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WEIGHTED f -GINI MEAN DIFFERENCE FOR CONVEX AND SYMMETRIC FUNCTIONS IN LINEAR SPACES

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ABSTRACT. The concept of *weighted f -Gini mean difference* for convex and symmetric functions in linear spaces is introduced. Some fundamental inequalities and applications for norms are also provided.

1. INTRODUCTION

The *Gini mean difference* of the sample $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ is defined by

$$G(\mathbf{a}) = \frac{1}{2n^2} \sum_{j=1}^n \sum_{i=1}^n |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|$$

and

$$R(\mathbf{a}) = \frac{1}{\bar{a}} G(\mathbf{a})$$

is the Gini index of \mathbf{a} , provided the sample mean \bar{a} is not zero [7, p. 257].

The Gini index of \mathbf{a} equals the Gini mean difference of the “scaled down” sample $\tilde{\mathbf{a}} = (\frac{a_1}{\bar{a}}, \dots, \frac{a_n}{\bar{a}})$ ($\bar{a} \neq 0$)

$$R(a_1, \dots, a_n) = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n \left| \frac{a_i}{\bar{a}} - \frac{a_j}{\bar{a}} \right|.$$

The following elementary properties of the Gini index for an empirical distribution of nonnegative data hold [7, p. 257]:

(i) Let $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n a_i > 0$. Then

$$0 = R(\bar{a}, \dots, \bar{a}) \leq R(a_1, \dots, a_n) \leq R\left(0, \dots, 0, \sum_{i=1}^n a_i\right) = 1 - \frac{1}{n} < 1,$$

$$R(\beta a_1, \dots, \beta a_n) = R(a_1, \dots, a_n) \quad \text{for every } \beta > 0$$

and

$$R(a_1 + \lambda, \dots, a_n + \lambda) = \frac{\bar{a}}{\bar{a} + \lambda} R(a_1, \dots, a_n) \quad \text{for } \lambda > 0.$$

(ii) R is a continuous function on \mathbb{R}_+^n .

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These and other properties have been investigated in [7], [4] and [5].

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability sequence, meaning that $p_i \geq 0$ ($i \in \{1, \dots, n\}$) and $\sum_{i=1}^n p_i = 1$, we considered in [1] the *weighted Gini mean difference* defined by formula

$$(1.1) \quad G(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n p_i p_j |a_i - a_j| = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|,$$

and proved that

$$(1.2) \quad \frac{1}{2} K(\mathbf{p}, \mathbf{a}) \leq G(\mathbf{p}, \mathbf{a}) \leq \inf_{\gamma \in \mathbb{R}} \left[\sum_{i=1}^n p_i |a_i - \gamma| \right] \leq K(\mathbf{p}, \mathbf{a}),$$

where $K(\mathbf{p}, \mathbf{a})$ is the *mean absolute deviation*, namely

$$(1.3) \quad K(\mathbf{p}, \mathbf{a}) := \sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right|.$$

We have also shown that if more information on the sampling data $\mathbf{a} = (a_1, \dots, a_n)$ is available, i.e., there exists the real numbers a and A such that $a \leq a_i \leq A$ for each $i \in \{1, \dots, n\}$, then

$$(1.4) \quad G(\mathbf{p}, \mathbf{a}) \leq (A - a) \max_{J \subseteq \{1, \dots, n\}} [P_J (1 - P_J)] \quad \left(\leq \frac{1}{4} (A - a) \right),$$

where $P_J := \sum_{j \in J} p_j$. Also, we have shown that

$$(1.5) \quad G(\mathbf{p}, \mathbf{a}) \leq \sum_{i=1}^n p_i \left| a_i - \frac{A + a}{2} \right| \quad \left(\leq \frac{1}{2} (A - a) \right).$$

Notice that in general the bounds for the weighted Gini mean difference $G(\mathbf{p}, \mathbf{a})$ provided by (1.4) and (1.5) cannot be compared to conclude that one is always better than the other [1].

