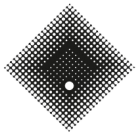
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New Inequalities of the Kantorovich Type for Bounded Linear Operators in Hilbert Spaces

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New Inequalities of the Kantorovich Type for Bounded Linear Operators in Hilbert Spaces

S.S. Dragomir

Abstract. Some new inequalities of the Kantorovich type are established. They hold for larger classes of operators and subsets of complex numbers than considered before in the literature and provide refinements of the classical results in the case when the involved operator satisfies the usual conditions. Several new reverse inequalities for the numerical radius of a bounded linear operator are obtained as well.

1. Introduction

Let $(H, \langle .,. \rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$, $B(H)$ the $\mathbb{C}^*$-algebra of all bounded linear operators defined on H and $A \in B(H)$. If A is invertible, then we can define the *Kantorovich functional* as

$$K(A;x) := \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle \tag{1.1}$$

for any $x \in H, \|x\| = 1$.

As pointed out by Greub and Rheinboldt in their seminal paper [**22**], if $M > m > 0$ and for the selfadjoint operator A we have

$$MI \geq A \geq mI \tag{1.2}$$

in the partial operator order of $B(H)$, where I is the identity operator, then the *operator Kantorovich inequality* holds true

$$1 \leq K(A;x) \leq \frac{(M+m)^2}{4mM}, \tag{1.3}$$

for any $x \in H, \|x\| = 1$.

An equivalent *additive* form of this result is incorporated in:

$$0 \leq K(A;x) - 1 \leq \frac{(M-m)^2}{4mM}, \tag{1.4}$$

for any $x \in H, \|x\| = 1$.

For results that are related to the operator Kantorovich inequality we recommend the classical works of Strang [**41**], Diaz & Metcalf [**2**], and Householder [**24**], Mond [**29**] and Mond & Shisha [**32**]. Other results have been obtained by Mond & Pečarić [**30**], [**31**], Fujii et al. [**11**], [**12**], Spain [**38**], Nakamoto and Nakamura [**33**],

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Furuta [**15**], [**16**], Tsukada & Takahasi [**42**] and more recently by Yamazaki [**45**], Furuta & Giga [**17**], Fujii & Nakamura [**13**], [**14**] and others.

Due to the important applications of the original Kantorovich inequality for matrices [**25**] in Statistics [**26**], [**40**], [**27**], [**36**], [**43**], [**39**], [**46**], [**35**], [**44**], [**28**] and Numerical Analysis [**19**], [**20**], [**37**], [**1**], [**18**], any new inequality of this type will have a flow of consequences in the areas of applications.

Motivated by the interest in both pure and applied mathematics outlined above we establish in this paper some new inequalities of Kantorovich type. They are shown to hold for larger classes of operators and subsets of complex numbers than considered before in the literature and provide refinements of the classical result in the case when the involved operator A satisfies the usual condition (1.2). As natural tools in deriving the new results, the recent Grüss' type inequalities for vectors in inner product obtained by the author in [**3**]-[**8**] are utilized. In the process, several new reverse inequalities for the numerical radius of a bounded linear operator are derived as well.

2. Some Grüss' Type Inequalities

The following lemmas, that are of interest in their own as well, collect some Grüss' type inequalities for vectors in inner product spaces obtained earlier by the author:

LEMMA 1. *Let $(H,\langle .,.\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u,v,e\in H$, $\|e\|=1$, and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ such that either*

$$\operatorname{Re}\langle \beta e-u,u-\alpha e\rangle \geq 0,\operatorname{Re}\langle \delta e-v,v-\gamma e\rangle \geq 0 \tag{2.1}$$

or, equivalently,

$$\left\|u-\frac{\alpha+\beta}{2}e\right\|\leq \frac{1}{2}\left|\beta-\alpha\right|,\left\|v-\frac{\gamma+\delta}{2}e\right\|\leq \frac{1}{2}\left|\delta-\gamma\right|. \tag{2.2}$$

Then

$$\begin{aligned}&\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right| \\ &\leq \frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|-\begin{cases}\left[\operatorname{Re}\langle \beta e-u,u-\alpha e\rangle \operatorname{Re}\langle \delta e-v,v-\gamma e\rangle\right]^{1/2}, \\ \left|\langle u,e\rangle-\frac{\alpha+\beta}{2}\right|\left|\langle v,e\rangle-\frac{\gamma+\delta}{2}\right|.\end{cases}\end{aligned} \tag{2.3}$$

The first inequality has been obtained in [**4**] (see also [**10**, p. 44]) while the second result was established in [**5**] (see also [**10**, p. 90]). They provide refinements of the earlier result from [**3**] where only the first part of the bound, i.e., $\frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|$ has been given. Notice that, as pointed out in [**5**], the upper bounds for the Grüss' functional incorporated in (2.3) cannot be compared in general meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

LEMMA 2. *With the assumptions in Lemma 1 and if* $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ *then*

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \tag{2.4}$$

$$\leq \begin{cases} \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}}|\langle u,e\rangle\langle e,v\rangle|, \\ \left[\left(|\alpha+\beta|-2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right)\left(|\delta+\gamma|-2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{1/2}\right)\right]^{1/2} \\ \times\left[|\langle u,e\rangle\langle e,v\rangle|\right]^{1/2}. \end{cases}$$

The first inequality has been established in [**6**] (see [**10**, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [**7**]. The details are omitted.

