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Refinements of the Cauchy-Bunyakovsky-Schwarz Inequality for Functions of Selfadjoint Operators in Hilbert Spaces

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ABSTRACT. Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces that improve the Cauchy-Bunyakovsky-Schwarz inequality, are given.

1. Introduction

In [1], Daykin, Elizer and Carlitz obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality, which, in the version from [5, p. 87], can be stated as:

$$(DEC) \quad \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n \varphi(a_i, b_i) \sum_{i=1}^n \psi(a_i, b_i) \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2,$$

where and $a_i, b_i \in [0, \infty)$ for each $i \in \{1, \dots, n\}$ and (φ, ψ) is a pair of functions defined on $[0, \infty) \times [0, \infty)$ and satisfying the conditions

- (i) $\varphi(a, b) \psi(a, b) = a^2 b^2$ for any $a, b \in [0, \infty)$;
- (ii) $\varphi(ka, kb) = k^2 \varphi(a, b)$ for any $a, b, k \in [0, \infty)$;
- (iii) $\frac{b\varphi(a,1)}{a\varphi(b,1)} + \frac{a\varphi(b,1)}{b\varphi(a,1)} \leq \frac{a}{b} + \frac{b}{a}$ for any $a, b \in (0, \infty)$.

As examples of such pairs of functions, which will be called for simplicity *(DEC)-pairs*, we can indicate the following functions: $\varphi(a, b) = a^2 + b^2$, $\psi(a, b) = \frac{a^2 b^2}{a^2 + b^2}$ and $\varphi(a, b) = a^{1+\alpha} b^{1-\alpha}$, $\psi(a, b) = a^{1-\alpha} b^{1+\alpha}$ with $\alpha \in [0, 1]$. The first pair generates the famous *Milne's inequality*:

$$(1.1) \quad \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n (a_i^2 + b_i^2) \sum_{i=1}^n \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2,$$

while the second generates the *Callebaut's inequality*:

$$(1.2) \quad \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^n a_i^{1-\alpha} b_i^{1+\alpha} \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2.$$

It is an open problem for the author to find other nice and simple examples of such pair of functions.

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In order to state the operator version of this result we recall the Gelfand functional calculus.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The *Gelfand map* establishes a $*$ -isometric isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A , denoted $Sp(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [4, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- (i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- (ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- (iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
- (iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A .

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H . Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

- (P) $f(t) \geq g(t)$ for any $t \in Sp(A)$ implies that $f(A) \geq g(A)$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [4] and the references therein.

For other results concerning functions of selfadjoint operators, see [2], [3], [7], [6] and [8].

2. A Two Operators Version

The following result may be stated:

THEOREM 1. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A, B are selfadjoint operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(2.1) \quad \begin{aligned} 2 \langle f(A)g(A)x, x \rangle \langle f(B)g(B)y, y \rangle \\ \leq \langle \varphi(f(A), g(A))x, x \rangle \langle \psi(f(B), g(B))y, y \rangle \\ + \langle \psi(f(A), g(A))x, x \rangle \langle \varphi(f(B), g(B))y, y \rangle \\ \leq \langle f^2(A)x, x \rangle \langle g^2(B)y, y \rangle + \langle g^2(A)x, x \rangle \langle f^2(B)y, y \rangle \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. We observe that from the property (iii) we have the inequality

$$2 \leq \frac{b\varphi(a, 1)}{a\varphi(b, 1)} + \frac{a\varphi(b, 1)}{b\varphi(a, 1)} \leq \frac{a}{b} + \frac{b}{a},$$

for any $a, b > 0$.

