Some Reverses of the Jensen Inequality for Functions of Selfadjoint Operators in Hilbert Spaces

SOME REVERSES OF THE JENSEN INEQUALITY FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [4, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f)\ \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\ge 0$ for any $t\in Sp(A)$ implies that $f(A)\ge 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\ge g(t)\ \text{ for any } t\in Sp(A)\ \text{ implies that } f(A)\ge g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [4] and the references therein. For other results, see [11], [5] and [7].

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [9] (see also [4, p. 5]):

Theorem 1 (Mond- Pečarić, 1993, [9])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with*

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$m<M$. If f is a convex function on $[m,M]$, then

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle \tag{MP}$$

for each $x\in H$ with $\|x\|=1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [6])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x,x\rangle\geq\langle Ax,x\rangle^r$ for all $r>1$ and $x\in H$ with $\|x\|=1$;
(ii) $\langle A^r x,x\rangle\leq\langle Ax,x\rangle^r$ for all $0<r<1$ and $x\in H$ with $\|x\|=1$;
(iii) If A is invertible, then $\langle A^r x,x\rangle\geq\langle Ax,x\rangle^r$ for all $r<0$ and $x\in H$ with $\|x\|=1$.

The following theorem is a multiple operator version of Theorem 1 (see for instance [4, p. 5]):

Theorem 3. *Let A_j be selfadjoint operators with $Sp(A_j)\subseteq[m,M]$, $j\in\{1,...,n\}$ for some scalars $m<M$ and $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^n\|x_j\|^2=1$. If f is a convex function on $[m,M]$, then*

$$f\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)\leq\sum_{j=1}^n\langle f(A_j)x_j,x_j\rangle. \tag{1.1}$$

The following particular case is of interest. Apparently it has not been stated before either in the monograph [4] or in the research papers cited therein.

Corollary 1. *Let A_j be selfadjoint operators with $Sp(A_j)\subseteq[m,M]$, $j\in\{1,...,n\}$ for some scalars $m<M$. If $p_j\geq0$, $j\in\{1,...,n\}$ with $\sum_{j=1}^n p_j=1$, then*

$$f\left(\left\langle\sum_{j=1}^n p_jA_jx,x\right\rangle\right)\leq\left\langle\sum_{j=1}^n p_jf(A_j)x,x\right\rangle, \tag{1.2}$$

for any $x\in H$ with $\|x\|=1$.

Proof. Follows from Theorem 3 by choosing $x_j=\sqrt{p_j}\cdot x$, $j\in\{1,...,n\}$ where $x\in H$ with $\|x\|=1$. ∎

Remark 1. *The above inequality can be used to produce some norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0,M]$. Namely, we have:*

$$f\left(\left\|\sum_{j=1}^n p_jA_j\right\|\right)\leq\left\|\sum_{j=1}^n p_jf(A_j)\right\|. \tag{1.3}$$

The inequality (1.3) reverses if the function is concave on $[0,M]$.

As particular cases we can state the following inequalities:

$$\left\|\sum_{j=1}^n p_jA_j\right\|^p\leq\left\|\sum_{j=1}^n p_jA_j^p\right\|, \tag{1.4}$$

for $p \geq 1$ *and*

$$\left\|\sum_{j=1}^{n} p_j A_j\right\|^p \geq \left\|\sum_{j=1}^{n} p_j A_j^p\right\| \tag{1.5}$$

for $0 < p < 1$.

If A_j *are positive definite for each* $j \in \{1, ..., n\}$ *then (1.4) also holds for* $p < 0$.

If one uses the inequality (1.3) for the exponential function, that one obtains the inequality

$$\exp\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \leq \left\|\sum_{j=1}^{n} p_j \exp(A_j)\right\|, \tag{1.6}$$

where A_j *are positive operators for each* $j \in \{1, ..., n\}$.