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability sequence, meaning that $p_i \geq 0$ ($i \in \{1, \dots, n\}$) and $\sum_{i=1}^n p_i = 1$, define the *r-weighted Gini mean difference*, for $r \in [1, \infty)$, by the formula [1, p. 291]:

$$(1.6) \quad G_r(\mathbf{p}, \mathbf{a}) := \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n p_i p_j |a_i - a_j|^r = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|^r.$$

For $r = 1$ we have the *weighted Gini mean difference* $G(\mathbf{p}, \mathbf{a})$ of (1.1) which becomes, for the uniform probability distribution $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$ the *Gini mean difference*

$$G(\mathbf{a}) := \frac{1}{2n^2} \sum_{j=1}^n \sum_{i=1}^n |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|.$$

For the uniform probability distribution $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$ we denote

$$G_r(\mathbf{a}) := G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n |a_i - a_j|^r = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|^r.$$

Now, if we define $\Delta := \{(i, j) \mid i, j \in \{1, \dots, n\}\}$, then we can simply write from (1.6)

$$(1.7) \quad G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^r, \quad r \geq 1.$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p}, \mathbf{a})$ may be stated (see [2]):

Theorem 1. *For any $p_i \in (0, 1)$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, we have the inequalities*

$$(1.8) \quad \frac{1}{2} \max_{(i,j) \in \Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} |a_i - a_j|^r \right\} \leq G_r(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|^r,$$

where $r \in [1, \infty)$.

Remark 1. *The case $r = 2$ is of interest, since*

$$G_2(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i \right)^2,$$

for which we can obtain from Theorem 1 the following bounds:

$$(1.9) \quad \frac{1}{2} \max_{(i,j) \in \Delta} \left\{ \frac{p_i p_j}{1 - p_i p_j} (a_i - a_j)^2 \right\} \leq G_2(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} (a_i - a_j)^2.$$

Remark 2. *Since the function*

$$h_r(t) := \frac{t^r + t(1-t)^{r-1}}{(1-t)^{r-1}} = t + t^r(1-t)^{1-r}$$

defined for $t \in [0, 1)$ and $r > 1$ is strictly increasing on $[0, 1)$ from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the r -weighted Gini mean difference, namely (see [2]):

$$(1.10) \quad G_r(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}} \cdot \max_{(i,j) \in \Delta} |a_i - a_j|^r,$$

where p_m is defined above.

For $r = 2$, we then have:

$$(1.11) \quad G_2(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^2}{1 - p_m^2} \cdot \max_{(i,j) \in \Delta} (a_i - a_j)^2.$$

For other results related to the above, see the recent paper [2]. For various inequalities concerning $G_2(\mathbf{p}, \mathbf{a})$, see the book [3] and the references therein.

In this paper, motivated by the above results, we introduce the more general concept of *weighted f -Gini mean difference* for convex and symmetric functions in linear spaces. Moreover, we provide some fundamental inequalities for the new quantity and apply them for norms that naturally extend to vectors the results obtained for real numbers.

2. WEIGHTED f -GINI MEAN DIFFERENCE

Consider $f : X \rightarrow \mathbb{R}$ be a convex function on the linear space X . Assume that $f(0) = 0$ and f is symmetric, i.e., $f(x) = f(-x)$ for any $x \in X$. In these circumstances we have

$$f(x) = \frac{f(x) + f(-x)}{2} \geq f\left(\frac{x-x}{2}\right) = f(0) = 0$$

showing that f is nonnegative on the entire space X .

For $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we define the *weighted f -Gini mean difference* of the n -tuple \mathbf{x} with the probability distribution \mathbf{p} the positive quantity

$$(2.1) \quad G_f(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^n p_i p_j f(x_i - x_j) = \sum_{1 \leq i < j \leq n} p_i p_j f(x_i - x_j) \geq 0.$$

For the uniform distribution $\mathbf{u} = (\frac{1}{n}, \dots, \frac{1}{n}) \in \mathbf{P}^n$ we have the *f -Gini mean difference* defined by

$$G_f(\mathbf{x}) := \frac{1}{2n^2} \sum_{i,j=1}^n f(x_i - x_j) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} f(x_i - x_j).$$

A natural example of such *f -Gini mean difference* can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$G_r(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^r = \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^r.$$

Further on, we need to consider another quantity that is naturally related with *f -Gini mean difference*. For a convex function $f : X \rightarrow \mathbb{R}$ defined on the linear space X with the properties that $f(0) = 0$ define the *mean f -deviation* of an n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ by the non-negative quantity

$$(2.2) \quad K_f(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^n p_i f\left(x_i - \sum_{k=1}^n p_k x_k\right).$$