If in this inequality we choose $a = \frac{u}{v}$ and $b = \frac{z}{w}$, then we get

$$(2.2) \quad 2 \leq \frac{zv\varphi\left(\frac{u}{v}, 1\right)}{uw\varphi\left(\frac{z}{w}, 1\right)} + \frac{uw\varphi\left(\frac{z}{w}, 1\right)}{zv\varphi\left(\frac{u}{v}, 1\right)} \leq \frac{uw}{vz} + \frac{vz}{uw}.$$

From the property (ii) we have

$$zv\varphi\left(\frac{u}{v}, 1\right) = \frac{z}{v}\varphi(u, v) \quad \text{and} \quad uw\varphi\left(\frac{z}{w}, 1\right) = \frac{u}{w}\varphi(z, w)$$

which give from (2.2) that

$$(2.3) \quad 2 \leq \frac{zw\varphi(u, v)}{uv\varphi(z, w)} + \frac{uv\varphi(z, w)}{zw\varphi(u, v)} \leq \frac{uw}{vz} + \frac{vz}{uw},$$

for any $u, v, z, w > 0$.

Utilising the property (i) we have

$$\varphi(z, w) = \frac{z^2w^2}{\psi(z, w)} \quad \text{and} \quad \varphi(u, v) = \frac{u^2v^2}{\psi(u, v)},$$

which, from (2.3), produces the inequality

$$2 \leq \frac{\varphi(u, v)\psi(z, w)}{z w u v} + \frac{\varphi(z, w)\psi(u, v)}{u v z w} \leq \frac{uw}{vz} + \frac{vz}{uw},$$

i.e., the inequality

$$(2.4) \quad 2uvwz \leq \varphi(u, v)\psi(z, w) + \varphi(z, w)\psi(u, v) \leq u^2w^2 + v^2z^2,$$

for any $u, v, z, w \geq 0$.

Now, if we choose $u = f(s)$, $v = g(s)$, $z = f(t)$ and $w = g(t)$ in (2.4) then we get

$$(2.5) \quad \begin{aligned} 2f(s)g(s)f(t)g(t) \\ \leq \varphi(f(s), g(s))\psi(f(t), g(t)) + \varphi(f(t), g(t))\psi(f(s), g(s)) \\ \leq f^2(s)g^2(t) + g^2(s)f^2(t) \end{aligned}$$

for any $s, t \in [m, M]$.

Further, if we fix $t \in [m, M]$ and apply the property (P) for the operator A , then we get the inequality

$$(2.6) \quad \begin{aligned} 2f(t)g(t)\langle f(A)g(A)x, x \rangle \\ \leq \psi(f(t), g(t))\langle \varphi(f(A), g(A))x, x \rangle + \varphi(f(t), g(t))\langle \psi(f(A), g(A))x, x \rangle \\ \leq g^2(t)\langle f^2(A)x, x \rangle + f^2(t)\langle g^2(A)x, x \rangle \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we fix $x \in H$ with $\|x\| = 1$ and apply the same property (P) for the inequality (2.6) and the operator B , then we get the desired inequality (2.1). ■

The following particular case is of interest:

COROLLARY 1. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A is a selfadjoint operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A) \subseteq$*

$[m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality

$$(2.7) \quad \langle f(A)g(A)x, x \rangle^2 \\ \leq \langle \varphi(f(A), g(A))x, x \rangle \langle \psi(f(A), g(A))x, x \rangle \leq \langle f^2(A)x, x \rangle \langle g^2(A)x, x \rangle$$

for any $x \in H, \|x\| = 1$.

REMARK 1. a. If A is a selfadjoint operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality

$$(2.8) \quad \langle f(A)g(A)x, x \rangle^2 \\ \leq \langle [f^{1+\alpha}(A)g^{1-\alpha}(A)]x, x \rangle \langle [f^{1-\alpha}(A)g^{1+\alpha}(A)]x, x \rangle \\ \leq \langle f^2(A)x, x \rangle \langle g^2(A)x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$, where $\alpha \in [0, 1]$.

b. If A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ with values in $[0, \infty)$ and such that $f^2(A) + g^2(A)$ is invertible, then we have the inequality

$$(2.9) \quad \langle f(A)g(A)x, x \rangle^2 \\ \leq \langle [f^2(A) + g^2(A)]x, x \rangle \left\langle \left[[f^2(A)g^2(A)] [f^2(A) + g^2(A)]^{-1} \right] x, x \right\rangle \\ \leq \langle f^2(A)x, x \rangle \langle g^2(A)x, x \rangle$$

for any $x \in H, \|x\| = 1$.