In Section 2.4 of the monograph [4] there are numerous and interesting converses of the Jensen's type inequality from which we would like to mention one of the simplest (see [7] and [4, p. 61]):

Theorem 4. *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ *for some scalars* $m < M$ *and* $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$. *If* f *is a strictly convex function twice differentiable on* $[m, M]$, *then for any positive real number* α *we have*

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \leq \alpha f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) + \beta, \tag{1.7}$$

where

$$\beta = \mu_f t_0 + \nu_f - \alpha f(t_0),$$

$$\mu_f = \frac{f(M) - f(m)}{M - m}, \nu_f = \frac{Mf(m) - mf(M)}{M - m}$$

and

$$t_0 = \begin{cases} f'^{-1}\left(\frac{\mu_f}{\alpha}\right) & \text{if } m < f'^{-1}\left(\frac{\mu_f}{\alpha}\right) < M \\ M & \text{if } M \leq f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \\ m & \text{if } f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \leq m. \end{cases}$$

The case of equality was also analyzed but will be not stated in here.

The main aim of the present paper is to provide different reverses of the Jensen inequality where some upper bounds for the nonnegative difference

$$\langle f(A) x, x\rangle - f(\langle Ax, x\rangle), \ x \in H \text{ with } \|x\| = 1,$$

will be provided. Applications for some particular convex functions of interest are also given.

2. Reverses of the Jensen Inequality

The following result holds:

Theorem 5. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable, we have that

$$f(t) - f(s) \leq f'(t) \cdot (t - s)$$

for any $t, s \in [m, M]$.

Now, if we chose in this inequality $s = \langle Ax, x\rangle \in [m, M]$ for any $x \in H$ with $\|x\| = 1$ since $Sp(A) \subseteq [m, M]$, then we have

$$f(t) - f(\langle Ax, x\rangle) \leq f'(t) \cdot (t - \langle Ax, x\rangle) \tag{2.2}$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.2) and apply the property (P) then we get

$$\langle [f(A) - f(\langle Ax, x\rangle) 1_H] x, x\rangle \leq \langle f'(A) \cdot (A - \langle Ax, x\rangle 1_H) x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent to the desired inequality (2.1). ∎

Corollary 2. *Assume that f is as in the Theorem 5. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then*

$$\begin{aligned}(0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \\ \leq \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j\rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j\rangle .\end{aligned} \tag{2.3}$$

Proof. As in [4, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle, \left\langle \widetilde{A}\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle$$

and so on.

Applying Theorem 5 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (2.3). ∎

Corollary 3. *Assume that f is as in the Theorem 5. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$(2.4)\quad (0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \leq \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle.$$

for each $x \in H$ with $\|x\| = 1$.

Remark 2. *The inequality (2.4), in the scalar case, namely*

$$(2.5)\quad (0 \leq) \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j \right) \leq \sum_{j=1}^{n} p_j f'(x_j) x_j - \sum_{j=1}^{n} p_j x_j \cdot \sum_{j=1}^{n} p_j f'(x_j),$$

where $x_j \in \mathring{I}$, $j \in \{1, ..., n\}$, has been obtained by the first time in 1994 by Dragomir & Ionescu, see [3].

The following particular cases are of interest:

Example 1. ***a.*** *Let A be a positive definite operator on the Hilbert space H. Then we have the following inequality:*

$$(2.6)\quad (0 \leq) \ln(\langle Ax, x \rangle) - \langle \ln(A) x, x \rangle \leq \langle Ax, x \rangle \cdot \left\langle A^{-1} x, x \right\rangle - 1,$$

for each $x \in H$ with $\|x\| = 1$.

b. *If A is a selfadjoint operator on H, then we have the inequality:*

$$(2.7)\quad (0 \leq) \langle \exp(A) x, x \rangle - \exp(\langle Ax, x \rangle) \leq \langle A \exp(A) x, x \rangle - \langle Ax, x \rangle \cdot \langle \exp(A) x, x \rangle,$$

for each $x \in H$ with $\|x\| = 1$.

c. *If $p \geq 1$ and A is a positive operator on H, then*

$$(2.8)\quad (0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \leq p \left[\langle A^p x, x \rangle - \langle Ax, x \rangle \cdot \left\langle A^{p-1} x, x \right\rangle \right],$$

for each $x \in H$ with $\|x\| = 1$. If A is positive definite, then the inequality (2.8) also holds for $p < 0$.

