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INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS

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ABSTRACT. In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

1. INTRODUCTION

Let A be a unital normed algebra over the complex number field \mathbb{C} and let $a \in A$. Recall that the numerical radius of a is given by (see [2, p. 15])

$$(1.1) \quad v(a) = \sup \{|f(a)|, f \in A', \|f\| \leq 1 \text{ and } f(1) = 1\},$$

where A' denotes the dual space of A , i.e., the Banach space of all continuous linear functionals on A .

It is known that $v(\cdot)$ is a norm on A that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds true:

$$(1.2) \quad \frac{1}{e} \|a\| \leq v(a) \leq \|a\|$$

for any $a \in A$.

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $\|x\| \leq e\Phi(x)$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $\frac{1}{4}$ in place of $\frac{1}{e}$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M.J. Crabb [3] which proved that

$$(1.3) \quad \|a^n\| \leq n! \left(\frac{e}{n}\right)^n [v(a)]^n, \quad n = 1, 2, \dots$$

for any $a \in A$.

In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra A are given via the celebrated representation result of Lumer from [5].

2. SOME SUBSETS IN A

Let $D(1) := \{f \in A' \mid \|f\| \leq 1 \text{ and } f(1) = 1\}$. For $\lambda \in \mathbb{C}$ and $r > 0$, we define the subset of A by

$$\bar{\Delta}(\lambda, r) := \{a \in A \mid |f(a) - \lambda| \leq r \text{ for each } f \in D(1)\}.$$

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The following result holds.

Proposition 1. *Let $\lambda \in \mathbb{C}$ and $r > 0$. Then $\bar{\Delta}(\lambda, r)$ is a closed convex subset of A and*

$$(2.1) \quad \bar{B}(\lambda, r) \subseteq \bar{\Delta}(\lambda, r),$$

where $\bar{B}(\lambda, r) := \{a \in A \mid \|a - \lambda\| \leq r\}$.

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$\bar{U}(\gamma, \Gamma) := \left\{ a \in A \mid \operatorname{Re} \left[(\Gamma - f(a)) \left(\overline{f(a)} - \bar{\gamma} \right) \right] \geq 0 \text{ for each } f \in D(1) \right\}.$$

The following representation result may be stated.

Proposition 2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have:*

$$(2.2) \quad \bar{U}(\gamma, \Gamma) = \bar{\Delta} \left(\frac{\gamma + \Gamma}{2}, \frac{1}{2} |\Gamma - \gamma| \right).$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple conclusion of this fact. \square

Making use of some obvious properties in \mathbb{C} and for continuous linear functionals, we can state the following corollary as well.

Corollary 1. *For any $\gamma, \Gamma \in \mathbb{C}$, we have*

$$(2.3) \quad \begin{aligned} \bar{U}(\gamma, \Gamma) &= \left\{ a \in A \mid \operatorname{Re} \left[f(\Gamma - a) \overline{f(a - \gamma)} \right] \geq 0 \text{ for each } f \in D(1) \right\} \\ &= \left\{ a \in A \mid (\operatorname{Re} \Gamma - \operatorname{Re} f(a)) (\operatorname{Re} f(a) - \operatorname{Re} \gamma) \right. \\ &\quad \left. + (\operatorname{Im} \Gamma - \operatorname{Im} f(a)) (\operatorname{Im} f(a) - \operatorname{Im} \gamma) \geq 0 \text{ for each } f \in D(1) \right\}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following subset of A :

$$(2.4) \quad \bar{S}(\gamma, \Gamma) := \{a \in A \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(a) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(a) \geq \operatorname{Im}(\gamma) \text{ for each } f \in D(1)\}.$$

One can easily observe that $\bar{S}(\gamma, \Gamma)$ is closed, convex and

$$(2.5) \quad \bar{S}(\gamma, \Gamma) \subseteq \bar{U}(\gamma, \Gamma).$$

3. SEMI-INNER PRODUCTS AND LUMER'S THEOREM

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . The mapping $f : X \rightarrow \mathbb{R}$, $f(x) = \frac{1}{2} \|x\|^2$ is obviously convex and then there exists the following limits:

$$\begin{aligned}\langle x, y \rangle_i &= \lim_{t \rightarrow 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t}, \\ \langle x, y \rangle_s &= \lim_{t \rightarrow 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}\end{aligned}$$

for every two elements $x, y \in X$. The mapping $\langle \cdot, \cdot \rangle_s$ ($\langle \cdot, \cdot \rangle_i$) will be called the *superior semi-inner product* (the *interior semi-inner product*) associated to the norm $\|\cdot\|$.