The above two inequalities provide various particular cases that are of interest. We give here some examples as follows:

EXAMPLE 1. a. Assume that A is a positive operator on the Hilbert space H and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality

$$(2.10) \quad \langle A^{p+q}x, x \rangle^2 \leq \left\langle A^{p+q+\alpha(p-q)}x, x \right\rangle \left\langle A^{p+q-\alpha(p-q)}x, x \right\rangle \\ \leq \langle A^{2p}x, x \rangle \langle A^{2q}x, x \rangle$$

where $\alpha \in [0, 1]$.

If A is positive definite then the inequality (2.10) also holds for $p, q < 0, p > 0, q < 0$ or $p < 0, q > 0$.

b. Assume that A is a selfadjoint operator and $n, r \in \mathbb{R}$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality

$$(2.11) \quad \langle \exp[(n+r)A]x, x \rangle^2 \\ \leq \langle \exp[n+r+\alpha(n-r)]Ax, x \rangle \langle \exp[n+r-\alpha(n-r)]Ax, x \rangle \\ \leq \langle \exp(2nA)x, x \rangle \langle \exp(2rA)x, x \rangle$$

where $\alpha \in [0, 1]$.

Another example concerning the trigonometric operators $\sin(A)$ and $\cos(A)$ is as follows:

EXAMPLE 2. Let A be a selfadjoint operator with $Sp(A) \subseteq [0, \frac{\pi}{2}]$. Then we have the inequality

$$(2.12) \quad \langle \sin(A) \cos(A) x, x \rangle^2 \leq \langle [\sin^2(A) \cos^2(A)] x, x \rangle \\ \leq \langle \sin^2(A) x, x \rangle \langle \cos^2(A) x, x \rangle$$

for any $x \in H, \|x\| = 1$.

3. Some Versions for $2n$ Operators

The following multiple operator version of Theorem 1 holds:

THEOREM 2. Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j, B_j are selfadjoint operators with $Sp(A_j), Sp(B_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality

$$(3.1) \quad 2 \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle f(B_j) g(B_j) y_j, y_j \rangle \\ \leq \sum_{j=1}^n \langle \varphi(f(A_j), g(A_j)) x_j, x_j \rangle \sum_{j=1}^n \langle \psi(f(B_j), g(B_j)) y_j, y_j \rangle \\ + \sum_{j=1}^n \langle \psi(f(A_j), g(A_j)) x_j, x_j \rangle \sum_{j=1}^n \langle \varphi(f(B_j), g(B_j)) y_j, y_j \rangle \\ \leq \sum_{j=1}^n \langle f^2(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle g^2(B_j) y_j, y_j \rangle + \sum_{j=1}^n \langle g^2(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle f^2(B_j) y_j, y_j \rangle$$

for each $x_j, y_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$.

PROOF. As in [4, p. 6], if we put

$$\tilde{A} := \begin{pmatrix} A_1 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & A_n \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} B_1 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & B_n \end{pmatrix} \\ \tilde{x} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

then we have $Sp(\tilde{A}), Sp(\tilde{B}) \subseteq [m, M], \|\tilde{x}\| = \|\tilde{y}\| = 1$,

$$\langle f(\tilde{A}) g(\tilde{A}) \tilde{x}, \tilde{x} \rangle = \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle,$$

$$\langle f(\tilde{A}) g(\tilde{A}) \tilde{y}, \tilde{y} \rangle = \sum_{j=1}^n \langle f(A_j) g(A_j) y_j, y_j \rangle$$

and so on.