The fact that $K_f(\mathbf{p}, \mathbf{x})$ is non-negative follows by Jensen's inequality, namely

$$K_f(\mathbf{p}, \mathbf{x}) \geq f\left(\sum_{i=1}^n p_i \left(x_i - \sum_{k=1}^n p_k x_k\right)\right) = f(0) = 0.$$

A natural example of such deviations can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$(2.3) \quad K_r(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^n p_i \left\| x_i - \sum_{k=1}^n p_k x_k \right\|^r$$

and call it the *mean r -absolute deviation* of the n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$.

The following connection between the *f -Gini mean difference* and the mean *f -deviation* holds true:

Theorem 2. *If $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have the inequalities*

$$(2.4) \quad G_f(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} K_f(\mathbf{p}, \mathbf{x}) \geq G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right).$$

Both inequalities in (2.4) are sharp and the constant $\frac{1}{2}$ best possible.

Proof. By the Jensen inequality we have

$$\begin{aligned} G_f(\mathbf{p}, \mathbf{x}) &= \frac{1}{2} \sum_{i=1}^n p_i \left(\sum_{j=1}^n p_j f(x_i - x_j) \right) \\ &\geq \frac{1}{2} \sum_{i=1}^n p_i f\left(\sum_{j=1}^n p_j (x_i - x_j) \right) = \frac{1}{2} \sum_{i=1}^n p_i f\left(x_i - \sum_{j=1}^n p_j x_j \right) \\ &= \frac{1}{2} K_f(\mathbf{p}, \mathbf{x}) \end{aligned}$$

which proves the first part of (2.4).

By the convexity and symmetry of f we also have

$$(2.5) \quad \begin{aligned} f\left(\frac{1}{2}x_i - \frac{1}{2}x_j\right) &= f\left(\frac{1}{2}x_i - \frac{1}{2}\sum_{k=1}^n p_k x_k + \frac{1}{2}\sum_{k=1}^n p_k x_k - \frac{1}{2}x_j\right) \\ &= f\left[\frac{1}{2}\left(x_i - \sum_{k=1}^n p_k x_k\right) + \frac{1}{2}\left(\sum_{k=1}^n p_k x_k - x_j\right)\right] \\ &\leq \frac{1}{2}\left[f\left(x_i - \sum_{k=1}^n p_k x_k\right) + f\left(x_j - \sum_{k=1}^n p_k x_k\right)\right] \end{aligned}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying the inequality (2.5) by $p_i p_j$ and summing over i and j from 1 to n we get

$$\begin{aligned} 2G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right) &\leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j \left[f\left(x_i - \sum_{k=1}^n p_k x_k\right) + f\left(x_j - \sum_{k=1}^n p_k x_k\right) \right] \\ &= K_f(\mathbf{p}, \mathbf{x}) \end{aligned}$$

which proves the last part of (2.4).

Now, if assume that $(X, \|\cdot\|)$ is a normed space and consider $f(x) = \|x\|$, then for $n = 2$ and $p_1, p_2 \in [0, 1]$ with $p_1 + p_2 = 1$ we have

$$G_1(\mathbf{p}, \mathbf{x}) = p_1 p_2 \|x_1 - x_2\| \quad \text{and} \quad K_1(\mathbf{p}, \mathbf{x}) = 2p_1 p_2 \|x_1 - x_2\|$$

which shows that the equality can be realized in (2.4) for a nonzero quantity when $x_1 \neq x_2$ and $p_1, p_2 \in (0, 1)$. \square

The following particular case for norms is of interest due to its natural generalization for the scalar case that is used in applications:

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have*

$$(2.6) \quad G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} K_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2^r} G_r(\mathbf{p}, \mathbf{x})$$

or, equivalently,

$$(2.7) \quad \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^r \geq \sum_{i=1}^n p_i \left\| x_i - \sum_{k=1}^n p_k x_k \right\|^r \geq \frac{1}{2^{r-1}} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^r$$

for any $r \geq 1$.