If $0 < p < 1$ and A is a positive definite operator then the reverse inequality also holds

$$(2.9)\quad \langle A^p x, x \rangle - \langle Ax, x \rangle^p \geq p \left[\langle A^p x, x \rangle - \langle Ax, x \rangle \cdot \left\langle A^{p-1} x, x \right\rangle \right] \geq 0,$$

for each $x \in H$ with $\|x\| = 1$.

Similar results can be stated for sequences of operators, however the details are omitted.

3. Further Reverses

In applications would be perhaps more useful to find upper bounds for the quantity

$$\langle f(A)x,x\rangle - f(\langle Ax,x\rangle), \ x\in H \text{ with } \|x\|=1,$$

that are in terms of the spectrum margins m, M and of the function f.

The following result may be stated:

Theorem 6. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m,M] \subset \mathring{I}$, then*

$$(3.1)\quad (0\le)\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \le \begin{cases} \frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \end{cases} \le \frac{1}{4}(M-m)(f'(M)-f'(m)),$$

for any $x\in H$ with $\|x\|=1$.

We also have the inequality

$$(3.2)\quad (0\le)\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \le \frac{1}{4}(M-m)(f'(M)-f'(m)) - \begin{cases} \left[\langle Mx-Ax, Ax-mx\rangle\langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \le \frac{1}{4}(M-m)(f'(M)-f'(m)),$$

for any $x\in H$ with $\|x\|=1$.

Moreover, if $m>0$ and $f'(m)>0$, then we also have

$$(3.3)\quad (0\le)\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \le \begin{cases} \frac{1}{4}\cdot\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)\left[\langle Ax,x\rangle\langle f'(A)x,x\rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x\in H$ with $\|x\|=1$.

Proof. We use the following Grüss' type result we obtained in [1]:

Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If $h$and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} h(t)$ and $\Gamma := \max_{t\in[m,M]} h(t)$, then

$$(3.4)\quad |\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\cdot\langle g(A)x,x\rangle| \le \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2}\left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)$$

for each $x\in H$ with $\|x\|=1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Therefore, we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \frac{1}{2}\cdot (M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x,x\rangle^2\right]^{1/2} \le \frac{1}{4}(M-m)(f'(M)-f'(m)) \tag{3.5}$$

and

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \frac{1}{2}\cdot (f'(M)-f'(m))\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \le \frac{1}{4}(M-m)(f'(M)-f'(m)) \tag{3.6}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (3.1).

On making use of the inequality obtained in [2]:

$$|\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\langle g(A)x,x\rangle| \le \frac{1}{4}\cdot(\Gamma-\gamma)(\Delta-\delta) - \begin{cases} [\langle \Gamma x - h(A)x, f(A)x - \gamma x\rangle \langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle h(A)x,x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \tag{3.7}$$

for each $x \in H$ with $\|x\| = 1$, we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \frac{1}{4}(M-m)(f'(M)-f'(m)) - \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right|. \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (3.2).

Further, in order to proof the third inequality, we make use of the following result of Grüss' type we obtained in [2]:

If γ and δ are positive, then

$$|\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\langle g(A)x,x\rangle| \le \begin{cases} \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\langle h(A)x,x\rangle\langle g(A)x,x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle h(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}}. \end{cases} \tag{3.8}$$

for each $x \in H$ with $\|x\| = 1$.

