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# APPROXIMATING THE STIELTJES INTEGRAL OF BOUNDED FUNCTIONS AND APPLICATIONS FOR THREE POINT QUADRATURE RULES

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ABSTRACT. Sharp error estimates in approximating the Stieltjes integral with bounded integrands and bounded integrators respectively, are given. Applications for three point quadrature rules of  $n$ -time differentiable functions are also provided.

## 1. INTRODUCTION

In order to approximate the *Stieltjes integral*  $\int_a^b f(t) du(t)$  with the simpler expression

$$(1.1) \quad \frac{1}{b-a} [u(b) - u(a)] \cdot \int_a^b f(t) dt,$$

S.S. Dragomir and I. Fedotov [8] introduced in 1998 the following *error functional*

$$(1.2) \quad D(f, u; a, b) := \int_a^b f(t) du(t) - \frac{1}{b-a} [u(b) - u(a)] \cdot \int_a^b f(t) dt,$$

provided that both the Stieltjes integral  $\int_a^b f(t) du(t)$  and the *Riemann integral*  $\int_a^b f(t) dt$  exist.

If the *integrand*  $f$  is *Riemann integrable* on  $[a, b]$  and the *integrator*  $u : [a, b] \rightarrow \mathbb{R}$  is *L-Lipschitzian*, i.e.,

$$(1.3) \quad |u(t) - u(s)| \leq L |t - s| \quad \text{for each } t, s \in [a, b],$$

then the Stieltjes integral  $\int_a^b f(t) du(t)$  exists and, as pointed out in [8],

$$(1.4) \quad |D(f, u; a, b)| \leq L \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.$$

The inequality (1.4) is sharp in the sense that the multiplicative constant  $C = 1$  in front of  $L$  cannot be replaced by a smaller quantity. Moreover, if there exist the constants  $m, M \in \mathbb{R}$  such that

$$(1.5) \quad m \leq f(t) \leq M \quad \text{for a.e. } t \in [a, b],$$

then [8]

$$(1.6) \quad |D(f, u; a, b)| \leq \frac{1}{2} L (M - m) (b - a).$$

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The constant  $\frac{1}{2}$  is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, see [9], where they proved that

$$(1.7) \quad |D(f, u; a, b)| \leq \max_{t \in [a, b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| \bigvee_a^b(u),$$

provided that  $f$  is continuous and  $u$  is of bounded variation. Here  $\bigvee_a^b(u)$  denotes the total variation of  $u$  on  $[a, b]$ . The inequality (1.7) is also sharp.

If we assume that  $f$  is  $K$ -Lipschitzian, then [9]

$$(1.8) \quad |D(f, u; a, b)| \leq \frac{1}{2} K (b-a) \bigvee_a^b(u),$$

with  $\frac{1}{2}$  the best possible constant in (1.8).

For various bounds on the *error functional*  $D(f, u; a, b)$  where  $f$  and  $u$  belong to different classes of functions for which the Stieltjes integral exists, see [2], [5], [6] and [7] and the references therein.

The main aim of the present paper is to estimate the error of approximating the Stieltjes integral  $\int_a^b f(t) du(t)$  with the simpler expression

$$(1.9) \quad \frac{m+M}{2} \cdot [u(b) - u(a)]$$

provided the integrand  $f$  is bounded below by  $m$  and above by  $M$ .

In the dual case, i.e., when  $n \leq u(t) \leq M$  on  $[a, b]$ , the problem under consideration consists of approximating the same Stieltjes integral  $\int_a^b f(t) du(t)$  with the quantity

$$(1.10) \quad \left[ u(b) - \frac{n+M}{2} \right] f(b) + \left[ \frac{n+M}{2} - u(a) \right] f(a).$$

Applications for the three point quadrature rule of  $n$ -differentiable functions are also given.

## 2. INEQUALITIES FOR THE STIELTJES INTEGRAL

The following result may be stated.