3. A LOWER BOUND FOR $G_f(\mathbf{p}, \mathbf{x})$

As in the introduction, consider $\Delta := \{(i, j) \mid i, j \in \{1, \dots, n\}\}$ and denote by Γ a nonempty subset of Δ , namely $\Gamma = I \times J$ where I and J are nonempty parts of $\{1, \dots, n\}$. We denote by \mathcal{G}_n the set of all nonempty Γ as above.

We define, for a given probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ and $\Gamma = I \times J$ the following quantities

$$P_\Gamma := \sum_{(i,j) \in \Gamma} p_i p_j = \sum_{i \in I} p_i \sum_{j \in J} p_j = P_I P_J, \text{ and } \bar{P}_\Gamma := P_{\bar{\Gamma}} \text{ where } \bar{\Gamma} := \Delta \setminus \Gamma.$$

Since $P_\Delta = \sum_{(i,j) \in \Delta} p_i p_j = \sum_{i,j=1}^n p_i p_j = 1$, then obviously $P_\Gamma \in [0, 1]$ and $\bar{P}_\Gamma = 1 - P_\Gamma = 1 - P_I P_J$.

We can state the following result:

Theorem 3. *If $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ with all terms nonzero we have the inequality*

$$(3.1) \quad G_f(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} \max_{\mathbf{I} \times \mathbf{J} \in \mathcal{G}_n} L_f(\mathbf{p}, \mathbf{x}, \mathbf{I} \times \mathbf{J}) (\geq 0)$$

where

$$(3.2) \quad L_f(\mathbf{p}, \mathbf{x}, \mathbf{I} \times \mathbf{J}) := P_I P_J f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i - \frac{1}{P_J} \sum_{j \in J} p_j x_j \right) \\ + (1 - P_I P_J) f \left[\frac{P_I P_J}{1 - P_I P_J} \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i - \frac{1}{P_J} \sum_{j \in J} p_j x_j \right) \right].$$

Proof. On applying Jensen's inequality for multiple sums we have for $\Gamma = I \times J$ that

$$\begin{aligned}
(3.3) \quad G_f(\mathbf{p}, \mathbf{x}) &= \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j f(x_i - x_j) \\
&= \frac{1}{2} \sum_{(i,j) \in \Gamma} p_i p_j f(x_i - x_j) + \frac{1}{2} \sum_{(i,j) \in \bar{\Gamma}} p_i p_j f(x_i - x_j) \\
&\geq \frac{1}{2} \sum_{(i,j) \in \Gamma} p_i p_j f \left[\frac{\sum_{(i,j) \in \Gamma} p_i p_j (x_i - x_j)}{\sum_{(i,j) \in \Gamma} p_i p_j} \right] \\
&\quad + \frac{1}{2} \sum_{(i,j) \in \bar{\Gamma}} p_i p_j f \left[\frac{\sum_{(i,j) \in \bar{\Gamma}} p_i p_j (x_i - x_j)}{\sum_{(i,j) \in \bar{\Gamma}} p_i p_j} \right].
\end{aligned}$$

However

$$\begin{aligned}
\sum_{(i,j) \in \Gamma} p_i p_j &= P_I P_J, \quad \sum_{(i,j) \in \bar{\Gamma}} p_i p_j = 1 - P_I P_J \\
\sum_{(i,j) \in \Gamma} p_i p_j (x_i - x_j) &= \sum_{(i,j) \in I \times J} p_i p_j (x_i - x_j) \\
&= P_J \sum_{i \in I} p_i x_i - P_I \sum_{j \in J} p_j x_j
\end{aligned}$$

and

$$\begin{aligned}
\sum_{(i,j) \in \bar{\Gamma}} p_i p_j (x_i - x_j) &= \sum_{(i,j) \in \Delta \setminus \Gamma} p_i p_j (x_i - x_j) \\
&= \sum_{(i,j) \in \Delta} p_i p_j (x_i - x_j) - \sum_{(i,j) \in \Gamma} p_i p_j (x_i - x_j) \\
&= - \sum_{(i,j) \in \Gamma} p_i p_j (x_i - x_j) = - \left(P_J \sum_{i \in I} p_i x_i - P_I \sum_{j \in J} p_j x_j \right).
\end{aligned}$$

