}{4}\right) + 2f\left(\frac{x+y+a+b}{4}\right) \right. \\ &\qquad \left. + f\left(\frac{x+y+2b}{4}\right)\right] g(x)\,g(y)\,dxdy. \end{aligned} \tag{2.2}$$

(3) *We have*

$$I(t)\int_a^b g(x)\,dx \le K(t) \tag{2.3}$$

and

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 \le K\left(\frac{1}{2}\right) \tag{2.4}$$

for all $t\in[0,1]$.

Proof. (1) It is easily observed from the convexity of f that K is convex on $[0,1]$. By changing the variable, we have that

$$K(t)=K(1-t), \qquad t\in[0,1],$$

from which we get that K is symmetric about $\frac{1}{2}$.

(2) Let $t_1<t_2$ in $\left[0,\frac{1}{2}\right]$. Using the symmetry of K, we have

$$K(t_1)=\frac{1}{2}\left[K(t_1)+K(1-t_1)\right], \tag{2.5}$$

$$K(t_2)=\frac{1}{2}\left[K(t_2)+K(1-t_2)\right] \tag{2.6}$$

and, by Lemma 2, we have

$$\frac{1}{2}\left[K(t_2)+K(1-t_2)\right]\le\frac{1}{2}\left[K(t_1)+K(1-t_1)\right]. \tag{2.7}$$

From (2.5) – (2.7), we obtain that K is decreasing on $\left[0,\frac{1}{2}\right]$. Since K is symmetric about $\frac{1}{2}$ and K is decreasing on $\left[0,\frac{1}{2}\right]$, we get that K is increasing on $\left[\frac{1}{2},1\right]$. Using the symmetry and monotonicity of K, we derive (2.1) and (2.2).

(3) Using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned} K(t)=\int_a^b\int_a^b\frac{1}{4}\Bigg[&f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)\\ &+f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right)\\ &+f\left(t\frac{x+b}{2}+(1-t)\frac{a+2b-y}{2}\right)\Bigg]g(x)g(y)\,dydx \end{aligned} \tag{2.8}$$

for all $t\in[0,1]$.

By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $x\in[a,b]$ and $y\in[a,b]$. The inequality

$$\begin{aligned} &\frac{1}{2}f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\\ &\le\frac{1}{4}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right)\right] \end{aligned} \tag{2.9}$$

holds when

$$A = t\frac{x+a}{2} + (1-t)\frac{y+a}{2},$$
$$C = D = t\frac{x+a}{2} + (1-t)\frac{a+b}{2} \qquad \text{and}$$
$$B = t\frac{x+a}{2} + (1-t)\frac{a+2b-y}{2}$$

in Lemma 2. The inequality

$$\begin{aligned}(2.10)\quad & \frac{1}{2}f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right) \\ & \leq \frac{1}{4}\left[f\left(t\frac{x+b}{2} + (1-t)\frac{y+a}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}\right)\right]\end{aligned}$$

holds when

$$A = t\frac{x+b}{2} + (1-t)\frac{y+a}{2},$$
$$C = D = t\frac{x+b}{2} + (1-t)\frac{a+b}{2} \qquad \text{and}$$
$$B = t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}$$

in Lemma 2.

Multiplying the inequalities (2.9) and (2.10) by $g(x)g(y)$, integrating them over x on $[a,b]$, over y on $[a,b]$ and using identities (2.8), we derive the inequality (2.3).

From the inequality (2.3) and the monotonicity of I, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &= I(0)\int_a^b g(x)\,dx \\ &\leq I\left(\frac{1}{2}\right)\int_a^b g(x)\,dx \leq K\left(\frac{1}{2}\right)\end{aligned}$$

from which we derive the inequality (2.4).

This completes the proof. □

Remark 4. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 3. Then* $I(t) = H(t)$, $K(t) = F(t)$ $(t \in [0,1])$ *and Theorem 3 reduces to Theorem D.*

Remark 5. *From Theorem E and Theorem 3, we obtain the following Fejér-type inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &\leq I(t)\int_a^b g(x)\,dx \leq K(t) \\ &\leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \cdot \int_a^b g(x)\,dx.\end{aligned}$$

Theorem 6. *Let* f, g, I, K, S_g *be defined as above. Then we have the inequality*

$$(2.11)\qquad 0 \leq K(t) - I(t)\int_a^b g(x)\,dx \leq S_g(1-t)\int_a^b g(x)\,dx - K(t),$$

for all $t \in [0,1]$.