**Theorem 1.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation and  $f : [a, b] \rightarrow \mathbb{R}$  a function such that there exists the constants  $m, M \in \mathbb{R}$  with*

$$(2.1) \quad m \leq f(t) \leq M \quad \text{for each } t \in [a, b],$$

*and the Stieltjes integral  $\int_a^b f(t) du(t)$  exists. Then, by defining the error functional*

$$\Delta(f, u, m, M; a, b) := \int_a^b f(t) du(t) - \frac{m+M}{2} [u(b) - u(a)],$$

*we have the bound*

$$(2.2) \quad |\Delta(f, u, m, M; a, b)| \leq \frac{1}{2} (M - m) \bigvee_a^b(u).$$

*The constant  $\frac{1}{2}$  is best possible in (2.2) in the sense that it cannot be replaced by a smaller quantity.*

*Proof.* Since, obviously, the function  $f - \frac{m+M}{2}$  satisfies the inequality

$$\left| f(t) - \frac{m+M}{2} \right| \leq \frac{1}{2} (M - m) \quad \text{for any } t \in [a, b]$$

and the Stieltjes integral  $\int_a^b (f(t) - \frac{m+M}{2}) du(t)$  exists, then

$$\begin{aligned} \left| \int_a^b \left( f(t) - \frac{m+M}{2} \right) du(t) \right| &\leq \sup_{t \in [a, b]} \left| f(t) - \frac{m+M}{2} \right| \bigvee_a^b(u) \\ &\leq \frac{1}{2} (M - m) \bigvee_a^b(u) \end{aligned}$$

and the inequality (2.2) is proved.

Now, assume that (2.2) holds with a positive constant  $C$ , i.e.,

$$(2.3) \quad |\Delta(f, u, m, M; a, b)| \leq C (M - m) \bigvee_a^b(u),$$

provided  $u$  is of bounded variation on  $[a, b]$  and  $f$  satisfies (2.1).

If we consider the function  $f_0(t) := \operatorname{sgn}(t - \frac{a+b}{2})$  and  $u_0(t) = \frac{1}{2}(t - \frac{a+b}{2})^2$ , then we observe that the Stieltjes integral  $\int_a^b f_0(t) du_0(t)$  exists,  $f_0$  is bounded above by  $M_0 = 1$  and below by  $m_0 = -1$ ,  $u_0$  is of bounded variation and

$$\bigvee_a^b(u_0) = \int_a^b |u_0'(t)| dt = \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{(b-a)^2}{4}.$$

Also

$$\begin{aligned} \int_a^b f_0(t) du_0(t) &= \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) \left(t - \frac{a+b}{2}\right) dt \\ &= \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{(b-a)^2}{4} \end{aligned}$$

and replacing  $f_0$  and  $u_0$  in (2.3) produces the inequality

$$\frac{(b-a)^2}{4} \leq 2C \cdot \frac{(b-a)^2}{4}$$

which implies that  $C \geq \frac{1}{2}$ . ■

The following corollary provides a natural example of functions  $f$  that can be chosen to fulfill the conditions in the above theorem.

**Corollary 1.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$  and  $f$  a continuous function on  $[a, b]$ . Then*

$$(2.4) \quad \left| \tilde{\Delta}(f, u; a, b) \right| \leq \frac{1}{2} \left[ \max_{t \in [a, b]} f(t) - \min_{t \in [a, b]} f(t) \right] \bigvee_a^b(u),$$

where

$$\tilde{\Delta}(f, u; a, b) := \int_a^b f(t) du(t) - \frac{\min_{t \in [a, b]} f(t) + \max_{t \in [a, b]} f(t)}{2} [u(b) - u(a)].$$

The constant  $\frac{1}{2}$  is best possible.

*Proof.* For the sharpness of the constant, we cannot use the above example since  $f_0$  was not continuous on  $[a, b]$ .