Utilising the inequality (3.3) and the symmetry of the function f we get

$$\begin{aligned}
G_f(\mathbf{p}, \mathbf{x}) &\geq \frac{1}{2} P_I P_J f \left(\frac{P_J \sum_{i \in I} p_i x_i - P_I \sum_{j \in J} p_j x_j}{P_I P_J} \right) \\
&\quad + \frac{1}{2} (1 - P_I P_J) f \left(\frac{P_J \sum_{i \in I} p_i x_i - P_I \sum_{j \in J} p_j x_j}{1 - P_I P_J} \right).
\end{aligned}$$

which is equivalent with the desired inequality (3.2). \square

A particular case of interest is for $I = \{i\}$ and $J = \{j\}$ giving the following

Corollary 2. *If $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then we have the inequality*

$$(3.4) \quad G_f(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} \max_{(i,j) \in \Delta} L_f(\mathbf{p}, \mathbf{x}, i, j) (\geq 0)$$

where

$$(3.5) \quad L_f(\mathbf{p}, \mathbf{x}, i, j) := p_i p_j f(x_i - x_j) + (1 - p_i p_j) f \left[\frac{p_i p_j}{1 - p_i p_j} (x_i - x_j) \right].$$

Remark 3. Let $(X, \|\cdot\|)$ be a normed space. Then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ with all terms nonzero we have from (3.1) that

$$(3.6) \quad G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} \left[P_I P_J + (1 - P_I P_J)^{1-r} P_I^r P_J^r \right] \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i - \frac{1}{P_J} \sum_{j \in J} p_j x_j \right\|^r$$

for any $I \times J \in \mathcal{G}_n$.

In particular, we have

$$(3.7) \quad G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} \left[p_i p_j + (1 - p_i p_j)^{1-r} p_i^r p_j^r \right] \|x_i - x_j\|^r$$

for any $(i, j) \in \Delta$, which extends and improves the scalar case obtained in [2].

4. ANOTHER LOWER BOUND

Before we provide another lower bound for the weighted f -Gini mean difference we need to introduce some notations.

For the vector $\mathbf{e} = (1, 2, \dots, n) \in \mathbb{R}^n$ and the probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we define the weighted arithmetic mean

$$A_j(\mathbf{p}, \mathbf{e}) := \sum_{k=1}^j k p_k$$

and, similarly, for the n -tuple $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and the probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we define

$$A_j(\mathbf{p}, \mathbf{x}) := \sum_{k=1}^j p_k x_k.$$

Theorem 4. If $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have the inequality

$$(4.1) \quad G_f(\mathbf{p}, \mathbf{x}) \geq G(\mathbf{p}, \mathbf{e}) f\left(\frac{G(\mathbf{p}, \mathbf{x})}{G(\mathbf{p}, \mathbf{e})}\right) (\geq 0)$$

where

$$G(\mathbf{p}, \mathbf{e}) := \sum_{j=1}^n p_j [j P_j - A_j(\mathbf{p}, \mathbf{e})] > 0$$

and

$$G(\mathbf{p}, \mathbf{x}) := \sum_{j=1}^n p_j [P_j x_j - A_j(\mathbf{p}, \mathbf{x})].$$

Proof. By the convexity of f and the fact that $f(0) = 0$ we have that

$$f(tx) = f[(1-t) \cdot 0 + t \cdot x] \leq (1-t)f(0) + tf(x) = tf(x)$$

for any $t \in [0, 1]$ and $x \in X$.

Now, if $1 \leq i < j \leq n$ then $t := \frac{1}{j-i} \in [0, 1]$ and writing the above inequality for this t and for $x = x_j - x_i$ we get

$$(4.2) \quad f(x_j - x_i) \geq (j-i) f\left(\frac{x_j - x_i}{j-i}\right)$$

for any $1 \leq i < j \leq n$.