Proof. Using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned}
K(t) &= \int_a^b \int_a^b \frac{1}{4}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)\right. \\
&\quad + f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right) \\
&\quad \left.+ f\left(t\frac{x+b}{2}+(1-t)\frac{a+2b-y}{2}\right)\right] g(x)\,g(y)\,dydx \\
&= \int_a^b \int_a^{\frac{a+b}{2}} \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\,y\right)\right. \\
&\quad + f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right)+f\left(t\frac{x+b}{2}+(1-t)\,y\right) \\
&\quad \left.+ f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right)\right] g(x)\,g(2y-a)\,dydx \\
&= \frac{1}{2}\int_a^b \int_a^{\frac{3a+b}{4}} \left[f\left(t\frac{x+a}{2}+(1-t)\,y\right)\right. \\
&\quad + f\left(t\frac{x+a}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \\
&\quad + f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right) \\
&\quad + f\left(t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right) \\
&\quad + f\left(t\frac{x+b}{2}+(1-t)\,y\right)+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \\
&\quad + f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right) \\
&\quad \left.+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)\right] g(x)\,g(2y-a)\,dydx \qquad (2.12)
\end{aligned}$$

for all $t\in[0,1]$.

By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $x\in[a,b]$ and $y\in\left[a,\frac{3a+b}{4}\right]$. The inequality

$$\begin{aligned}
&f\left(t\frac{x+a}{2}+(1-t)\,y\right)+f\left(t\frac{x+a}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \\
&\qquad \le f\left(t\frac{x+a}{2}+(1-t)\,a\right)+f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right) \qquad (2.13)
\end{aligned}$$

holds when

$$\begin{aligned}
&A=t\frac{x+a}{2}+(1-t)\,a, \qquad C=t\frac{x+a}{2}+(1-t)\,y, \\
&D=t\frac{x+a}{2}+(1-t)\left(\frac{3a+b}{2}-y\right) \quad \text{and} \quad B=t\frac{x+a}{2}+(1-t)\frac{a+b}{2}
\end{aligned}$$

in Lemma 2. The inequality

$$f\left(t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)+f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right) \leq f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)b\right) \tag{2.14}$$

holds when

$$A=t\frac{x+a}{2}+(1-t)\frac{a+b}{2}, \qquad C=t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right),$$
$$D=t\frac{x+a}{2}+(1-t)(a+b-y) \quad \text{and} \quad B=t\frac{x+a}{2}+(1-t)b$$

in Lemma 2. The inequality

$$f\left(t\frac{x+b}{2}+(1-t)y\right)+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \leq f\left(t\frac{x+b}{2}+(1-t)a\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right) \tag{2.15}$$

holds when

$$A=t\frac{x+b}{2}+(1-t)a, \quad C=t\frac{x+b}{2}+(1-t)y,$$
$$D=t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right) \qquad \text{and} \qquad B=t\frac{x+b}{2}+(1-t)\frac{a+b}{2}$$

in Lemma 2. The inequality

$$f\left(t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)+f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right) \leq f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)b\right) \tag{2.16}$$

holds when

$$A=t\frac{x+b}{2}+(1-t)\frac{a+b}{2}, \qquad C=t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right),$$
$$D=t\frac{x+b}{2}+(1-t)(a+b-y) \qquad \text{and} \qquad B=t\frac{x+b}{2}+(1-t)b$$

in Lemma 2.

Multiplying the inequalities (2.13) and (2.16) by $g(x)\,g(2y-a)$, integrating them over x on $[a,b]$, over y on $\left[a,\frac{3a+b}{4}\right]$ and using identity (2.12), we have the inequality

$$2K(t) \leq \left[I(t)+S_g(1-t)\right]\int_a^b g(x)\,dx, \tag{2.17}$$

for all $t\in[0,1]$. Using (2.3) and (2.17), we derive (2.11). This completes the proof. □

Remark 7. *Let $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ in Theorem 6. Then $K(t)=F(t)$, $I(t)=H(t)$, $S_g(1-t)=L(1-t)$ $(t\in[0,1])$ and Theorem 6 reduces to Theorem I.*

The following two Fejér-type inequalities are natural consequences of Theorems 3, 6, E, G, H and we omit their proofs.

Theorem 8. *Let f, g, G, I, K, L_g, S_g be defined as above. Then, for all $t \in [0,1]$, we have*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &\le I(t)\int_a^b g(x)\,dx \le K(t) \\
&\le \frac{1}{2}\left[I(t)+S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\left[G(t)\int_a^b g(x)\,dx + S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\left[L_g(t)+S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\Bigg((1-t)\int_a^b f(x)\,g(x)\,dx \\
&\qquad + t\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\qquad + \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\Bigg)\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\left(\int_a^b g(x)\,dx\right)^2 \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &\le I(t)\int_a^b g(x)\,dx \le K(t) \\
&\le \frac{1}{2}\left[I(t)+S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\left[G(t)\int_a^b g(x)\,dx + S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\left[S_g(t)+S_g(1-t)\right]\int_a^b g(x)\,dx \\
&\le \frac{1}{2}\Bigg(\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\qquad + \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\Bigg)\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\left(\int_a^b g(x)\,dx\right)^2. \qquad (2.19)
\end{aligned}
$$

Let $g(x) = \frac{1}{b-a}\ (x \in [a,b])$. Then we have the following Hermite-Hadamard-type inequality which is a natural consequence of Theorem 8.