Applying Theorem 1 for \widetilde{A} , \widetilde{B} , \widetilde{x} and \widetilde{y} we deduce the desired result (3.1). ■

As a particular case of interest we can state the following corollary:

COROLLARY 2. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(3.2) \quad \left(\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle \right)^2 \\ \leq \sum_{j=1}^n \langle \varphi(f(A_j), g(A_j)) x_j, x_j \rangle \sum_{j=1}^n \langle \psi(f(A_j), g(A_j)) x_j, x_j \rangle \\ \leq \sum_{j=1}^n \langle f^2(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle g^2(A_j) x_j, x_j \rangle$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

REMARK 2. a. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(3.3) \quad \left(\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle \right)^2 \\ \leq \sum_{j=1}^n \langle [f^{1+\alpha}(A_j) g^{1-\alpha}(A_j)] x_j, x_j \rangle \sum_{j=1}^n \langle [f^{1-\alpha}(A_j) g^{1+\alpha}(A_j)] x_j, x_j \rangle \\ \leq \sum_{j=1}^n \langle f^2(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle g^2(A_j) x_j, x_j \rangle$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, where $\alpha \in [0, 1]$.

b. *If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ with values in $[0, \infty)$ and such that $f^2(A_j) + g^2(A_j)$ are invertible for each, $j \in \{1, \dots, n\}$ then we*

have the inequality

$$\begin{aligned}
(3.4) \quad & \left(\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle \right)^2 \\
& \leq \sum_{j=1}^n \langle [f^2(A_j) + g^2(A_j)] x_j, x_j \rangle \\
& \quad \times \sum_{j=1}^n \langle [[f^2(A_j) g^2(A_j)] [f^2(A_j) + g^2(A_j)]^{-1}] x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f^2(A) x_j, x_j \rangle \sum_{j=1}^n \langle g^2(A) x_j, x_j \rangle
\end{aligned}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Some particular inequalities similar to those from Example 1 and Example 2 may be stated, however we do not mention them in here.

Another version for n operators is the following one:

THEOREM 3. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j, B_j are selfadjoint operators with $Sp(A_j), Sp(B_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M, p_j \geq 0, q_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$ and if f and g are continuous on $[m, M]$ with values in $[0, \infty)$, then we have the inequality*

$$\begin{aligned}
(3.5) \quad & 2 \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n q_j f(B_j) g(B_j) y, y \right\rangle \\
& \leq \left\langle \sum_{j=1}^n p_j \varphi(f(A_j), g(A_j)) x, x \right\rangle \left\langle \sum_{j=1}^n q_j \psi(f(B_j), g(B_j)) y, y \right\rangle \\
& \quad + \left\langle \sum_{j=1}^n p_j \psi(f(A_j), g(A_j)) x, x \right\rangle \left\langle \sum_{j=1}^n q_j \varphi(f(B_j), g(B_j)) y, y \right\rangle \\
& \leq \left\langle \sum_{j=1}^n p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n q_j g^2(B_j) y, y \right\rangle \\
& \quad + \left\langle \sum_{j=1}^n p_j g^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n q_j f^2(B_j) y, y \right\rangle
\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. Follows from Theorem 2 on choosing $x_j = \sqrt{p_j} \cdot x, y_j = \sqrt{q_j} \cdot y, j \in \{1, \dots, n\}$, where $p_j \geq 0, q_j \geq 0, j \in \{1, \dots, n\}, \sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. ■

COROLLARY 3. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M, p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ and if f and g are*

continuous on $[m, M]$ with values in $[0, \infty)$, then we have the inequality

$$(3.6) \quad \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle^2 \\ \leq \left\langle \sum_{j=1}^n p_j \varphi(f(A_j), g(A_j)) x, x \right\rangle \left\langle \sum_{j=1}^n p_j \psi(f(A_j), g(A_j)) x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n p_j g^2(A_j) x, x \right\rangle,$$

for each $x \in H$, with $\|x\| = 1$.