Multiplying (4.2) with $p_i p_j \geq 0$, summing over i, j with $1 \leq i < j \leq n$ and applying the Jensen inequality for multiple sums we get successively

$$\begin{aligned}
 (4.3) \quad G_f(\mathbf{p}, \mathbf{x}) &= \sum_{1 \leq i < j \leq n} p_i p_j f(x_i - x_j) \geq \sum_{1 \leq i < j \leq n} p_i p_j (j - i) f\left(\frac{x_i - x_j}{j - i}\right) \\
 &\geq \sum_{1 \leq i < j \leq n} p_i p_j (j - i) f\left[\sum_{1 \leq i < j \leq n} p_i p_j (j - i) \left(\frac{x_i - x_j}{j - i}\right)\right] \\
 &= \sum_{1 \leq i < j \leq n} p_i p_j (j - i) f\left[\sum_{1 \leq i < j \leq n} p_i p_j (x_i - x_j)\right].
 \end{aligned}$$

However

$$\begin{aligned}
 (0 <) \quad \sum_{1 \leq i < j \leq n} p_i p_j (j - i) &= \sum_{j=1}^n p_j \sum_{i=1}^j p_i (j - i) \\
 &= \sum_{j=1}^n p_j \left(P_j - \sum_{i=1}^j i p_i\right) = G(\mathbf{p}, \mathbf{e})
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{1 \leq i < j \leq n} p_i p_j (x_i - x_j) &= \sum_{j=1}^n p_j \sum_{i=1}^j p_i (x_j - x_i) \\
 &= \sum_{j=1}^n p_j \left(P_j x_j - \sum_{i=1}^j i x_i\right) = G(\mathbf{p}, \mathbf{x})
 \end{aligned}$$

which together with (4.3) produces the desired result (4.1). \square

The case of normed linear spaces is of interest. We have:

Corollary 3. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have*

$$(4.4) \quad G_r(\mathbf{p}, \mathbf{x}) \geq G^{1-r}(\mathbf{p}, \mathbf{e}) \|G(\mathbf{p}, \mathbf{x})\|^r (\geq 0)$$

or, equivalently,

$$\begin{aligned}
 (4.5) \quad \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^r \\
 \geq \left(\sum_{j=1}^n p_j [j P_j - A_j(\mathbf{p}, \mathbf{e})] \right)^{1-r} \left\| \sum_{j=1}^n p_j [P_j x_j - A_j(\mathbf{p}, \mathbf{x})] \right\|^r
 \end{aligned}$$

for any $r \geq 1$.

5. SOME UPPER BOUNDS

From the definition of the weighted f -Gini mean difference we have

$$(5.1) \quad G_f(\mathbf{p}, \mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^n p_i p_j f(x_i - x_j) \\ \leq \max_{1 \leq i < j \leq n} f(x_i - x_j) \frac{1}{2} \sum_{i,j=1}^n p_i p_j = \frac{1}{2} \max_{1 \leq i < j \leq n} f(x_i - x_j)$$

and

$$(5.2) \quad G_f(\mathbf{p}, \mathbf{x}) = \sum_{1 \leq i < j \leq n} p_i p_j f(x_i - x_j) \leq \max_{1 \leq i < j \leq n} f(x_i - x_j) \sum_{1 \leq i < j \leq n} p_i p_j \\ = \frac{1}{2} \max_{1 \leq i < j \leq n} f(x_i - x_j) \left(1 - \sum_{i=1}^n p_i^2 \right) = \frac{1}{2} \max_{1 \leq i < j \leq n} f(x_i - x_j) \sum_{i=1}^n p_i (1 - p_i)$$

since, obviously,

$$\sum_{1 \leq i < j \leq n} p_i p_j = \frac{1}{2} \left(\sum_{i,j=1}^n p_i p_j - \sum_{i=1}^n p_i^2 \right) = \frac{1}{2} \left(1 - \sum_{i=1}^n p_i^2 \right) = \frac{1}{2} \sum_{i=1}^n p_i (1 - p_i).$$

Observe that the second approach provides a better inequality, therefore we can state the following result:

Proposition 1. *If $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have the inequality*

$$(5.3) \quad \frac{1}{2} \max_{1 \leq i < j \leq n} f(x_i - x_j) \sum_{i=1}^n p_i (1 - p_i) \geq G_f(\mathbf{p}, \mathbf{x}).$$

The following particular case for norms holds true:

Corollary 4. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have*

$$\frac{1}{2} \max_{1 \leq i < j \leq n} \|x_i - x_j\|^r \sum_{i=1}^n p_i (1 - p_i) \geq G_r(\mathbf{p}, \mathbf{x})$$

for any $r \geq 1$.