Let us now consider  $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$  and  $f_0(t) = \left|t - \frac{a+b}{2}\right|$ . The Stieltjes integral  $\int_a^b f_0(t) du_0(t)$  exists and

$$\begin{aligned} & \int_a^b f_0(t) du_0(t) \\ &= f_0(t) u_0(t) \Big|_a^b - \int_a^b u_0(t) df_0(t) \\ &= \frac{b-a}{2} + \frac{b-a}{2} - \left[ \int_a^{\frac{a+b}{2}} (-1) d\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (1) d\left(t - \frac{a+b}{2}\right) \right] \\ &= 0 \end{aligned}$$

we have then

$$\left| \tilde{\Delta}(f_0, u_0; a, b) \right| = \frac{b-a}{2}.$$

Also

$$\frac{1}{2} \left[ \max_{t \in [a, b]} f_0(t) - \min_{t \in [a, b]} f_0(t) \right] \bigvee_a^b(u_0) = \frac{b-a}{2},$$

which shows that the equality case holds in (2.4). ■

The following result providing bounds for the Lipschitzain integrators may be stated as well:

**Theorem 2.** *If  $u : [a, b] \rightarrow \mathbb{R}$  is  $L$ -Lipschitzian and  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable and satisfies the condition (2.1), then*

$$(2.5) \quad |\Delta(f, u, m, M; a, b)| \leq \frac{1}{2} (M - m) L (b - a).$$

*The constant  $\frac{1}{2}$  is best possible.*

*Proof.* It is well known that if  $p$  is Riemann integrable on  $[a, b]$  and  $v$  is  $L$ -Lipschitzian on  $[a, b]$ , then the Stieltjes integral  $\int_a^b p(t) dv(t)$  exists and

$$(2.6) \quad \left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| dt.$$

Now, taking into account that  $f - \frac{m+M}{2}$  is Riemann integrable, by making use of (2.6) we have

$$\begin{aligned} \left| \int_a^b \left( f(t) - \frac{m+M}{2} \right) du(t) \right| &\leq L \int_a^b \left| f(t) - \frac{m+M}{2} \right| dt \\ &\leq \frac{1}{2} (M - m) L (b - a) \end{aligned}$$

and the desired inequality (2.5) is obtained.

To prove the sharpness of the constant  $\frac{1}{2}$ , assume that the inequality (2.5) holds with a positive constant  $D$ , i.e.,

$$(2.7) \quad |\Delta(f, u, m, M; a, b)| \leq D (M - m) L (b - a),$$

provided  $f$  is Riemann integrable and satisfies (2.1) while  $u$  is Lipschitz continuous with the constant  $L > 0$ .

Consider the functions  $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$  and  $u_0(t) = \left|t - \frac{a+b}{2}\right|$ . It is obvious that  $f_0$  is Riemann integrable and  $M_0 = 1$ ,  $m_0 = -1$ . Since, by the triangle inequality we have

$$|u_0(t) - u_0(s)| = \left| \left|t - \frac{a+b}{2}\right| - \left|s - \frac{a+b}{2}\right| \right| \leq |t - s|,$$

for any  $t, s \in [a, b]$ , hence  $u_0$  is Lipschitzian with the constant  $L = 1$ . Now, observe that

$$\begin{aligned} \int_a^b f_0(t) du_0(t) &= \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) d\left(\left|t - \frac{a+b}{2}\right|\right) \\ &= \int_a^{\frac{a+b}{2}} (-1) d\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (1) d\left(t - \frac{a+b}{2}\right) \\ &= b - a, \end{aligned}$$

and introducing the above values in (2.7) we deduce

$$b - a \leq 2D(b - a),$$

which implies that  $D \geq \frac{1}{2}$ . ■

**Corollary 2.** *If  $f$  is continuous on  $[a, b]$  and  $u$  is  $L$ -Lipschitzian, then:*

$$(2.8) \quad \left| \tilde{\Delta}(f, u; a, b) \right| \leq \frac{1}{2} \left[ \max_{t \in [a, b]} f(t) - \min_{t \in [a, b]} f(t) \right] L(b - a).$$

The constant  $\frac{1}{2}$  is best possible.

*Proof.* In order to prove the sharpness of the constant, we cannot use the example from Theorem 2 since  $f_0$  was not continuous.