Finally for the section, we can state the following particular inequalities of interest:

REMARK 3. a. Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality

$$(3.7) \quad \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle^2 \\ \leq \left\langle \sum_{j=1}^n p_j [f^{1+\alpha}(A_j) g^{1-\alpha}(A_j)] x, x \right\rangle \left\langle \sum_{j=1}^n p_j [f^{1-\alpha}(A_j) g^{1+\alpha}(A_j)] x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n p_j g^2(A_j) x, x \right\rangle$$

for each $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ and $x \in H$ with $\|x\| = 1$ where $\alpha \in [0, 1]$.

b. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ with values in $[0, \infty)$ and such that $f^2(A_j) + g^2(A_j)$ are invertible for each $j \in \{1, \dots, n\}$ then we have the inequality

$$(3.8) \quad \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle^2 \\ \leq \left\langle \sum_{j=1}^n p_j [f^2(A_j) + g^2(A_j)] x, x \right\rangle \\ \times \left\langle \sum_{j=1}^n p_j [f^2(A_j) g^2(A_j)] [f^2(A_j) + g^2(A_j)]^{-1} x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n p_j g^2(A_j) x, x \right\rangle,$$

for each $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ and $x \in H$ with $\|x\| = 1$.

4. Related Results for Two Operators

The following result that provides another refinement for the Cauchy-Bunyakovsky-Schwarz inequality may be stated as well:

THEOREM 4. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A, B are selfadjoint operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(4.1) \quad 2 \langle f(A)g(A)x, x \rangle \langle f(B)g(B)y, y \rangle \\ \leq \langle \Gamma_1(B)(A, x)y, y \rangle + \langle \Gamma_2(B)(A, x)y, y \rangle \\ \leq \langle f^2(A)g^2(A)x, x \rangle + \langle f^2(B)g^2(B)y, y \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ where

$$\Gamma_1(t)(A, x) := \langle \varphi(f(A), g(t))\psi(f(t), g(A))x, x \rangle$$

and

$$\Gamma_2(t)(A, x) := \langle \varphi(f(t), g(A))\psi(f(A), g(t))x, x \rangle$$

for $t \in [m, M]$.

PROOF. We know that the following inequality holds

$$(4.2) \quad 2uvwz \leq \varphi(u, v)\psi(z, w) + \varphi(z, w)\psi(u, v) \leq u^2w^2 + v^2z^2$$

for any $u, v, z, w \geq 0$.

Now, if we choose $u = f(s), v = g(t), z = f(t)$ and $w = g(s)$ in (4.2) then we get

$$(4.3) \quad 2f(s)g(s)f(t)g(t) \\ \leq \varphi(f(s), g(t))\psi(f(t), g(s)) + \varphi(f(t), g(s))\psi(f(s), g(t)) \\ \leq f^2(s)g^2(s) + g^2(t)f^2(t)$$

for any $s, t \in [m, M]$.

Further, if we fix $t \in [m, M]$ and apply the property (P) for the operator A , then we get the inequality

$$(4.4) \quad 2f(t)g(t)\langle f(A)g(A)x, x \rangle \\ \leq \langle \varphi(f(A), g(t))\psi(f(t), g(A))x, x \rangle + \langle \varphi(f(t), g(A))\psi(f(A), g(t))x, x \rangle \\ \leq \langle f^2(A)g^2(A)x, x \rangle + g^2(t)f^2(t)$$

for any $x \in H$ with $\|x\| = 1$. This inequality can be written in terms of the functions $\Gamma_1(\cdot)(A, x)$ and $\Gamma_2(\cdot)(A, x)$ as

$$(4.5) \quad 2f(t)g(t)\langle f(A)g(A)x, x \rangle \\ \leq \Gamma_1(t)(A, x) + \Gamma_2(t)(A, x) \\ \leq \langle f^2(A)g^2(A)x, x \rangle + g^2(t)f^2(t)$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Now, if we fix $x \in H$ with $\|x\| = 1$ and apply the same property (P) for the inequality (4.5) for the operator B then we get the desired inequality (4.1). ■

The following particular case is of interest

COROLLARY 4. Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A is a selfadjoint operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality

$$(4.6) \quad \langle f(A)g(A)x, x \rangle^2 \leq \langle \Gamma(B)(A, x)x, x \rangle \leq \langle f^2(A)g^2(A)x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$ where

$$\Gamma(t)(A, x) := \langle \varphi(f(A), g(t))\psi(f(t), g(A))x, x \rangle$$

for $t \in [m, M]$.

