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This is the Published version of the following publication

Dragomir, Sever S and Sofo, Anthony (2007) Approximating the Stieltjes Integral via the Darst-Pollard Inequality. Research report collection, 10 (Supp).

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# APPROXIMATING THE STIELTJES INTEGRAL VIA THE DARST-POLLARD INEQUALITY

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ABSTRACT. An approximation of the Stieltjes integral of bounded integrals and continuous integrators via the Darst-Pollard inequality is given. Applications for the generalised trapezoid formula and the Ostrowski inequality for functions of bounded variation are also provided.

## 1. INTRODUCTION

In 1970, R. Darst and H. Pollard [2] obtained by elementary arguments the following interesting inequality for the Stieltjes integral.

**Theorem 1** (Darst-Pollard, 1970). *If  $h$  is real and of bounded variation on the interval  $[a, b]$  and  $g$  is real and continuous there, then*

$$(1.1) \quad \int_a^b h(t) dg(t) \leq \inf_{t \in [a, b]} h(t) [g(b) - g(a)] + S(g; a, b) \cdot \bigvee_a^b(h),$$

where  $\bigvee_a^b(h)$  is the total variation of  $h$  on  $[a, b]$  and

$$(1.2) \quad S(g; a, b) := \sup_{a \leq \alpha < \beta \leq b} [g(\beta) - g(\alpha)].$$

In the recent paper [4], in order to approximate the Stieltjes integral  $\int_a^b f(t) du(t)$  by the quadrature rule

$$\frac{m + M}{2} [u(b) - u(a)],$$

where  $m \leq f(t) \leq M$  for each  $t \in [a, b]$ , the first author defined the *error functional*

$$\Delta(f, u, m, M; a, b) := \int_a^b f(t) du(t) - \frac{m + M}{2} [u(b) - u(a)]$$

and showed that

$$(1.3) \quad |\Delta(f, u, m, M; a, b)| \leq \begin{cases} \frac{1}{2} (M - m) \bigvee_a^b(u) & \text{if } u \text{ is of bounded variation,} \\ \frac{1}{2} (M - m) L (b - a) & \text{if } u \text{ is } L\text{-Lipschitzian,} \\ \int_a^b |f(t) - \frac{m+M}{2}| du(t) & \text{if } u \text{ is monotonic nondecreasing.} \end{cases}$$

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*Date:* 20th February, 2007.

*2000 Mathematics Subject Classification.* 26D15, 41A55.

*Key words and phrases.* Stieltjes integral, Darst-Pollard inequality, Generalised trapezoid rule, Ostrowski inequality.

The constant  $\frac{1}{2}$  is best possible in both inequalities. The last inequality in (1.3) is also sharp.

In the same paper [4], in order to approximate the Stieltjes integral  $\int_a^b f(t) du(t)$  in terms of the *generalised trapezoid type* rule

$$\left[ u(b) - \frac{n+N}{2} \right] f(b) + \left[ \frac{n+N}{2} - u(a) \right] f(a),$$

the first author also considered the *error functional*

$$\nabla(f, u, n, N; a, b) := \left[ u(b) - \frac{n+N}{2} \right] f(b) + \left[ \frac{n+N}{2} - u(a) \right] f(a) - \int_a^b f(t) du(t),$$

where  $-\infty < n \leq u(t) \leq N < \infty$  for  $t \in [a, b]$  and showed that

$$(1.4) \quad |\nabla(f, u, n, N; a, b)| \leq \begin{cases} \frac{1}{2}(N-n) \bigvee_a^b(f) & \text{if } f \text{ is of bounded variation,} \\ \frac{1}{2}(N-n) K(b-a) & \text{if } f \text{ is } K\text{-Lipschitzian,} \\ \int_a^b |u(t) - \frac{n+N}{2}| df(t) & \text{if } f \text{ is monotonic nondecreasing.} \end{cases}$$

The constant  $\frac{1}{2}$  is best possible in both inequalities above and the last one is sharp.

The main aim of the present paper is to provide other error estimates for the functionals  $\Delta$  and  $\nabla$  defined above by utilising as a main tool the Darst-Pollard inequality stated in (1.1). Applications for the generalised trapezoid formula and the Ostrowski inequality for functions of bounded variation are also provided.