Now, on making use of (3.8) we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \begin{cases} \frac{1}{4}\cdot\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle\langle f'(A)x,x\rangle]^{\frac{1}{2}}. \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (3.3). ∎

Corollary 4. *Assume that f is as in the Theorem 6. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$, then*

$$(3.9)\quad (0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\sum_{j=1}^{n} \|f'(A_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle\right)^2\right]^{1/2}, \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\sum_{j=1}^{n} \|A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^2\right]^{1/2}, \end{cases}$$
$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We also have the inequality

$$(3.10)\quad (0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$
$$- \begin{cases} \left[\sum_{j=1}^{n} \langle M x_j - A_j x, A_j x_j - m x_j \rangle\right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} \langle f'(M) x_j - f'(A_j) x_j, f'(A_j) x_j - f'(m) x_j \rangle\right]^{1/2}, \\ \\ \left|\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - \frac{M+m}{2}\right| \left|\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases}$$
$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$(3.11)\quad (0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right) \left(\sqrt{f'(M)} - \sqrt{f'(m)}\right) \\ \times \left[\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following corollary also holds:

Corollary 5. *Assume that f is as in the Theorem 5. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$(3.12)\quad (0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M-m) \left[\sum_{j=1}^{n} p_j \|f'(A_j) x\|^2 - \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle^2 \right]^{1/2}, \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\sum_{j=1}^{n} p_j \|A_j x\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right]^{1/2}, \end{cases}$$

$$\leq \frac{1}{4} (M-m) (f'(M) - f'(m)),$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$(3.13)\quad (0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \frac{1}{4} (M-m) (f'(M) - f'(m))$$

$$- \begin{cases} \left[\sum_{j=1}^{n} p_j \langle Mx - A_j x, A_j x - mx \rangle \right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} p_j \langle f'(M) x - f'(A_j) x, f'(A_j) x - f'(m) x \rangle \right]^{1/2}, \\ \\ \left| \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \frac{M+m}{2} \right| \left| \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle - \frac{f'(M)+f'(m)}{2} \right| \end{cases}$$

$$\leq \frac{1}{4} (M-m) (f'(M) - f'(m)),$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$(3.14)\quad (0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M) - f'(m)\right)}{\sqrt{M m f'(M) f'(m)}} \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right) \\ \times \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle \right]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 3. *Some of the inequalities in Corollary 5 can be used to produce reverse norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0, M]$.*

For instance, if we use the inequality (3.12), then we have

$$(0\leq)\left\|\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)\right\|-f\left(\left\|\sum_{j=1}^{n}p_{j}A_{j}\right\|\right)\leq\frac{1}{4}\left(M-m\right)\left(f'\left(M\right)-f'\left(m\right)\right). \tag{3.15}$$

Moreover, if we use the inequality (3.14), then we obtain

$$(0\leq)\left\|\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)\right\|-f\left(\left\|\sum_{j=1}^{n}p_{j}A_{j}\right\|\right) \tag{3.16}$$

$$\leq\begin{cases}\frac{1}{4}\cdot\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}}\left\|\sum_{j=1}^{n}p_{j}A_{j}\right\|\left\|\sum_{j=1}^{n}p_{j}f'\left(A_{j}\right)\right\|,\\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'\left(M\right)}-\sqrt{f'\left(m\right)}\right)\left[\left\|\sum_{j=1}^{n}p_{j}A_{j}\right\|\left\|\sum_{j=1}^{n}p_{j}f'\left(A_{j}\right)\right\|\right]^{\frac{1}{2}}.\end{cases}$$

4. Some Particular Inequalities of Interest

1. Consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$. On utilising the inequality (3.1), then for any positive definite operator A on the Hilbert space H, we have the inequality

$$(0\leq)\ln\left(\langle Ax,x\rangle\right)-\langle\ln\left(A\right)x,x\rangle \tag{4.1}$$

$$\leq\begin{cases}\frac{1}{2}\cdot\left(M-m\right)\left[\left\|A^{-1}x\right\|^{2}-\left\langle A^{-1}x,x\right\rangle^{2}\right]^{1/2}\\ \\ \frac{1}{2}\cdot\frac{M-m}{mM}\left[\left\|Ax\right\|^{2}-\left\langle Ax,x\right\rangle^{2}\right]^{1/2}\end{cases}\quad\left(\leq\frac{1}{4}\cdot\frac{\left(M-m\right)^{2}}{mM}\right)$$

for any $x\in H$ with $\|x\|=1$.