REMARK 4. If $\varphi(a, b) = a^{1+\alpha}b^{1-\alpha}$, $\psi(a, b) = a^{1-\alpha}b^{1+\alpha}$ with $\alpha \in [0, 1]$ then

$$\Gamma_1(t)(A, x) = f^{1-\alpha}(t)g^{1-\alpha}(t)\langle f^{1+\alpha}(A)g^{1+\alpha}(A)x, x \rangle$$

and

$$\Gamma_2(t)(A, x) := f^{1+\alpha}(t)g^{1+\alpha}(t)\langle f^{1-\alpha}(A)g^{1-\alpha}(A)x, x \rangle$$

and from (4.1) we get the inequality

$$(4.7) \quad \begin{aligned} 2\langle f(A)g(A)x, x \rangle \langle f(B)g(B)y, y \rangle \\ \leq \langle f^{1+\alpha}(A)g^{1+\alpha}(A)x, x \rangle \langle f^{1-\alpha}(B)g^{1-\alpha}(B)y, y \rangle \\ + \langle f^{1-\alpha}(A)g^{1-\alpha}(A)x, x \rangle \langle f^{1+\alpha}(B)g^{1+\alpha}(B)y, y \rangle \\ \leq \langle f^2(A)g^2(A)x, x \rangle + \langle f^2(B)g^2(B)y, y \rangle \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ provided that A is a selfadjoint operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$.

In particular we have the inequality

$$(4.8) \quad \langle f(A)g(A)x, x \rangle^2 \leq \langle f^{1+\alpha}(A)g^{1+\alpha}(A)x, x \rangle \langle f^{1-\alpha}(A)g^{1-\alpha}(A)x, x \rangle \\ \leq \langle f^2(A)g^2(A)x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

The above two inequalities provide various particular cases that are of interest. We give here some examples as follows:

EXAMPLE 3. **a.** Assume that A is a positive operator on the Hilbert space H and $p > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality

$$(4.9) \quad \langle A^p x, x \rangle^2 \leq \langle A^{(1+\alpha)p} x, x \rangle \langle A^{(1-\alpha)p} x, x \rangle \leq \langle A^{2p} x, x \rangle$$

where $\alpha \in [0, 1]$.

If A is positive definite then the inequality (4.9) also holds for $p < 0$.

b. Assume that A is a selfadjoint operator and $r \in \mathbb{R}$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality

$$(4.10) \quad \langle \exp(rA)x, x \rangle^2 \leq \langle \exp[(1+\alpha)rA]x, x \rangle \langle \exp[(1-\alpha)rA]x, x \rangle \\ \leq \langle \exp(2rA)x, x \rangle$$

where $\alpha \in [0, 1]$.

Similar results can be stated for $2n$ operators, however the details are omitted. The following different inequality may be stated as well:

THEOREM 5. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A, B are selfadjoint operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(4.11) \quad \begin{aligned} & 2 \langle f(A)g(A)x, x \rangle \langle f(B)g(B)y, y \rangle \\ & \leq \langle \varphi(f(A), g(A))x, x \rangle \langle \psi(f(B), g(B))y, y \rangle \\ & \quad + \langle \psi(f(A), g(A))x, x \rangle \langle \varphi(f(B), g(B))y, y \rangle \\ & \leq \langle f^2(A)x, x \rangle \langle f^2(B)y, y \rangle + \langle g^2(A)x, x \rangle \langle g^2(B)y, y \rangle \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. We know that the following inequality holds

$$(4.12) \quad 2uvw \leq \varphi(u, v)\psi(z, w) + \varphi(z, w)\psi(u, v) \leq u^2w^2 + v^2z^2$$

for any $u, v, z, w \geq 0$.