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]):

- (i) $\langle x, x \rangle_p = \|x\|^2$; $\langle ix, x \rangle_p = \langle x, ix \rangle_p = 0$, $x \in X$;
- (ii) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_p$; $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_p$ for $\lambda \geq 0$, $x, y \in X$;
- (iii) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_q$; $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_q$ for $\lambda < 0$, $x, y \in X$;
- (iv) $\langle ix, y \rangle_p = -\langle x, iy \rangle_p$; $\langle \alpha x, \beta y \rangle_p = \alpha\beta \langle x, y \rangle$ if $\alpha\beta \geq 0$, $x, y \in X$;
- (v) $\langle -x, y \rangle_p = \langle x, -y \rangle_p = -\langle x, y \rangle_q$, $x, y \in X$;
- (vi) $|\langle x, y \rangle_p| \leq \|x\| \|y\|$, $x, y \in X$;
- (vii) $\langle x_1 + x_2, y \rangle_{s(i)} \leq (\geq) \langle x_1, y \rangle_{s(i)} + \langle x_2, y \rangle_{s(i)}$ for $x_1, x_2, y \in X$;
- (ix) $\langle \alpha x + y, x \rangle_p = \alpha \|x\|^2 + \langle y, x \rangle_p$, $\alpha \in \mathbb{R}$, $x, y \in X$;
- (x) $|\langle y + z, x \rangle_p - \langle z, x \rangle_p| \leq \|y\| \|x\|$, $x, y, z \in X$;
- (xi) The mapping $\langle \cdot, x \rangle_p$ is continuous on $(X, \|\cdot\|)$ for each $x \in X$, where $p, q \in \{s, i\}$ and $p \neq q$.

The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.

Theorem 1. *Let A be a unital normed algebra over \mathbb{K} ($\mathbb{K} = \mathbb{C}, \mathbb{R}$). For each $a \in A$,*

$$(3.1) \quad \max \{ \operatorname{Re} \lambda \mid \lambda \in V(a) \} = \inf_{\alpha > 0} \frac{1}{\alpha} [\|1 + \alpha a\| - 1] = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} [\|1 + \alpha a\| - 1],$$

where $V(a)$ is the numerical range of a (see for instance [2, p. 15]).

Remark 1. *In terms of semi-inner products, the above identity can be stated as:*

$$(3.2) \quad \max \{ \operatorname{Re} f(a) \mid f \in D(1) \} = \langle a, 1 \rangle_s.$$

The following result that provides more information may be stated.

Theorem 2. *For any $a \in A$, we have:*

$$(3.3) \quad \langle a, 1 \rangle_{v,s} = \langle a, 1 \rangle_s,$$

where

$$\langle a, b \rangle_{v,s} := \lim_{t \rightarrow 0^+} \frac{v^2(b + ta) - v^2(b)}{2t}$$

is the superior semi-inner product associated with the numerical radius.

Proof. Since $v(a) \leq \|a\|$, we have:

$$\begin{aligned} \langle a, 1 \rangle_{v,s} &= \lim_{t \rightarrow 0^+} \frac{v^2(1+ta) - v^2(1)}{2t} = \lim_{t \rightarrow 0^+} \frac{v^2(1+ta) - 1}{2t} \\ &\leq \lim_{t \rightarrow 0^+} \frac{\|1+ta\|^2 - 1}{2t} = \langle a, 1 \rangle_s. \end{aligned}$$

Now, let $f \in D(1)$. Then, for each $\alpha > 0$,

$$f(a) = \frac{1}{\alpha} [f(1 + \alpha a) - f(1)] = \frac{1}{\alpha} [f(1 + \alpha a) - 1],$$

giving

$$\begin{aligned} \operatorname{Re} f(a) &= \frac{1}{\alpha} [\operatorname{Re} f(1 + \alpha a) - f(1)] \leq \frac{1}{\alpha} [|f(1 + \alpha a)| - 1] \\ &\leq \frac{1}{\alpha} [v(1 + \alpha a) - 1]. \end{aligned}$$

Taking the infimum over $\alpha > 0$, we deduce

$$\begin{aligned} (3.4) \quad \operatorname{Re} f(a) &\leq \inf_{\alpha > 0} \left[\frac{1}{\alpha} [v(1 + \alpha a) - 1] \right] = \lim_{\alpha \rightarrow 0^+} \left[\frac{v^2(1 + \alpha a) - 1}{2\alpha} \right] \\ &= \lim_{\alpha \rightarrow 0^+} \frac{v(1 + \alpha a) - 1}{\alpha} = \langle a, 1 \rangle_{v,s}. \end{aligned}$$

