Sums of Series of Rogers Dilogarithm Functions

SUMS OF SERIES OF ROGERS DILOGARITHM FUNCTIONS

ABDOLHOSSEIN HOORFAR AND FENG QI

Abstract. Some sums of series of Rogers dilogarithm functions are established by Abel's functional equation.

1. Introduction

The dilogarithm is defined [2, p.102] by the series

$$\mathrm{Li}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \tag{1}$$

for $-1\le x\le 1$. The Rogers dilogarithm function $L_R(x)$ is defined in [8] and [13, p. 287] for $0\le x\le 1$ by

$$L_R(x)=\begin{cases}\mathrm{Li}_2(x)+\dfrac{1}{2}\ln x\ln(1-x), & 0<x<1,\\ 0, & x=0,\\ \dfrac{\pi^2}{6}, & x=1.\end{cases} \tag{2}$$

The function $L_R(x)$ satisfies the concise identity

$$L_R(x)+L_R(1-x)=\frac{\pi^2}{6} \tag{3}$$

for $0\le x\le 1$, see [7, pp. 110–113], and Abel's functional equation

$$L_R(x)+L_R(y)=L_R(xy)+L_R\left(\frac{x(1-y)}{1-xy}\right)+L_R\left(\frac{y(1-x)}{1-xy}\right) \tag{4}$$

for $0<x,y<1$, see [1, pp. 189–192] and [5]. The duplication formula for $L_R(x)$ follows from Abel's functional equation (4) and is given for $0\le x\le 1$ by

$$L_R(x)=\frac{1}{2}L_R\big(x^2\big)+L_R\left(\frac{x}{1+x}\right). \tag{5}$$

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The function $L_R(x)$ satisfies also the following identities:

$$L_R\left(\frac{1}{2}\right)=\frac{\pi^2}{12},\quad L_R(\rho)=\frac{\pi^2}{10},\quad L_R(\rho^2)=L_R(1-\rho)=\frac{\pi^2}{15},\tag{6}$$

where $\rho=\frac{\sqrt{5}-1}{2}$, and has the nice infinite series

$$\sum_{n=2}^{\infty}L_R\left(\frac{1}{n^2}\right)=\frac{\pi^2}{6}\tag{7}$$

obtained in [13, p. 298] and [14].

It is remarked that the formulas from (1) to (7) can be looked up at [18, 19].

For more information on its history, properties, identities, generalizations, applications and recent developments of the dilogarithms and Rogers dilogarithm functions, please refer to [1, pp. 189–192], [2, pp.102–107], [4, pp. 323–326], [7, pp. 110–113], [3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein.

The main aim of this paper is to generalize the series (7).

Our main results are the following four theorems.

Theorem 1. *For $p,q\in\mathbb{N}$ and $\alpha\ge0$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{p}{n+p+\alpha}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{q}{n+q+\alpha}\right).\tag{8}$$

Remark 1. The series (7) is a special case of (8) for $p=q=\alpha=1$.

Theorem 2. *For $p,q\in\mathbb{N}$ and $0<\theta,\beta<1$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1-\beta\theta^{n+p})(1-\beta\theta^{n+q})}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{\beta(1-\theta^p)\theta^n}{1-\beta\theta^{n+p}}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{1-\theta^q}{1-\beta\theta^{n+q}}\right)-pL_R(1-\theta^q).\tag{9}$$

Theorem 3. *For $p,q\in\mathbb{N}$, $0<\beta\le1$ and $0<\theta<1$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1+\beta\theta^n)(1+\beta\theta^{n+p+q})}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{\theta^p(1+\beta\theta^n)}{1+\beta\theta^{n+p}}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{\beta(1-\theta^q)\theta^n}{1+\beta\theta^n}\right)-qL_R(\theta^p).\tag{10}$$

Theorem 4. *For $r > 1$,*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} L_R\left(\frac{1}{r^{2^n}+1}\right) = L_R\left(\frac{1}{r}\right). \tag{11}$$

As straightforward consequences of above theorems, some sums of series of special Rogers dilogarithm functions are deduced as follows.