If  $u_0(t) = \left|t - \frac{a+b}{2}\right|$  and  $f_0$  is continuous, then

$$\int_a^b f_0(t) d\left|t - \frac{a+b}{2}\right| = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_0(t) dt.$$

Consider now the sequence of continuous functions

$$f_{0,n}(t) = \begin{cases} -1 & \text{if } t \in \left[a, \frac{a+b}{2} - \frac{1}{n}\right]; \\ -1 + n\left(t - \frac{a+b}{2} + \frac{1}{n}\right) & \text{if } t \in \left(\frac{a+b}{2} - \frac{1}{n}, \frac{a+b}{2} + \frac{1}{n}\right); \\ 1 & \text{if } t \in \left[\frac{a+b}{2} + \frac{1}{n}, b\right], \end{cases}$$

which coincides with  $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$  on  $\left[a, \frac{a+b}{2} - \frac{1}{n}\right] \cup \left[\frac{a+b}{2} + \frac{1}{n}, b\right]$  and connects the end segments of this function on  $\left[\frac{a+b}{2} - \frac{1}{n}, \frac{a+b}{2} + \frac{1}{n}\right]$  respectively. Obviously

$$\begin{aligned} &\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_{0,n}(t) dt \\ &= \int_a^{\frac{a+b}{2} - \frac{1}{n}} dt + \int_{\frac{a+b}{2} - \frac{1}{n}}^{\frac{a+b}{2} + \frac{1}{n}} \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_{0,n}(t) dt + \int_{\frac{a+b}{2} + \frac{1}{n}}^b dt \\ &= b - a + x_n, \end{aligned}$$

where

$$|x_n| = \left| \int_{\frac{a+b}{2} - \frac{1}{n}}^{\frac{a+b}{2} + \frac{1}{n}} \operatorname{sgn} \left( t - \frac{a+b}{2} \right) f_{0,n}(t) dt \right| \leq \frac{2}{n}.$$

Now, if (2.8) holds with a constant  $E > 0$ , i.e.,

$$\left| \tilde{\Delta}(f, u; a, b) \right| \leq E \left[ \max_{t \in [a, b]} f(t) - \min_{t \in [a, b]} f(t) \right] L(b-a),$$

then on choosing  $f_{0,n}$  and  $u_0$  as above, we get

$$b-a+x_n \leq 2E(b-a)$$

for each  $n \in \mathbb{N}$ . Letting  $n \rightarrow \infty$  and taking into account that  $\lim_{n \rightarrow \infty} x_n = 0$ , we deduce  $E \geq \frac{1}{2}$ , and the corollary is proved. ■

**Corollary 3.** *Let  $f, h : [a, b] \rightarrow \mathbb{R}$  be Riemann integrable functions,  $f$  satisfies (2.1) and  $|h(t)| \leq N$  for a.e.  $t \in [a, b]$ . Then*

$$(2.9) \quad \left| \int_a^b f(t) h(t) dt - \frac{m+M}{2} \int_a^b h(t) dt \right| \leq \frac{1}{2} (M-m) N (b-a).$$

The constant  $\frac{1}{2}$  is best possible.

The proof follows by (2.5) on choosing  $u(t) = \int_a^t h(s) ds$ . The details are omitted. Finally, we can state the following result as well.

**Theorem 3.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a monotonic nondecreasing function on  $[a, b]$  and  $f : [a, b] \rightarrow \mathbb{R}$  a bounded function satisfying (2.1) and such that  $\int_a^b f(t) du(t)$  exists. Then*

$$(2.10) \quad \begin{aligned} |\Delta(f, u, m, M; a, b)| &\leq \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) \\ &\leq \frac{1}{2} (M-m) [u(b) - u(a)]. \end{aligned}$$

The first inequality in (2.10) is sharp. The constant  $\frac{1}{2}$  is best possible.

*Proof.* The inequality

$$\left| \int_a^b \left( f(t) - \frac{m+M}{2} \right) du(t) \right| \leq \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t)$$

follows by the definition of Stieltjes integrals.

Since

$$\left| f(t) - \frac{m+M}{2} \right| \leq \frac{1}{2} (M-m) \quad \text{for each } t \in [a, b],$$

we also have that

$$\begin{aligned} \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) &\leq \frac{1}{2} (M-m) \int_a^b du(t) \\ &= \frac{1}{2} (M-m) [u(b) - u(a)] \end{aligned}$$

and the inequality (2.10) is thus proved.