If we now take the supremum over $f \in D(1)$ in (3.4), we obtain:

$$\sup \{ \operatorname{Re} f(a) \mid f \in D(1) \} \leq \langle a, 1 \rangle_{v,s}$$

which gives, by Lumer's identity that $\langle a, 1 \rangle_s \leq \langle a, 1 \rangle_{v,s}$. □

Corollary 2. *We have the inequality*

$$(3.5) \quad |\langle a, 1 \rangle_s| \leq v(a) \quad (\leq \|a\|).$$

Proof. Schwarz's inequality for the norm $v(\cdot)$ gives that

$$\left| \langle a, 1 \rangle_{v,s} \right| \leq v(a) v(1) = v(a),$$

and by (3.3), the inequality (3.5) is proved. □

4. REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS

Utilising the inequality (3.5) we observe that for any complex number β located in the closed disc centered in 0 and with radius 1 we have $|\langle \beta a, 1 \rangle_s|$ as a lower bound for the numerical radius $v(a)$. Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element a in the unital normed algebra A and the scalar β . A number of results answering this question are incorporated in the following theorems.

Theorem 3. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$(4.1) \quad v(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s + \frac{1}{2} \cdot \frac{r^2}{|\lambda|}.$$

Proof. Since $a \in \bar{\Delta}(\lambda, 1)$, then $|f(a) - \lambda|^2 \leq r^2$, for each $f \in D(1)$, giving that

$$(4.2) \quad |f(a)|^2 + |\lambda|^2 \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2$$

for each $f \in D(1)$.

Taking the supremum of $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce

$$(4.3) \quad v^2(a) + |\lambda|^2 \leq 2 \langle \bar{\lambda}a, 1 \rangle_s + r^2$$

which is an inequality of interest in itself.

On the other hand, we have the elementary inequality

$$(4.4) \quad 2v(a)|\lambda| \leq v^2(a) + |\lambda|^2,$$

which, together with (4.3) implies the desired result (4.1). \square

Remark 2. Notice that, by the inclusion (2.1) a sufficient condition for (4.1) to holds is that $a \in \bar{B}(\lambda, r)$.

Corollary 3. Let $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \pm\gamma$. If $a \in \bar{U}(\gamma, \Gamma)$, then

$$(4.5) \quad v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s + \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.$$

Remark 3. If $M > m \geq 0$ and $a \in \bar{U}(m, M)$, then

$$(4.6) \quad (0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M}.$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to holds is that $M \geq \operatorname{Re} f(a)$, $\operatorname{Im} f(a) \geq m$ for any $f \in D(1)$.

The following result may be stated as well.

Theorem 4. Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then

$$(4.7) \quad v(a) \leq \left\langle \frac{\bar{\lambda}}{\sqrt{|\lambda|^2 - r^2}} a, 1 \right\rangle_s$$

and, equivalently,

$$(4.8) \quad v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s^2 + \frac{r^2}{|\lambda|^2} \cdot v^2(a).$$

Proof. Since $|\lambda| > r$, hence by (4.3) we have, on dividing by $\sqrt{|\lambda|^2 - r^2} > 0$, that

$$(4.9) \quad \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda}a, 1 \rangle_s.$$

On the other hand, we also have

$$2v(a) \leq \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}$$

which, together with (4.9), gives

$$(4.10) \quad v(a) \leq \frac{1}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda}a, 1 \rangle_s.$$

Taking the square in (4.10), we have

$$v^2(a) \left(|\lambda|^2 - r^2 \right) \leq \langle \bar{\lambda}a, 1 \rangle_s^2,$$

which is clearly equivalent to (4.7). \square

Corollary 4. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,*

$$(4.11) \quad v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} a, 1 \right\rangle_s.$$

Remark 4. *If $M \geq m > 0$ and $a \in \bar{U}(m, M)$, then*

$$(4.12) \quad v(a) \leq \frac{M + m}{2\sqrt{mM}} \langle a, 1 \rangle_s,$$

or, equivalently,

$$(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \langle a, 1 \rangle_s \quad \left(\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \|a\| \right).$$

The following result may be stated as well.