Corollary 1. *Let $t > 0$ and $\phi = \frac{\sqrt{5}+1}{2}$, then the following identities are valid:*

$$\sum_{n=2}^{\infty} L_R\left(\frac{2}{n(n+1)}\right) = \frac{\pi^2}{4}, \tag{12}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} L_R\left(\frac{1}{2^{2^n}+1}\right) = \frac{\pi^2}{12}, \tag{13}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2}{n^2+\sqrt{5}\,n+1}\right) = \frac{\pi^2}{6} + L_R(3-\sqrt{5}\,), \tag{14}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2}{(n+\sqrt{2}\,)(n+1+\sqrt{2}\,)}\right) = \frac{\pi^2}{6} + L_R\left(\frac{1}{2+\sqrt{2}}\right), \tag{15}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{4}{\left(2n-1+\sqrt{5}\,\right)^2}\right) = \frac{\pi^2}{5}, \tag{16}$$

$$\sum_{n=2}^{\infty} (-1)^n L_R\left(\frac{4}{n^2}\right) = \frac{\pi^2}{3} - 2L_R\left(\frac{2}{3}\right), \tag{17}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^n}{(2^{n+1}-1)^2}\right) = \frac{\pi^2}{12}, \tag{18}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{\phi^{n-2}}{(\phi^{n+1}-1)^2}\right) = \frac{\pi^2}{10}, \tag{19}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^{n-1}}{(2^{n-1}+1)(2^{n+1}+1)}\right) = \frac{3}{2} L_R\left(\frac{1}{4}\right), \tag{20}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^2 3^{n-1}}{(3^{n-1}+1)(3^{n+1}+1)}\right) = \frac{\pi^2}{12}, \tag{21}$$

$$\sum_{n=2}^{\infty} L_R\left(\frac{\sinh^2 t}{\sinh^2(nt)}\right) = L_R(e^{-2t}), \tag{22}$$

and

$$\sum_{n=1}^{\infty} L_R\left(\frac{\sinh^2 t}{\cosh[(n-1)t]\cosh[(n+1)t]}\right) = L_R\left(\frac{e^{-t}}{\cosh t}\right) + L_R(e^{-t}\sinh t) - L_R(e^{-2t}). \tag{23}$$

2. Proofs of theorems and corollary

Proof of Theorem 1. Let

$$x_n = \frac{p}{n+p+\alpha} \qquad \text{and} \qquad y_n = \frac{q}{n+q+\alpha}$$

for $n = 0, 1, 2, \dots$. It is clear that

$$\frac{x_n(1-y_n)}{1-x_ny_n} = \frac{p}{(n+q)+p+\alpha} = x_{n+q},$$

and

$$\frac{y_n(1-x_n)}{1-x_ny_n} = \frac{q}{(n+p)+q+\alpha} = y_{n+p}.$$

Taking $x = x_n$ and $y = y_n$ in (4) leads to

$$L_R(x_n) + L_R(y_n) = L_R\biggl(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\biggr) + L_R\bigl(x_{n+q}\bigr) + L_R\bigl(y_{n+p}\bigr)$$

for $n = 0, 1, 2, \dots$. Summing up on both sides of above equality for n from 0 to $N \ge \max\{p, q\}$ gives

$$\sum_{n=0}^{q-1} L_R(x_n) + \sum_{n=0}^{p-1} L_R(y_n) = \sum_{n=0}^{N} L_R\biggl(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\biggr) + \sum_{n=N+1-q}^{N} L_R\bigl(x_{n+q}\bigr) + \sum_{n=N+1-p}^{N} L_R\bigl(y_{n+p}\bigr).$$