Corollary 9. *Let f, g, G, H, F, L be defined as above. Then, for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H(t) \leq F(t) \leq \frac{1}{2}[H(t) + L(1-t)] \\ &\leq \frac{1}{2}[G(t) + L(1-t)] \leq \frac{1}{2}[L(t) + L(1-t)] \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.20}$$

References

[1] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.* 58 (1893), 171-215.

[2] S. S. Dragomir, Two Mappings in Connection to Hadamard's Inequalities, *J. Math. Anal. Appl.* 167 (1992), 49-56.

[3] S. S. Dragomir, A Refinement of Hadamard's Inequality for Isotonic Linear Functionals, *Tamkang. J. Math.*, 24 (1993), 101-106.

[4] S. S. Dragomir, On the Hadamard's Inequality for Convex Functions on the Co-ordinates in a Rectangle from the Plane, *Taiwanese J. Math.*, 5 (4) (2001), 775-788.

[5] S. S. Dragomir, Further Properties of Some Mappings Associated with Hermite-Hadamard Inequalities, *Tamkang. J. Math.* 34 (1) (2003), 45-57.

[6] S. S. Dragomir, Y.-J. Cho and S.-S. Kim, Inequalities of Hadamard's type for Lipschitzian Mappings and Their Applications, *J. Math. Anal. Appl.* 245 (2000),489-501.

[7] S. S. Dragomir, D. S. Milošević and József Sándor, On Some Refinements of Hadamard's Inequalities and Applications, *Univ. Belgrad. Publ. Elek. Fak. Sci. Math.* 4 (1993), 3-10.

[8] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.* 24 (1906),369-390.(In Hungarian).

[9] D.-Y. Hwang, K.-L. Tseng and G.-S. Yang, Some Hadamard's Inequalities for Co-ordinated Convex Functions in a Rectangle from the Plane, *Taiwanese J. Math.*, 11 (1) (2007), 63-73.

[10] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir, On Some New Inequalities of Hermite-Hadamard-Fejér Type Involving Convex Functions, *Demonstratio Math.* XL (1) (2007), 51-64.

[11] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir, Fejér-type Inequalities (I). (Submitted), Preprint *RGMIA Res. Rep. Coll.* **12**(2009), No.4, Article 5. [Online http://www.staff.vu.edu.au/RGMIA/v12n4.asp].

[12] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir, Fejér-type Inequalities (II). (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 15.[Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp].

[13] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir, Some Companions of Fejer's Inequality for Convex Functions. (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 19.[Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp].

[14] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir,On Some Weighted Integral Inequalities for Convex Functions Related to Fejér's Result (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 20.[Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp].

[15] K.-L. Tseng, G.-S. Yang and K.-C. Hsu, On Some Inequalities of Hadamard's Type and Applications, *Taiwanese J. Math.*, 13.

[16] G.-S. Yang and M.-C. Hong, A Note on Hadamard's Inequality, *Tamkang. J. Math.* 28 (1) (1997), 33-37.

[17] G.-S. Yang and K.-L. Tseng, On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities, *J. Math. Anal. Appl.* 239 (1999), 180-187.

[18] G.-S. Yang and K.-L. Tseng, Inequalities of Hadamard's Type for Lipschitzian Mappings, *J. Math. Anal. Appl.* 260 (2001),230-238.

[19] G.-S. Yang and K.-L. Tseng, On Certain Multiple Integral Inequalities Related to Hermite-Hadamard Inequalities, *Utilitas Math.,* 62 (2002), 131-142.

[20] G.-S. Yang and K.-L. Tseng, Inequalities of Hermite-Hadamard-Fejér Type for Convex Functions and Lipschitzian Functions, *Taiwanese J. Math.*, 7 (3) (2003), 433-440.

Department of Mathematics, Aletheia University, Tamsui, Taiwan 25103.
E-mail address: kltseng@email.au.edu.tw

China University of Science and Technology, Nankang, Taipei, Taiwan 11522
E-mail address: hsru@cc.cust.edu.tw

School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: sever.dragomir@.vu.edu.au
URL: http://www.staff.vu.edu.au/RGMIA/dragomir/