## 2. THE RESULTS

We can state the following result in estimating the error functional  $\Delta(f, u, m, M; a, b)$  :

**Theorem 2.** *Let  $u$  be continuous on  $[a, b]$  and  $f : [a, b] \rightarrow \mathbb{R}$  of bounded variation such that*

$$(2.1) \quad -\infty < m = \inf_{t \in [a, b]} f(t), \quad \sup_{t \in [a, b]} f(t) = M < \infty.$$

*Then:*

$$(2.2) \quad |\Delta(f, u, m, M; a, b)| \leq \bigvee_a^b(f) \cdot S(u; a, b) - \frac{1}{2}(M-m)[u(b) - u(a)].$$

*The inequality (2.2) is sharp.*

*Proof.* Let us denote  $K(a, b) := \bigvee_a^b(f) \cdot S(u; a, b)$ .

If we apply the Darst-Pollard inequality (1.1) for  $h = f$  and  $g = u$ , we have

$$(2.3) \quad \int_a^b f(t) du(t) \leq m[u(b) - u(a)] + K(a, b).$$

Now assume that  $h := M - f$ . Then  $h$  is of bounded variation on  $[a, b]$ ,

$$\bigvee_a^b(h) = \bigvee_a^b(M - f) = \bigvee_a^b(f)$$

and  $\inf_{t \in [a, b]} h(t) = 0$ . On applying the Darst-Pollard inequality (1.1), we obtain

$$\int_a^b (M - f(t)) du(t) \leq K(a, b)$$

which is clearly equivalent with

$$(2.4) \quad M[u(b) - u(a)] - K(a, b) \leq \int_a^b f(t) du(t).$$

Now, subtracting the same quantity  $\frac{m+M}{2}[u(b) - u(a)]$  in both (2.3) and (2.4), we deduce

$$(2.5) \quad \begin{aligned} & \frac{1}{2}(M - m)[u(b) - u(a)] - K(a, b) \\ & \leq \int_a^b f(t) du(t) - \frac{m+M}{2}[u(b) - u(a)] \\ & \leq -\frac{1}{2}(M - m)[u(b) - u(a)] + K(a, b) \end{aligned}$$

which is clearly equivalent with the desired inequality (2.2).

For the sharpness of the inequality, let us assume that  $u(t) = t$ ,  $t \in [a, b]$ . Then (1.2) becomes

$$(2.6) \quad \left| \int_a^b f(t) dt - \frac{m+M}{2}(b-a) \right| \leq (b-a) \left[ \bigvee_a^b(f) - \frac{1}{2}(M-m) \right].$$

Now, if we choose the function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a, b] \\ k & \text{if } t = b, k > 0, \end{cases}$$

then we have  $m = 0$ ,  $M = k$ ,  $\int_a^b f(t) dt = 0$ ,  $\bigvee_a^b(f) = k$  and in both sides of (2.6) we obtain the same quantity  $\frac{1}{2}k(b-a)$ . ■

The following particular cases are of interest.

**Corollary 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be as in Theorem 2. If  $u : [a, b] \rightarrow \mathbb{R}$  is of  $r - H - \text{H\"older type}$  ( $L - \text{Lipschitzian}$ ), i.e.,*

$$(2.7) \quad |u(t) - u(s)| \leq H|t - s|^r \quad (\leq L|t - s|)$$

for any  $t, s \in [a, b]$ , where  $H > 0$ ,  $r \in (0, 1)$  ( $L > 0$ ) are given, then:

$$(2.8) \quad \begin{aligned} |\Delta(f, u, m, M; a, b)| & \leq H(b-a)^r \bigvee_a^b(f) - \frac{1}{2}(M-m)[u(b) - u(a)] \\ & \left( \leq L(b-a) \bigvee_a^b(f) - \frac{1}{2}(M-m)[u(b) - u(a)] \right). \end{aligned}$$

The case for Lipschitzian function  $u$  provides a sharp inequality.

The proof is obvious from the above theorem since

$$S(u; a, b) \leq H(b-a)^r \quad (L(b-a))$$

and the sharpness of the inequality has been clearly proven for the function  $u(t) = t$ .

The case of absolutely continuous integrators  $u : [a, b] \rightarrow \mathbb{R}$  is incorporated in the following corollary.