However, if we use the inequality (3.2), then we have the following result as well

$$(0\leq)\ln\left(\langle Ax,x\rangle\right)-\langle\ln\left(A\right)x,x\rangle\leq\frac{1}{4}\cdot\frac{\left(M-m\right)^{2}}{mM} \tag{4.2}$$

$$-\begin{cases}\left[\langle Mx-Ax,Ax-mx\rangle\left\langle M^{-1}x-A^{-1}x,A^{-1}x-m^{-1}x\right\rangle\right]^{\frac{1}{2}},\\ \\ \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\left\langle A^{-1}x,x\right\rangle-\frac{M+m}{2mM}\right|\end{cases}$$

$$\left(\leq\frac{1}{4}\cdot\frac{\left(M-m\right)^{2}}{mM}\right)$$

for any $x\in H$ with $\|x\|=1$.

2. Now consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x\ln x$. On utilising the inequality (3.1), then for any positive definite operator A on the Hilbert space

H, we have the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\|\ln (eA) x\|^2 - \langle \ln (eA) x, x \rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \left(\leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}} \right) \tag{4.3}$$

for any $x \in H$ with $\|x\| = 1$.

If we apply now the inequality (3.2), then we have the following result as well

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}} - \begin{cases} \left[\langle Mx - Ax, Ax - mx \rangle \langle \ln (M) x - \ln (A) x, \ln (A) x - \ln (m) x \rangle \right]^{\frac{1}{2}}, \\ \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle \ln (A) x, x \rangle - \ln \sqrt{mM} \right| \end{cases} \left(\leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}} \right) \tag{4.4}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if we assume that $m > e^{-1}$, then, by utilising the inequality (3.3) we can state the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot \frac{(M-m) \ln \sqrt{\frac{M}{m}}}{\sqrt{Mm \ln (eM) \ln (em)}} \langle Ax, x \rangle \langle \ln (eA) x, x \rangle, \\ \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{\ln (eM)} - \sqrt{\ln (em)} \right) \left[\langle Ax, x \rangle \langle \ln (eA) x, x \rangle \right]^{\frac{1}{2}}. \end{cases} \tag{4.5}$$

for any $x \in H$ with $\|x\| = 1$.

3. Consider now the following convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(\alpha x)$ with $\alpha > 0$. If we apply the inequalities (3.1), (3.2) and (3.3) for $f(x) = \exp(\alpha x)$ and for a selfadjoint operator A, then we get the following results

$$(0 \leq) \langle \exp (\alpha A) x, x \rangle - \exp (\alpha \langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot \alpha (M - m) \left[\|\exp (\alpha A) x\|^2 - \langle \exp (\alpha A) x, x \rangle^2 \right]^{1/2} \\ \\ \frac{1}{2} \cdot \alpha (\exp (\alpha M) - \exp (\alpha m)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \left(\leq \frac{1}{4} \alpha (M - m) (\exp (\alpha M) - \exp (\alpha m)) \right), \tag{4.6}$$

and

$$(4.7)\quad (0\leq)\left\langle \exp\left(\alpha A\right)x,x\right\rangle -\exp\left(\alpha\left\langle Ax,x\right\rangle\right)$$
$$\leq\frac{1}{4}\alpha\left(M-m\right)\left(\exp\left(\alpha M\right)-\exp\left(\alpha m\right)\right)$$
$$-\alpha\times\left\{\begin{array}{l}\left[\left\langle Mx-Ax,Ax-mx\right\rangle\right]^{1/2}\\ \times\left[\left\langle \exp\left(\alpha M\right)x-\exp\left(\alpha A\right)x,\exp\left(\alpha A\right)x-\exp\left(\alpha m\right)x\right\rangle\right]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle \exp\left(\alpha A\right)x,x\right\rangle-\frac{\exp(\alpha M)+\exp(\alpha m)}{2}\right|\end{array}\right.$$
$$\left(\leq\frac{1}{4}\alpha\left(M-m\right)\left(\exp\left(\alpha M\right)-\exp\left(\alpha m\right)\right)\right)$$