Now, assume that  $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ ,  $t \in [a, b]$ . Then for any continuous and monotonic nondecreasing function  $u_0 : [a, b] \rightarrow \mathbb{R}$  we can state that

$$\begin{aligned} \Delta(f_0, u_0, m_0, M_0; a, b) &= \int_a^{\frac{a+b}{2}} (-1) du_0(t) + \int_{\frac{a+b}{2}}^b (1) du_0(t) \\ &= u_0(a) + u_0(b) - 2u_0\left(\frac{a+b}{2}\right). \end{aligned}$$

Also,

$$\int_a^b \left| f_0(t) - \frac{m_0 + M_0}{2} \right| du_0(t) = u_0(b) - u_0(a)$$

and

$$\frac{1}{2} (M_0 - m_0) [u_0(b) - u_0(a)] = u_0(b) - u_0(a),$$

which shows that the last inequality holds with equality in (1.9).

Finally, to have equality in the first part of (2.10) it is sufficient selecting  $u_0$  to vanish in  $[a, \frac{a+b}{2}]$  and being continuous and monotonic nondecreasing on  $[\frac{a+b}{2}, b]$ . In this situation we get in all terms of (2.10) the same quantity  $u_0(b)$ . ■

**Corollary 4.** *If  $f$  is continuous on  $[a, b]$  and  $u$  is monotonic nondecreasing, then*

$$\begin{aligned} (2.11) \quad \left| \tilde{\Delta}(f, u; a, b) \right| &\leq \int_a^b \left| f(t) - \frac{\min_{t \in [a, b]} f(t) + \max_{t \in [a, b]} f(t)}{2} \right| du(t) \\ &\leq \frac{1}{2} \left[ \max_{t \in [a, b]} f(t) - \min_{t \in [a, b]} f(t) \right] [u(b) - u(a)]. \end{aligned}$$

To prove the sharpness of the inequality we use the functions  $f_0(t) = |t - \frac{a+b}{2}|$  and  $u_0(t) = \operatorname{sgn}(t - \frac{a+b}{2})$  which produce in all terms of (2.11) the quantity  $\frac{b-a}{2}$ .

**Corollary 5.** *If  $f, w$  are Riemann integrable on  $[a, b]$  and  $f$  satisfies (2.1) while  $w$  is nonnegative, then*

$$\begin{aligned} (2.12) \quad \left| \int_a^b f(t) w(t) dt - \frac{m+M}{2} \int_a^b w(t) dt \right| &\leq \int_a^b \left| f(t) - \frac{m+M}{2} \right| w(t) dt \\ &\leq \frac{1}{2} (M - m) \int_a^b w(t) dt. \end{aligned}$$

The dual case, i.e., when the integrator is bounded below and above, is incorporated in the following result.

**Theorem 4.** *Assume that  $u$  is Riemann integrable on  $[a, b]$  and*

$$(2.13) \quad -\infty < n \leq u(t) \leq N < \infty \quad \text{for a.e. } t \in [a, b].$$

*Define the error functional of generalised trapezoid type*

$$\nabla(f, u, n, N; a, b) := \left[ u(b) - \frac{n+N}{2} \right] f(b) + \left[ \frac{n+N}{2} - u(a) \right] f(a) - \int_a^b f(t) du(t).$$



- (i) If  $f$  is of bounded variation and such that the Stieltjes integral  $\int_a^b f(t) du(t)$  exists, then

$$(2.14) \quad |\nabla(f, u, n, N; a, b)| \leq \frac{1}{2} (N - n) \bigvee_a^b(f).$$

The constant  $\frac{1}{2}$  is best possible in (2.14).

- (ii) If  $f$  is  $K$ -Lipschitzian on  $[a, b]$ , then

$$(2.15) \quad |\nabla(f, u, n, N; a, b)| \leq \frac{1}{2} (N - n) K (b - a).$$

The constant  $\frac{1}{2}$  is best possible in (2.15).