Theorem 5. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$(4.13) \quad v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s^2 + 2 \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right) \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s.$$

Proof. Since (by (4.2)) $\operatorname{Re}[f(\bar{\lambda}a)] > 0$, then dividing by it in (4.2) gives:

$$\frac{|f(a)|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} + \frac{|\lambda|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} \leq 2 + \frac{r^2}{\operatorname{Re}[f(\bar{\lambda}a)]},$$

which is clearly equivalent to:

$$(4.14) \quad \frac{|f(a)|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} \leq 2 + \frac{r^2}{\operatorname{Re}[f(\bar{\lambda}a)]} - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{|\lambda|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} =: I.$$

Since

$$(4.15) \quad \begin{aligned} I &= 2 - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{(|\lambda|^2 - r^2)}{\operatorname{Re}[f(\bar{\lambda}a)]} \\ &= 2 - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|} - \left[\frac{\sqrt{\operatorname{Re}[f(\bar{\lambda}a)]}}{|\lambda|} - \frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{\operatorname{Re}[f(\bar{\lambda}a)]}} \right]^2 \\ &\leq 2 \left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|} \right)^2} \right), \end{aligned}$$

hence by (4.14) and (4.15) we have

$$(4.16) \quad |f(a)|^2 \leq \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} + 2 \left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|} \right)^2} \right) \operatorname{Re}[f(\bar{\lambda}a)].$$

Taking the supremum in $f \in D(1)$ and utilising Lumer's result, we deduce the desired inequality (4.13). \square

Corollary 5. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,*

$$v^2(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s^2 + 2 \left(\left| \frac{\gamma + \Gamma}{2} \right| - \sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right) \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s.$$

Remark 5. *If $M > m \geq 0$ and $a \in \bar{U}(m, M)$, then*

$$(0 \leq) v^2(a) - \langle a, 1 \rangle_s^2 \leq \left(\sqrt{M} - \sqrt{m} \right)^2 \langle a, 1 \rangle_s \left(\leq \left(\sqrt{M} - \sqrt{m} \right)^2 \|a\| \right).$$

Finally, the following result can be stated as well.

Theorem 6. *Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$(4.17) \quad v(a) \leq \left(|\lambda| + \sqrt{|\lambda|^2 - r^2} \right) \left\langle \frac{\bar{\lambda}}{r^2} a, 1 \right\rangle_s + \frac{|\lambda| \left(|\lambda| + \sqrt{|\lambda|^2 - r^2} \right) \left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right)}{2r^2}.$$

Proof. From the proof of Theorem 3 above, we have

$$(4.18) \quad |f(a)|^2 + |\lambda|^2 \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2$$

which is equivalent with

$$(4.19) \quad \begin{aligned} |f(a)|^2 + \left(|\lambda| + \sqrt{|\lambda|^2 - r^2} \right)^2 \\ \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2 - |\lambda|^2 + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2 \\ = 2 \operatorname{Re}[f(\bar{\lambda}a)] + |\lambda|^2 - 2|\lambda| \sqrt{|\lambda|^2 - r^2}. \end{aligned}$$

Taking the supremum in (4.19) over $f \in D(1)$ and utilising Lumer's representation theorem, we get:

$$(4.20) \quad v^2(a) + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2 \leq 2 \langle \bar{\lambda}a, 1 \rangle_s + |\lambda| \left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right).$$

Since $r \neq 0$, then $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$, giving

$$(4.21) \quad 2 \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right) v(a) \leq v^2(a) + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right)^2.$$

Now, utilising (4.20) and (4.21), we deduce

$$v(a) \leq \frac{1}{|\lambda| - \sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda}a, 1 \rangle_s + \frac{|\lambda| \left(|\lambda| - 2\sqrt{|\lambda|^2 - r^2} \right)}{2 \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right)},$$

which is clearly equivalent with the desired result (4.17). \square

Remark 6. If $M > m \geq 0$ and $a \in \bar{U}(m, M)$, then

$$v(a) \leq \frac{M + m}{(\sqrt{M} - \sqrt{m})^2} \left[\langle a, 1 \rangle_s + \frac{1}{2} \left(\frac{m + M}{2} - 2\sqrt{mM} \right) \right].$$

In particular, if $a \in \bar{U}(0, \delta)$ with $\delta > 0$, then we have the following reverse inequality as well

$$(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4} \delta.$$

REFERENCES

- [1] H.F. BOHNENBLUST and S. KARLIN, Geometrical properties of the unit sphere of Banach algebras, *Ann. of Math.*, **62** (1955), 217-229.
- [2] F.F. BONSALL and J. DUNCAN, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge University Press, 1971.
- [3] M.J. CRABB, Numerical range estimates for the norm of iterated operators, *Glasgow Math. J.*, **11** (1970), 85-87.
- [4] S.S. DRAGOMIR, *Semi-inner Products and Applications*. Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+222 pp.
- [5] G. LUMER, Semi-inner product spaces, *Trans. Amer. Math. Soc.*, **100** (1961), 29-43.
- [6] B.W. GLICKFELD, On an inequality of Banach algebra geometry and semi-inner product space theory, *Illinois J. Math.*, **14** (1970), 76-81.

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