and

$$(4.8)\quad (0\leq)\left\langle \exp\left(\alpha A\right)x,x\right\rangle -\exp\left(\alpha\left\langle Ax,x\right\rangle\right)$$
$$\leq\alpha\times\left\{\begin{array}{l}\frac{1}{4}\cdot\frac{(M-m)(\exp(\alpha M)-\exp(\alpha m))}{\sqrt{Mm}\exp\left[\frac{\alpha(M+m)}{2}\right]}\left\langle Ax,x\right\rangle\left\langle \exp\left(\alpha A\right)x,x\right\rangle,\\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\exp\left(\frac{\alpha M}{2}\right)-\exp\left(\frac{\alpha m}{2}\right)\right)\\ \times\left[\left\langle Ax,x\right\rangle\left\langle \exp\left(\alpha A\right)x,x\right\rangle\right]^{\frac{1}{2}}.\end{array}\right.$$

for any $x\in H$ with $\|x\|=1$, respectively.

Now, consider the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(-\beta x)$ with $\beta>0$. If we apply the inequalities (3.1) and (3.2) for $f(x)=\exp(-\beta x)$ and for a selfadjoint operator A, then we get the following results

$$(4.9)\quad (0\leq)\left\langle \exp\left(-\beta A\right)x,x\right\rangle -\exp\left(-\beta\left\langle Ax,x\right\rangle\right)$$
$$\leq\beta\times\left\{\begin{array}{l}\frac{1}{2}\cdot\left(M-m\right)\left[\left\|\exp\left(-\beta A\right)x\right\|^{2}-\left\langle \exp\left(-\beta A\right)x,x\right\rangle^{2}\right]^{1/2}\\ \\ \frac{1}{2}\cdot\left(\exp\left(-\beta m\right)-\exp\left(-\beta M\right)\right)\left[\left\|Ax\right\|^{2}-\left\langle Ax,x\right\rangle^{2}\right]^{1/2}\end{array}\right.$$
$$\left(\leq\frac{1}{4}\beta\left(M-m\right)\left(\exp\left(-\beta m\right)-\exp\left(-\beta M\right)\right)\right)$$

and

$$(4.10)\quad (0\leq)\left\langle \exp\left(-\beta A\right)x,x\right\rangle -\exp\left(-\beta\left\langle Ax,x\right\rangle\right)$$
$$\leq\frac{1}{4}\beta\left(M-m\right)\left(\exp\left(-\beta m\right)-\exp\left(-\beta M\right)\right)$$
$$-\beta\times\left\{\begin{array}{l}\left[\left\langle Mx-Ax,Ax-mx\right\rangle\right]^{1/2}\\ \times\left[\left\langle \exp\left(-\beta M\right)x-\exp\left(-\beta A\right)x,\exp\left(-\beta A\right)x-\exp\left(-\beta m\right)x\right\rangle\right]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle \exp\left(-\beta A\right)x,x\right\rangle-\frac{\exp(-\beta M)+\exp(-\beta m)}{2}\right|\end{array}\right.$$
$$\left(\leq\frac{1}{4}\beta\left(M-m\right)\left(\exp\left(-\beta m\right)-\exp\left(-\beta M\right)\right)\right)$$

for any $x\in H$ with $\|x\|=1$, respectively.