- (iii) If  $f$  is monotonic nondecreasing on  $[a, b]$  such that the Stieltjes integrals,  $\int_a^b f(t) du(t)$ ,  $\int_a^b |u(t) - \frac{n+N}{2}| df(t)$  exist, then

$$(2.16) \quad \begin{aligned} |\nabla(f, u, n, N; a, b)| &\leq \int_a^b \left| u(t) - \frac{n+N}{2} \right| df(t) \\ &\leq \frac{1}{2} (N - n) [f(b) - f(a)]. \end{aligned}$$

The first inequality is sharp and the constant  $\frac{1}{2}$  is best possible in (2.16).

*Proof.* The proof follows by Theorems 1 – 3 on utilising the integral identity:

$$\begin{aligned} &\left[ u(b) - \frac{n+N}{2} \right] f(b) + \left[ \frac{n+N}{2} - u(a) \right] f(a) - \int_a^b f(t) du(t) \\ &= \int_a^b \left[ u(t) - \frac{n+N}{2} \right] df(t) \end{aligned}$$

and the details are omitted. ■

**Remark 1.** The above inequalities also hold for continuous functions  $u : [a, b] \rightarrow \mathbb{R}$  when  $n$  is replaced by  $\min_{t \in [a, b]} u(t)$  and  $N$  is replaced by  $\max_{t \in [a, b]} u(t)$ . The details are left to the interested reader.

### 3. APPLICATIONS FOR THREE POINT QUADRATURE RULES

In [1] (see also [10, p. 223]) P. Cerone and S.S. Dragomir established the following three point quadrature rule for  $n$ -times differentiable functions:

$$(3.1) \quad \begin{aligned} \int_a^b f(t) dt &= \sum_{k=1}^n \frac{1}{k!} \left\{ (1-\gamma)^k \left[ (b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right. \\ &\quad \left. + \gamma^k \left[ (x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\} \\ &\quad + (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt, \end{aligned}$$

where

$$(3.2) \quad C_n(x, t) = \begin{cases} \frac{[t - (\gamma x + (1-\gamma)a)]^n}{n!} & \text{if } t \in [a, x]; \\ \frac{[t - (\gamma x + (1-\gamma)b)]^n}{n!} & \text{if } t \in (x, b], \end{cases}$$

and  $\gamma \in [0, 1]$ ,  $x \in (a, b)$ .

This representation comprises amongst others the interior point quadrature rule obtained by Cerone et al. [3] in 1999 for  $\gamma = 0$  and the trapezoid quadrature rule obtained by Cerone et al. [4] in 2000 for  $\gamma = 1$ .

Consider the function:

$$(3.3) \quad K_n(x, t) := (-1)^n \begin{cases} \frac{[t - (\gamma x + (1 - \gamma)a)]^{n+1}}{(n+1)!} & \text{if } t \in [a, x]; \\ \frac{[t - (\gamma x + (1 - \gamma)b)]^{n+1}}{(n+1)!} & \text{if } t \in (x, b]. \end{cases}$$

The function  $K_n(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ , for each fixed  $x \in [a, b]$ , is of bounded variation and

$$\begin{aligned} \bigvee_a^b(K_n(x, \cdot)) &= \int_a^x \left| \frac{dK_n(x, t)}{dt} \right| dt + \int_x^b \left| \frac{dK_n(x, t)}{dt} \right| dt \\ &= \int_a^x \frac{|t - (\gamma x + (1 - \gamma)a)|^n}{n!} dt + \int_x^b \frac{|\gamma x + (1 - \gamma)b - t|^n}{n!} dt. \end{aligned}$$