**Corollary 2.** *Let  $f$  be as in Theorem 2. If  $u : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous on  $[a, b]$ , then*

$$(2.9) \quad |\Delta(f, u, m, M; a, b)| \leq \begin{cases} (b-a) \|u'\|_{\infty, [a, b]} V_a^b(f) - \frac{1}{2} (M-m) [u(b) - u(a)] & \text{if } u' \in L_\infty[a, b]; \\ (b-a)^{\frac{1}{q}} \|u'\|_{p, [a, b]} V_a^b(f) - \frac{1}{2} (M-m) [u(b) - u(a)] & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|u'\|_{1, [a, b]} V_a^b(f) - \frac{1}{2} (M-m) [u(b) - u(a)], & \end{cases}$$

where  $\|\cdot\|_{p, [a, b]}$  are the usual Lebesgue norms,  $p \in [1, \infty]$ .

*Proof.* Since  $u$  is absolutely continuous, hence for  $a \leq \alpha < \beta \leq b$  we have

$$u(\beta) - u(\alpha) = \int_\alpha^\beta u'(s) ds \leq \begin{cases} (\beta - \alpha) \|u'\|_{\infty, [a, b]} & \text{if } u' \in L_\infty[a, b]; \\ (\beta - \alpha)^{\frac{1}{q}} \|u'\|_{p, [a, b]} & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|u'\|_{1, [a, b]}; & \end{cases}$$

which gives

$$S(u; a, b) \leq \begin{cases} (b-a) \|u'\|_{\infty, [a, b]} & \text{if } u' \in L_\infty[a, b]; \\ (b-a)^{\frac{1}{q}} \|u'\|_{p, [a, b]} & \text{if } u' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|u'\|_{1, [a, b]}. & \end{cases}$$

These together with (2.2) produces (2.9). ■

The case of monotonic integrators is considered in the following.

**Corollary 3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be as in Theorem 2 and  $u : [a, b] \rightarrow \mathbb{R}$  a continuous and monotonic function on  $[a, b]$ . Then*

$$(2.10) \quad |\Delta(f, u, m, M; a, b)| \leq [u(b) - u(a)] \left[ V_a^b(f) - \frac{1}{2} (M-m) \right] \left( \leq \frac{1}{2} (M-m) [u(b) - u(a)] \right).$$

*The inequality is sharp.*

The proof is obvious from the fact that for monotonic functions  $u : [a, b] \rightarrow \mathbb{R}$  one can derive that  $S(u; a, b) = u(b) - u(a)$ .

The following lemma may be stated (see also [4]).

**Lemma 1.** Let  $u : [a, b] \rightarrow \mathbb{R}$  and  $\varphi, \phi \in \mathbb{R}$  with  $\phi > \varphi$ . The following statements are equivalent:

- (i) The function  $u - \frac{\varphi + \Phi}{2} \cdot e$ , where  $e(t) = t$ ,  $t \in [a, b]$  is  $\frac{1}{2}(\Phi - \varphi)$ -Lipschitzian;
- (ii) We have the inequality:

$$(2.11) \quad \varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s.$$

- (iii) We have the inequality:

$$(2.12) \quad \varphi(t - s) \leq u(t) - u(s) \leq \Phi(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s.$$

Following [5], we can introduce the concept

**Definition 1.** The function  $u : [a, b] \rightarrow \mathbb{R}$  which satisfies one of the equivalent conditions (i) – (iii) is said to be  $(\varphi, \Phi)$ -Lipschitzian on  $[a, b]$ .

Notice that in [5], the definition was introduced on utilising the statement (iii) and only the equivalence (i)  $\iff$  (iii) was considered.

Utilising Lagrange's mean value theorem, we can state the following result that provides practical examples of  $(\varphi, \Phi)$ -Lipschitzian functions:

**Proposition 1.** Let  $u : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $[a, b]$ . If

$$-\infty < \gamma = \inf_{t \in (a, b)} u'(t), \quad \sup_{t \in (a, b)} u'(t) = \Gamma < \infty,$$

then  $u$  is  $(\Gamma, \gamma)$ -Lipschitzian on  $[a, b]$ .

Now the following corollary can be stated as well.

**Corollary 4.** If  $f : [a, b] \rightarrow \mathbb{R}$  is as in Theorem 2 and  $u : [a, b] \rightarrow \mathbb{R}$  is  $(\varphi, \Phi)$ -Lipschitzian with  $\varphi > 0$ , then

$$(2.13) \quad |\Delta(f, u, m, M; a, b)| \leq \left[ \Phi \cdot \bigvee_a^b(f) - \frac{1}{2}(M - m) \cdot \varphi \right] (b - a).$$

The following result may be stated as well.