Further, if we choose $u = f(s), v = g(s), z = g(t)$ and $w = f(t)$ in (4.12) then we get

$$(4.13) \quad \begin{aligned} & 2f(s)g(s)f(t)g(t) \\ & \leq \varphi(f(s), g(s))\psi(f(t), g(t)) + \varphi(f(t), g(t))\psi(f(s), g(s)) \\ & \leq f^2(s)f^2(t) + g^2(s)g^2(t) \end{aligned}$$

for any $s, t \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A then we get the inequality

$$(4.14) \quad \begin{aligned} & 2f(t)g(t)\langle f(A)g(A)x, x \rangle \\ & \leq \psi(f(t), g(t))\langle \varphi(f(A), g(A))x, x \rangle + \varphi(f(t), g(t))\langle \psi(f(A), g(A))x, x \rangle \\ & \leq f^2(t)\langle f^2(A)x, x \rangle + g^2(t)\langle g^2(A)x, x \rangle \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we fix $x \in H$ with $\|x\| = 1$ and apply the same property (P) for the inequality (4.14) for the operator B then we get the desired inequality (4.11). ■

In particular, we have

COROLLARY 5. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0, \infty) \times [0, \infty)$. If A is a selfadjoint operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$(4.15) \quad \begin{aligned} & \left(2 \langle f(A)g(A)x, x \rangle \right)^2 \leq 2 \langle \varphi(f(A), g(A))x, x \rangle \langle \psi(f(A), g(A))x, x \rangle \\ & \leq \langle f^2(A)x, x \rangle^2 + \langle g^2(A)x, x \rangle^2 \end{aligned}$$

for any $x \in H, \|x\| = 1$.

REMARK 5. *We observe that the inequality (4.15) is not as good as the second inequality in (2.7).*

REMARK 6. Consider now the following two bounds

$$B_2 := \langle f^2(A)x, x \rangle \langle f^2(B)y, y \rangle + \langle g^2(A)x, x \rangle \langle g^2(B)y, y \rangle$$

and

$$B_1 := \langle f^2(A)x, x \rangle \langle g^2(B)y, y \rangle + \langle g^2(A)x, x \rangle \langle f^2(B)y, y \rangle$$

for the quantity

$$\begin{aligned} & \langle \varphi(f(A), g(A))x, x \rangle \langle \psi(f(B), g(B))y, y \rangle \\ & \quad + \langle \psi(f(A), g(A))x, x \rangle \langle \varphi(f(B), g(B))y, y \rangle \end{aligned}$$

that have been obtained in Theorem 5 and Theorem 1, respectively. We observe that

$$(4.16) \quad B_2 - B_1 = \langle [f^2(A) - g^2(A)]x, x \rangle (\langle [f^2(B) - g^2(B)]y, y \rangle),$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Utilising the equality (4.16) we can observe, for instance, that, if $f^2(A) \geq g^2(A)$ and $f^2(B) \geq g^2(B)$ in the operator order of $B(H)$, then B_1 is a better bound than B_2 . The conclusion is the other way around if, for instance, $f^2(A) \geq g^2(A)$ and $g^2(B) \geq f^2(B)$ in the operator order of $B(H)$.

Similar results can be stated for $2n$ operators, however the details are omitted.

REMARK 7. One can choose the variables $u, v, z, w \geq 0$ in other different ways in the inequality

$$(4.17) \quad 2uvwz \leq \varphi(u, v) \psi(z, w) + \varphi(z, w) \psi(u, v) \leq u^2w^2 + v^2z^2$$

to get similar results as those pointed out above. The details are left to the interested reader.

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