We have

$$\begin{aligned} I_1 &= \int_a^x \frac{|t - (\gamma x + (1 - \gamma)a)|^n}{n!} dt \\ &= \int_a^{\gamma x + (1 - \gamma)a} \frac{[\gamma x + (1 - \gamma)a - t]^n}{n!} dt + \int_{\gamma x + (1 - \gamma)a}^x \frac{[t - (\gamma x + (1 - \gamma)a)]^n}{n!} dt \\ &= - \left[ \frac{[\gamma x + (1 - \gamma)a - t]^{n+1}}{(n+1)!} \right]_a^{\gamma x + (1 - \gamma)a} + \left[ \frac{[t - (\gamma x + (1 - \gamma)a)]^{n+1}}{(n+1)!} \right]_{\gamma x + (1 - \gamma)a}^x \\ &= \frac{\gamma^{n+1} (x - a)^{n+1}}{(n+1)!} + \frac{(1 - \gamma)^{n+1} (x - a)^{n+1}}{(n+1)!} \\ &= \frac{1}{(n+1)!} (x - a)^{n+1} [\gamma^{n+1} + (1 - \gamma)^{n+1}] \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_x^b \frac{|\gamma x + (1 - \gamma)b - t|^n}{n!} dt \\ &= \int_x^{\gamma x + (1 - \gamma)b} \frac{[\gamma x + (1 - \gamma)b - t]^n}{n!} dt + \int_{\gamma x + (1 - \gamma)b}^b \frac{[t - (\gamma x + (1 - \gamma)b)]^n}{n!} dt \\ &= - \left[ \frac{[\gamma x + (1 - \gamma)b - t]^{n+1}}{(n+1)!} \right]_x^{\gamma x + (1 - \gamma)b} + \left[ \frac{[t - (\gamma x + (1 - \gamma)b)]^{n+1}}{(n+1)!} \right]_{\gamma x + (1 - \gamma)b}^b \\ &= \frac{(1 - \gamma)^{n+1} (b - x)^{n+1}}{(n+1)!} + \frac{\gamma^{n+1} (b - x)^{n+1}}{(n+1)!} \\ &= \frac{1}{(n+1)!} (b - x)^{n+1} [\gamma^{n+1} + (1 - \gamma)^{n+1}]. \end{aligned}$$

Therefore

$$(3.4) \quad \bigvee_a^b(K_n(x, \cdot)) = \frac{1}{(n+1)!} [\gamma^{n+1} + (1 - \gamma)^{n+1}] [(b - x)^{n+1} + (x - a)^{n+1}].$$

We also have

$$\begin{aligned}
 (3.5) \quad \int_a^b f^{(n)}(t) d(K_n(x, t)) &= \int_a^x f^{(n)}(t) d \left[ (-1)^n \frac{[t - (\gamma x + (1 - \gamma)a)]^{n+1}}{(n+1)!} \right] \\
 &\quad + \int_x^b f^{(n)}(t) d \left[ (-1)^n \frac{[t - (\gamma x + (1 - \gamma)b)]^{n+1}}{(n+1)!} \right] \\
 &= (-1)^n \int_a^b C_n(t, x) f^{(n)}(t) dt,
 \end{aligned}$$

with  $C_n(t, x)$  defined by (3.2).

We can state the following result in approximating the Riemann integral  $\int_a^b f(x) dx$  of  $n$ -times differentiable functions  $f$  in terms of three point quadrature rules.

**Theorem 5.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function such that for  $n \geq 1$  the derivative  $f^{(n-1)}$  is absolutely continuous and there exists the real constants  $\gamma_n, \Gamma_n$  such that*

$$(3.6) \quad \gamma_n \leq f^{(n)}(t) \leq \Gamma_n \quad \text{for a.e. } t \in [a, b].$$

Then

$$\begin{aligned}
 (3.7) \quad \int_a^b f(t) dt &= \sum_{k=1}^n \frac{1}{k!} \left\{ (1 - \gamma)^k [(b - x)^k + (-1)^{k-1} (x - a)^k] f^{(k-1)}(x) \right. \\
 &\quad \left. + \gamma^k [(x - a)^k f^{(k-1)}(a) + (-1)^{k-1} (b - x)^k f^{(k-1)}(b)] \right\} \\
 &\quad + \frac{\gamma_n + \Gamma_n}{2} [(-1)^n (b - x)^{n+1} + (x - a)^{n+1}] \frac{\gamma^{n+1}}{(n+1)!} + R_n,
 \end{aligned}$$

and the error  $R_n$  satisfies the bound

$$(3.8) \quad |R_n| \leq \frac{1}{2} (\Gamma_n - \gamma_n) \frac{1}{(n+1)!} [\gamma^{n+1} + (1 - \gamma)^{n+1}] [(b - x)^{n+1} + (x - a)^{n+1}]$$

for  $\gamma \in [0, 1]$  and  $x \in [a, b]$ .