For two vectors $x, y \in X$ we define the segment $[x, y]$ by $\{(1-t)x + ty, t \in [0, 1]\}$. If $0 \in [x, y]$, then there exists a unique $t \in [0, 1]$ such that $(1-t)x + ty = 0$.

The following result may be stated as well:

Theorem 5. *Assume that $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If x and y are two vectors and $t \in [0, 1]$ with $(1-t)x + ty = 0$ then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ with the property that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, \dots, n\}$ we have the inequality*

$$(5.4) \quad \frac{1}{2} [(1-t)f(x) + tf(y)] \geq G_f(\mathbf{p}, \mathbf{x})$$

for any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$.

Proof. Since $x_i - x_j \in [x, y]$ for $i, j \in \{1, \dots, n\}$, then there exists the numbers $t_{ij} \in [0, 1]$ such that $x_i - x_j = (1 - t_{ij})x + t_{ij}y$ for $i, j \in \{1, \dots, n\}$.

Let $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$. Then by the above equality we get that

$$p_i p_j (x_i - x_j) = (1 - t_{ij}) p_i p_j x + t_{ij} p_i p_j y$$

for any $i, j \in \{1, \dots, n\}$. If we sum over i, j from 1 to n , then we get

$$(5.5) \quad 0 = \sum_{i,j=1}^n p_i p_j (x_i - x_j) = \sum_{i,j=1}^n [(1 - t_{ij}) p_i p_j x + t_{ij} p_i p_j y] \\ = \left(1 - \sum_{i,j=1}^n t_{ij} p_i p_j\right) x + \left(\sum_{i,j=1}^n t_{ij} p_i p_j\right) y.$$

Now, due to the fact that $(1 - t)x + ty = 0$ and the representation is unique, we get that $t = \sum_{i,j=1}^n t_{ij} p_i p_j$.

On the other hand, due to the convexity of the function f we have that

$$G_f(\mathbf{p}, \mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^n p_i p_j f(x_i - x_j) \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j f[(1 - t_{ij})x + t_{ij}y] \\ \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j [(1 - t_{ij})f(x) + t_{ij}f(y)] \\ = \frac{1}{2} \left[\left(1 - \sum_{i,j=1}^n t_{ij} p_i p_j\right) f(x) + \left(\sum_{i,j=1}^n t_{ij} p_i p_j\right) f(y) \right] \\ = \frac{1}{2} [(1 - t)f(x) + tf(y)]$$

and the theorem is proved. \square

In applications one may be able to provide the "smallest" symmetric interval $[-z, z]$ containing all the differences $x_i - x_j$. In that situation we can state the following particular case of interest:

Corollary 5. *Assume that $f : X \rightarrow \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If z is a nonzero vector in X then for any n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ with the property that $x_i - x_j \in [-z, z]$ for all $i, j \in \{1, \dots, n\}$ we have the inequality*

$$(5.6) \quad \frac{1}{2} f(z) \geq G_f(\mathbf{p}, \mathbf{x})$$

for any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$.

Remark 4. *If X is a normed linear space and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and z satisfy the condition from Corollary 5, then we have the inequality*

$$(5.7) \quad \frac{1}{2} \|z\|^r \geq G_r(\mathbf{p}, \mathbf{x})$$

for each $r \geq 1$.

For an n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and a probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we consider the condition

$$(5.8) \quad x_i - \sum_{j=1}^n p_j x_j \in [x, y] \text{ for any } i \in \{1, \dots, n\}.$$

Since the segment $[x, y]$ is a convex set then $0 = \sum p_j (x_i - \sum_{j=1}^n p_j x_j) \in [x, y]$. Moreover, the fact that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, \dots, n\}$ also imply that the condition (5.8) holds true.

We can state the following result that provides an upper bound for the mean f -deviation $K_f(\mathbf{p}, \mathbf{x})$:

Theorem 6. *Let $f : X \rightarrow \mathbb{R}$ be a convex function defined on the linear space X with the properties that $f(0) = 0$. If, for an n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and a probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$ we have the condition (5.8), then*

$$(5.9) \quad (1-t)f(x) + tf(y) \geq K_f(\mathbf{p}, \mathbf{x}),$$

where t is the unique real number for which we have $(1-t)x + ty = 0$.

The argument is similar to that in the proof of Theorem 5 and the details are omitted.

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