Letting $N \to \infty$ yields

$$\lim_{N\to\infty} \sum_{n=N+1-q}^{N} L_R\bigl(x_{n+q}\bigr) = \lim_{N\to\infty} \sum_{n=N+1-p}^{N} L_R\bigl(y_{n+p}\bigr) = 0.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Now let us consider the sequences

$$x_n = \frac{\beta(1-\theta^p)\theta^n}{1-\beta\theta^{n+p}} \qquad \text{and} \qquad y_n = \frac{1-\theta^q}{1-\beta\theta^{n+q}}$$

for $n = 0, 1, 2, \dots$. It is obvious that $0 < x_n < 1$ and $0 < y_n < 1$. Straightforward computation gives

$$\frac{x_n(1-y_n)}{1-x_ny_n} = \frac{\beta(1-\theta^p)\theta^{n+q}}{1-\beta\theta^{(n+q)+p}} = x_{n+q}$$

and

$$\frac{y_n(1-x_n)}{1-x_ny_n} = \frac{1-\theta^q}{1-\beta\theta^{(n+p)+q}} = y_{n+p}.$$

Using identity (4) again gives

$$L_R(x_n)+L_R(y_n)=L_R(x_ny_n)+L_R\big(x_{n+q}\big)+L_R\big(y_{n+p}\big).$$

Summing up for n from 0 to $N\geq\max\{p,q\}$ leads to

$$\sum_{n=0}^{q-1}L_R(x_n)+\sum_{n=0}^{p-1}L_R(y_n)=\sum_{n=0}^{N}L_R(x_ny_n)+\sum_{n=N+1-q}^{N}L_R\big(x_{n+q}\big)+\sum_{n=N+1-p}^{N}L_R\big(y_{n+p}\big).$$

Since $\lim_{n\to\infty}L_R(x_n)=0$ and $\lim_{n\to\infty}L_R(y_n)=L_R(1-\theta^q)$, if taking $N\to\infty$ in above identity, then formula (9) follows. The proof of Theorem 3 is finished. □

Proof of Theorem 3. Let

$$x_n=\frac{\theta^p(1+\beta\theta^n)}{1+\beta\theta^{n+p}}\qquad\text{and}\qquad y_n=\frac{\beta(1-\theta^q)\theta^n}{1+\beta\theta^n}$$

for $n=0,1,2,\dots$. It is apparent that $0<x_n,y_n<1$. Direct calculation reveals

$$x_ny_n=\frac{\beta(1-\theta^q)\theta^{n+p}}{1+\beta\theta^{n+p}}=y_{n+p}$$

and

$$\frac{x_n(1-y_n)}{1-x_ny_n}=\frac{\theta^p(1+\beta\theta^{n+q})}{1+\beta\theta^{(n+q)+p}}=x_{n+q}$$

with

$$\frac{y_n(1-x_n)}{1-x_ny_n}=\frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1+\theta^n)(1+\theta^{n+p+q})}\triangleq z_n.$$

From identity (4), it follows that

$$L_R(x_n)+L_R(y_n)=L_R\big(y_{n+p}\big)+L_R\big(x_{n+q}\big)+L_R(z_n).$$

Therefore, for $N\geq\max\{p,q\}$,

$$\sum_{n=0}^{q-1}L_R(x_n)+\sum_{n=0}^{p-1}L_R(y_n)=\sum_{n=0}^{N}L_R(z_n)+\sum_{n=N+1-q}^{N}L_R(x_{n+q})+\sum_{n=N+1-p}^{N}L_R(y_{n+p}).$$

Since $\lim_{n\to\infty}L_R(x_n)=L_R(\theta^p)$ and $\lim_{n\to\infty}L_R(y_n)=0$, then formula (10) is deduced by taking $N\to\infty$. Theorem 3 is proved. □

Proof of Theorem 4. Applying (5) to $x = x_n = \frac{1}{r^{2^n}+1}$ for $n = 0, 1, 2, \dots$ gives

$$L_R(x_n) = \frac{1}{2}L_R(x_{n+1}) + L_R\left(\frac{1}{r^{2^n}+1}\right)$$

and

$$\frac{1}{2^n}L_R(x_n) = \frac{1}{2^{n+1}}L_R(x_{n+1}) + \frac{1}{2^n}L_R\left(\frac{1}{r^{2^n}+1}\right)$$

for $n = 0, 1, 2, \dots$. Summing up for n from 0 to ∞ yields

$$\sum_{n=0}^{\infty}\frac{1}{2^n}L_R\left(\frac{1}{r^{2^n}+1}\right) = L_R(x_0) = L_R\left(\frac{1}{r}\right).$$

The proof of Theorem 4 is complete. □

Proof of Corollary 1. Taking $p = 2$, $q = 1$ and $\alpha = 1, \frac{\sqrt{5}-1}{2}, \sqrt{2}$ in (8) and simplifying by employing (3) and (6) leads to the identities (12), (14) and (15) respectively.