**Theorem 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and  $u : [a, b] \rightarrow \mathbb{R}$  a function of bounded variation such that

$$-\infty < n = \inf_{t \in [a, b]} u(t), \quad \sup_{t \in [a, b]} u(t) = N < \infty.$$

Then we have the inequality:

$$(2.14) \quad |\nabla(f, u, n, N; a, b)| \leq \bigvee_a^b(u) S(f; a, b) - \frac{1}{2}(N - n)[f(b) - f(a)].$$

The inequality is sharp.

*Proof.* Follows from Theorem 2 on utilising the identity

$$\begin{aligned} & f(b) \left[ u(b) - \frac{n + N}{2} \right] + f(a) \left[ \frac{n + N}{2} - u(a) \right] - \int_a^b f(t) du(t) \\ &= \int_a^b \left[ u(t) - \frac{n + N}{2} \right] df(t) \\ &= \int_a^b u(t) df(t) - \frac{n + N}{2} [f(b) - f(a)]. \end{aligned}$$

The details are omitted. ■

Similar corollaries to Corollary 1 – Corollary 4 may be stated. We leave them to the interested reader.

### 3. APPLICATIONS FOR THE TRAPEZOID RULE

In this section we provide some applications in connection with the generalised trapezoid rule.

In [1], in order to approximate the integral  $\int_a^b f(t) dt$  for the function  $f : [a, b] \rightarrow \mathbb{R}$  of bounded variation with the generalised trapezoid rule

$$f(a)(x-a) + f(b)(b-x), \quad x \in [a, b]$$

the authors have considered the *generalised trapezoid error functional*

$$T(f; a, b, x) := \int_a^b f(t) dt - [f(a)(x-a) + f(b)(b-x)],$$

and obtained the following sharp bound

$$(3.1) \quad |T(f; a, b, x)| \leq \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

for each  $x \in [a, b]$ .

The best inequality we can derive from (3.1) is the following trapezoid inequality for functions of bounded variation:

$$(3.2) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \cdot (b-a) \right| \leq \frac{1}{2}(b-a) \bigvee_a^b(f),$$

where the constant  $\frac{1}{2}$  is best possible in the sense that it cannot be replaced by a smaller constant.

The version of (3.2) for continuous functions is incorporated in

**Proposition 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then*

$$(3.3) \quad \left| \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt \right| \leq (b-a) \left\{ S(f; a, b) - \frac{1}{2} [f(b) - f(a)] \right\},$$

where, as above

$$S(f; a, b) = \sup_{a \leq \alpha < \beta \leq b} [f(\beta) - f(\alpha)].$$

*Proof.* We use the identity [1]

$$(3.4) \quad f(b)(b-x) + f(a)(x-a) - \int_a^b f(t) dt = \int_a^b (t-x) df(t)$$

for any  $x \in [a, b]$ .

On applying Theorem 2 for the Stieltjes integral  $\int_a^b (t-x) df(t)$  we can write that

$$(3.5) \quad \left| \int_a^b (t-x) df(t) - \frac{b-x+a-x}{2} [f(b) - f(a)] \right| \\ \leq (b-a) S(f; a, b) - \frac{1}{2} (b-a) [f(b) - f(a)].$$

Finally, on utilising the identity (3.4) and the inequality (3.5), a simple calculation provides the desired inequality (3.3). ■

The following result may be stated as well.

**Proposition 3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous and such that the derivative  $f'$  is of bounded variation on  $[a, b]$ . If  $-\infty < \gamma = \inf_{t \in [a, b]} f'(t)$ ,  $\sup_{t \in [a, b]} f'(t) = \Gamma < \infty$ , then*

$$(3.6) \quad \left| f(b)(b-x) + f(a)(x-a) + \frac{\gamma + \Gamma}{2} (b-a) \left( x - \frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \\ \leq (b-a) \left[ \bigvee_a^b(f') \left[ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] - \frac{1}{2} (\Gamma - \gamma) \left( \frac{a+b}{2} - x \right) \right].$$

*Proof.* If  $f$  is absolutely continuous on  $[a, b]$ , then

$$f(b)(b-x) + f(a)(x-a) - \int_a^b f(t) dt = \int_a^b f'(t) d \left[ \frac{1}{2} (t-x)^2 \right]$$

for any  $x \in [a, b]$ .