4. Finally, if we consider the convex function $f:[0,\infty)\to[0,\infty)$, $f(x)=x^p$ with $p\geq 1$, then on applying the inequalities (3.1) and (3.2) for the positive operator A we have the inequalities

$$(0\leq)\left\langle A^p x,x\right\rangle-\left\langle Ax,x\right\rangle^p \leq p\times\begin{cases}\frac{1}{2}\cdot(M-m)\left[\left\|A^{p-1}x\right\|^2-\left\langle A^{p-1}x,x\right\rangle^2\right]^{1/2}\\ \\ \frac{1}{2}\cdot\left(M^{p-1}-m^{p-1}\right)\left[\left\|Ax\right\|^2-\left\langle Ax,x\right\rangle^2\right]^{1/2}\end{cases} \left(\leq\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\right) \tag{4.11}$$

and

$$(0\leq)\left\langle A^p x,x\right\rangle-\left\langle Ax,x\right\rangle^p\leq\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right) -p\times\begin{cases}\left[\left\langle Mx-Ax,Ax-mx\right\rangle\left\langle M^{p-1}x-A^{p-1}x,A^{p-1}x-m^{p-1}x\right\rangle\right]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle A^{p-1}x,x\right\rangle-\frac{M^{p-1}+m^{p-1}}{2}\right|\end{cases} \left(\leq\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\right) \tag{4.12}$$

for any $x\in H$ with $\|x\|=1$, respectively.

If the operator A is positive definite $(m>0)$ then, by utilising the inequality (3.3), we have

$$(0\leq)\left\langle A^p x,x\right\rangle-\left\langle Ax,x\right\rangle^p \leq p\times\begin{cases}\frac{1}{4}\cdot\frac{(M-m)\left(M^{p-1}-m^{p-1}\right)}{M^{p/2}m^{p/2}}\left\langle Ax,x\right\rangle\left\langle A^{p-1}x,x\right\rangle,\\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(M^{(p-1)/2}-m^{(p-1)/2}\right)\left[\left\langle Ax,x\right\rangle\left\langle A^{p-1}x,x\right\rangle\right]^{\frac{1}{2}},\end{cases} \tag{4.13}$$

for any $x\in H$ with $\|x\|=1$.

Now, if we consider the convex function $f:[0,\infty)\to[0,\infty)$, $f(x)=-x^p$ with $p\in(0,1)$, then from the inequalities (3.1) and (3.2) and for the positive definite operator A we have the inequalities

$$(0\leq)\left\langle Ax,x\right\rangle^p-\left\langle A^p x,x\right\rangle \leq p\times\begin{cases}\frac{1}{2}\cdot(M-m)\left[\left\|A^{p-1}x\right\|^2-\left\langle A^{p-1}x,x\right\rangle^2\right]^{1/2}\\ \\ \frac{1}{2}\cdot\left(m^{p-1}-M^{p-1}\right)\left[\left\|Ax\right\|^2-\left\langle Ax,x\right\rangle^2\right]^{1/2}\end{cases} \left(\leq\frac{1}{4}p(M-m)\left(m^{p-1}-M^{p-1}\right)\right) \tag{4.14}$$

and

$$(0 \leq) \langle Ax, x\rangle^p - \langle A^p x, x\rangle \leq \frac{1}{4} p (M - m) \left(m^{p-1} - M^{p-1}\right) \tag{4.15}$$

$$- p \times \begin{cases} \left[\langle Mx - Ax, Ax - mx\rangle \left\langle M^{p-1}x - A^{p-1}x, A^{p-1}x - m^{p-1}x\right\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\left\langle A^{p-1}x, x\right\rangle - \frac{M^{p-1}+m^{p-1}}{2}\right| \end{cases}$$

$$\left(\leq \frac{1}{4} p (M - m) \left(m^{p-1} - M^{p-1}\right)\right)$$

for any $x \in H$ with $\|x\| = 1$, respectively.

Similar results may be stated for the convex function $f : (0, \infty) \to (0, \infty)$, $f(x) = x^p$ with $p < 0$. However the details are left to the interested reader.

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