Identity (13) is a direct consequence of (11) for $r = 2$.

Letting $p = q = 1$ and $\alpha = \frac{\sqrt{5}-1}{2}$ in (8) yields (16).

It is easy to see that

$$\sum_{n=2}^{\infty}(-1)^n L_R\left(\frac{4}{n^2}\right) = \sum_{n=1}^{\infty}L_R\left(\frac{1}{n^2}\right) - \sum_{n=1}^{\infty}L_R\left(\frac{1}{(n+1/2)^2}\right).$$

Combining this with (8) for $p = q = 1$ and $\alpha = \frac{1}{2}$ leads to (17).

Identities (18) and (19) are special cases of (9) for $p = q = 1$, $\beta = \theta = \frac{1}{2}$ and $\beta = \theta = \frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$, respectively.

Applying $p = q = \beta = 1$ and $\theta = \frac{1}{2}$ in (10) gives

$$\sum_{n=1}^{\infty}L_R\left(\frac{2^{n-1}}{(2^{n-1}+1)(2^{n+1}+1)}\right) = L_R\left(\frac{2}{3}\right) + L_R\left(\frac{1}{4}\right) - L_R\left(\frac{1}{2}\right).$$

Taking $x = \frac{1}{2}$ in identity (5) yields

$$L_R\left(\frac{2}{3}\right) - L_R\left(\frac{1}{2}\right) = \frac{1}{2}L_R\left(\frac{1}{4}\right).$$

Thus, identity (20) is obtained.

Identity (21) is a direct consequence of (10) for $p = q = \beta = 1$ and $\theta = \frac{1}{3}$.

Taking $\theta = e^{-2t}$ and $\beta = e^{-2b}$ in (9) and (10) and simplifying gives

$$\sum_{n=0}^{\infty}L_R\left(\frac{\sinh(pt)\sinh(qt)}{\sinh((n+p)t+b)\sinh((n+q)t+b)}\right) = \sum_{n=0}^{q-1}L_R\left(\frac{e^{-(nt+b)}\sinh(pt)}{\sinh((n+p)t+b)}\right) + \sum_{n=0}^{p-1}L_R\left(\frac{e^{(nt+b)}\sinh(qt)}{\sinh((n+q)t+b)}\right) - pL_R\big(1-e^{2qt}\big) \quad (24)$$

for $t > 0$ and $b > 0$ and

$$\sum_{n=0}^{\infty} L_R\left(\frac{\sinh(pt)\sinh(qt)}{\cosh(nt+b)\cosh((n+p+q)t+b)}\right) = \sum_{n=0}^{q-1} L_R\left(\frac{e^{-pt}\cosh(nt+b)}{\cosh((n+p)t+b)}\right) + \sum_{n=0}^{p-1} L_R\left(\frac{e^{-(n+q)t-b}\sinh(qt)}{\cosh(nt+b)}\right) - qL_R\big(e^{-2t}\big) \quad (25)$$

for $t > 0$ and $b \geq 0$. Identities (22) and (23) are special cases of (24) and (25) for $p = q = 1$, $b = t$ and $b = 0$, respectively. □

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(A. Hoorfar) Department of Irrigation Engineering, College of Agriculture, Tehran University, Karaj, 31587-77871, Iran

E-mail address: hoorfar@ut.ac.ir

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@msn.com, qifeng618@qq.com, qifeng@hpu.edu.cn, fengqi618@member.ams.org

URL: http://rgmia.vu.edu.au/qi.html