For fixed  $x \in [a, b]$ , let  $u(t) = \frac{1}{2} (t-x)^2$ . Then  $u$  is differentiable and  $u'(t) = (t-x)$ . Also

$$\sup_{t \in [a, b]} |u'(t)| = \sup_{t \in [a, b]} |t-x| = \max \{x-a, b-x\} \\ = \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right|.$$

Now, if we apply Corollary 1 for the Lipschitzian case, we can write:

$$\left| \int_a^b f'(t) d \left[ \frac{1}{2} (t-x)^2 \right] - \frac{\gamma + \Gamma}{2} \left[ \frac{(b-x)^2 - (a-x)^2}{2} \right] \right| \\ \leq (b-a) \left[ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f') \\ - \frac{1}{2} (\Gamma - \gamma) \left[ \frac{(b-x)^2 - (a-x)^2}{2} \right],$$

which is equivalent with the desired inequality (3.6). ■

**Corollary 5.** *With the assumptions in Proposition 3, we have the trapezoid inequality:*

$$(3.7) \quad \left| \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a)^2 \bigvee_a^b(f').$$



## 4. APPLICATIONS FOR THE OSTROWSKI INEQUALITY

In 1938, A. Ostrowski proved the following inequality

$$(4.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M$$

for all  $x \in [a, b]$ , provided that  $f$  is differentiable on  $(a, b)$  and  $|f'(t)| \leq M$  for all  $t \in (a, b)$ .

Using the following representation, which has been obtained by Montgomery in an equivalent form [6, p. 565]

$$(4.2) \quad f(x) - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b p(t, x) f'(t) dt$$

for all  $x \in [a, b]$ , provided that  $f$  is absolutely continuous on  $[a, b]$  and

$$(4.3) \quad p(t, x) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}, (x, t) \in [a, b]^2,$$

we can put in place of  $M$ , i.e., in (4.1), the sup norm of  $f'$ , i.e.,  $\|f'\|_\infty$  where

$$\|f'\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)|,$$

provided that  $f' \in L_\infty[a, b]$ .

The following result related to the Ostrowski inequality can be stated.

**Proposition 4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function on  $[a, b]$ . Then*

$$(4.4) \quad \left| f(x)(b-a) - \left(x - \frac{a+b}{2}\right) [f(b) - f(a)] - \int_a^b f(t) dt \right| \leq (b-a) \left[ S(f; a, b) - \frac{1}{2} [f(b) - f(a)] \right],$$

where, as above,  $S(f; a, b) = \sup_{a \leq \alpha < \beta \leq b} [f(\beta) - f(\alpha)]$ .

*Proof.* We use the following Montgomery type identity [3]

$$\int_a^b p(t, x) df(t) = f(x)(b-a) - \int_a^b f(t) dt$$

for any  $x \in [a, b]$ , where the kernel  $p : [a, b]^2 \rightarrow \mathbb{R}$  is defined by (4.3).

For any fixed  $x \in [a, b]$ , the function  $p(\cdot, x)$  is of bounded variation, and

$$\bigvee_a^b p(\cdot, x) = \bigvee_a^x p(\cdot, x) + \bigvee_x^b p(\cdot, x) = x - a + b - x = b - a.$$

Also

$$\sup_{t \in [a, b]} p(t, x) = x - a \quad \text{and} \quad \inf_{t \in [a, b]} p(t, x) = x - b,$$

for any  $x \in [a, b]$ .

Now, if we apply Theorem 2 for the Stieltjes integral  $\int_a^b p(t, x) df(t)$ , then we can write:

$$\left| \int_a^b p(t, x) df(t) - \frac{1}{2}(x-a+x-b)[f(b)-f(a)] \right| \leq (b-a)S(f; a, b) - \frac{1}{2}(b-a)[f(b)-f(a)],$$

for any  $x \in [a, b]$ , which is equivalent with the desired result (4.4). ■

**Corollary 6.** *With the assumptions in Proposition 4 we have the midpoint inequality:*

$$(4.5) \quad \left| f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t) dt \right| \leq (b-a) \left[ S(f; a, b) - \frac{1}{2}[f(b)-f(a)] \right].$$

**Remark 1.** *The interested reader can apply Theorem 2 and Theorem 3 for other quadrature rules where the remainder can be written in the form of*

$$\int_a^b K_n(t, x) df^{(n)}(t)$$

where the  $n$ -th derivative of the integrand  $f$  is continuous and the Peano kernel  $K_n(\cdot, x)$  is of bounded variation on  $[a, b]$ . The details are omitted.

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