*Proof.* We apply Theorem 1 for the functions  $f^{(n)}$  and  $K(x, \cdot)$  to get:

$$\begin{aligned}
 \left| \int_a^b f^{(n)}(t) dK_n(x, t) - \frac{\gamma_n + \Gamma_n}{2} [K_n(x, b) - K_n(x, a)] \right| \\
 \leq \frac{1}{2} (\Gamma_n - \gamma_n) \bigvee_a^b (K_n(x, \cdot))
 \end{aligned}$$

for  $x \in [a, b]$ .

Since

$$\begin{aligned}
 K_n(x, b) &= (-1)^n \frac{[b - (\gamma x + (1 - \gamma)b)]^{n+1}}{(n+1)!} \\
 &= (-1)^n \frac{\gamma^{n+1} (b - x)^{n+1}}{(n+1)!}
 \end{aligned}$$

and

$$\begin{aligned} K_n(x, a) &= (-1)^n \frac{[a - (\gamma x + (1 - \gamma)a)]^{n+1}}{(n+1)!} \\ &= (-1)^n \frac{[\gamma(a-x)]^{n+1}}{(n+1)!} \\ &= -\frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!}, \end{aligned}$$

hence by (3.4) and (3.5) we deduce:

$$\begin{aligned} (3.9) \quad & \left| (-1)^n \int_a^b C_n(t, x) f^{(n)}(t) dt \right. \\ & \left. - \frac{\gamma_n + \Gamma_n}{2} \left[ (-1)^n \frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!} + \frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!} \right] \right| \\ & \leq \frac{1}{2} (\Gamma_n - \gamma_n) \frac{1}{(n+1)!} [\gamma^{n+1} + (1-\gamma)^{n+1}] [(b-x)^{n+1} + (x-a)^{n+1}] .. \end{aligned}$$

Finally, on utilising the identity (3.1) we deduce from (3.9) the representation (3.7) and the estimate (3.8). ■

**Remark 2.** The above approximation of the integral  $\int_a^b f(t) dt$  contains some particular cases of interest.

If  $\lambda = 0$ , then we have

$$(3.10) \quad \int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{k!} [(b-x)^k + (-1)^{k-1}(x-a)^k] f^{(k-1)}(x) + T_n,$$

with

$$|T_n| \leq \frac{1}{2} (\Gamma_n - \gamma_n) \frac{1}{(n+1)!} [(b-x)^{n+1} + (x-a)^{n+1}].$$

If  $\lambda = \frac{1}{2}$ , then we have

$$\begin{aligned} (3.11) \quad \int_a^b f(t) dt &= \sum_{k=1}^n \frac{1}{2^k k!} \left\{ [(b-x)^k + (-1)^{k-1}(x-a)^k] f^{(k-1)}(x) \right. \\ &\quad \left. + [(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)] \right\} \\ &\quad + \frac{\gamma_n + \Gamma_n}{2^{n+2}(n+1)!} [(-1)^n (b-x)^{n+1} + (x-a)^{n+1}] + M_n, \end{aligned}$$

with

$$|M_n| \leq \frac{1}{2^{n+1}(n+1)!} (\Gamma_n - \gamma_n) [(b-x)^{n+1} + (x-a)^{n+1}].$$

Finally, if  $\lambda = 1$ , then we have

$$\begin{aligned} (3.12) \quad \int_a^b f(t) dt &= \sum_{k=1}^n \frac{1}{k!} [(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)] \\ &\quad + \frac{\gamma_n + \Gamma_n}{2(n+1)!} [(-1)^n (b-x)^{n+1} + (x-a)^{n+1}] + Q_n, \end{aligned}$$

with

$$|Q_n| \leq \frac{1}{2(n+1)!} (\Gamma_n - \gamma_n) \left[ (b-x)^{n+1} + (x-a)^{n+1} \right].$$

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