Finally, another inequality of Grüss' type that has been obtained in [**8**] (see also [**10**, p. 65]) can be stated as:

LEMMA 3. *With the assumptions in Lemma 1 and if* $\beta \neq -\alpha$, $\delta \neq -\gamma$ *then*

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \tag{2.5}$$

$$\leq \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[|\beta+\alpha||\delta+\gamma|]^{1/2}}\left[\left(\|u\|+|\langle u,e\rangle|\right)\left(\|v\|+|\langle v,e\rangle|\right)\right]^{1/2}.$$

3. Operator Inequalities of Grüss' Type

For the complex numbers α, β and the bounded linear operator A we define the following transform

$$C_{\alpha,\beta}(A) := (A^* - \overline{\alpha}I)(\beta I - A) \tag{3.1}$$

where by A^* we denote the adjoint of A.

The following proposition collects some of the immediate properties of the transform $C_{\alpha,\beta}(\cdot)$ and can be easily proved by applying the definition (3.1):

PROPOSITION 1. *For any* $\alpha, \beta \in \mathbb{C}$ *and* $A \in B(H)$ *we have:*

$$C_{\alpha,\beta}(I) = (1-\overline{\alpha})(\beta-1)I, \quad C_{\alpha,\alpha}(A) = -(\alpha I - A)^*(\alpha I - A), \tag{3.2}$$

$$C_{\alpha,\beta}(\gamma A) = |\gamma|^2 C_{\frac{\alpha}{\gamma},\frac{\beta}{\gamma}}(A) \text{ for each } \gamma \in \mathbb{C}\setminus\{0\}, \tag{3.3}$$

$$[C_{\alpha,\beta}(A)]^* = C_{\beta,\alpha}(A) \tag{3.4}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(A^*) - C_{\alpha,\beta}(A) = A^*A - AA^*. \tag{3.5}$$

The following characterization of normal operators, i.e., we recall that the operator A is *normal* if and only if either $A^*A = AA^*$ or, equivalently, $\|Ax\| = \|A^*x\|$ for each $x \in H$, can be stated:

COROLLARY 1. *The operator* $A \in B(H)$ *is normal if and only if* $C_{\overline{\beta},\overline{\alpha}}(A^*) = C_{\alpha,\beta}(A)$ *for each* $\alpha, \beta \in \mathbb{C}$.

The connection between the transform of the inverse operator A^{-1} and the transform of the operator A is described in the following result:

PROPOSITION 2. *If $A \in B(H)$ is invertible and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$, then*

$$\left(A^{-1}\right)^* C_{\alpha,\beta}(A) A^{-1} = \overline{\alpha}\beta C_{\frac{1}{\alpha},\frac{1}{\beta}}\left(A^{-1}\right). \tag{3.6}$$

PROOF. Observe, for any $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ that

$$\begin{aligned} \left(A^{-1}\right)^* C_{\beta,\alpha}(A) A^{-1} &= (A^*)^{-1}\left(A^* - \overline{\beta}I\right)(\alpha I - A) A^{-1} \\ &= \alpha\overline{\beta}\left(\frac{1}{\overline{\beta}}I - (A^*)^{-1}\right)\left(A^{-1} - \frac{1}{\alpha}I\right). \end{aligned} \tag{3.7}$$

However,

$$\left[\left(A^{-1}\right)^* C_{\beta,\alpha}(A) A^{-1}\right]^* = \left(A^{-1}\right)^* C_{\alpha,\beta}(A) A^{-1}$$

and

$$\left[\alpha\overline{\beta}\left(\frac{1}{\overline{\beta}}I - (A^*)^{-1}\right)\left(A^{-1} - \frac{1}{\alpha}I\right)\right]^* = \overline{\alpha}\beta C_{\frac{1}{\alpha},\frac{1}{\beta}}\left(A^{-1}\right)$$

and by (3.7) we get the desired representation (3.6). ∎

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle .,. \rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

The following simple characterization result is useful in the following:

LEMMA 4. *For $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ the following statements are equivalent:*

(i) The transform $C_{\alpha,\beta}(A)$ is accretive;

(ii) The transform $C_{\overline{\alpha},\overline{\beta}}(A^*)$ is accretive;

(iii) We have the norm inequality

$$\left\| A - \frac{\alpha+\beta}{2}I \right\| \leq \frac{1}{2}|\beta - \alpha|. \tag{3.8}$$

PROOF. The proof of the equivalence "$(i) \Leftrightarrow (iii)$" is obvious by the equality

$$\operatorname{Re}\langle (A^* - \overline{\alpha}I)(\beta I - A) x, x\rangle = \frac{1}{4}|\beta - \alpha|^2 - \left\| \left(A - \frac{\alpha+\beta}{2}I\right) x \right\|^2 \tag{3.9}$$

that holds for any $\alpha, \beta \in \mathbb{C}$, $A \in B(H)$ and $x \in H, \|x\| = 1$.

Since, obviously

$$\left\| A^* - \frac{\overline{\alpha}+\overline{\beta}}{2}I \right\| = \left\| A - \frac{\alpha+\beta}{2}I \right\| \leq \frac{1}{2}|\beta - \alpha| = \frac{1}{2}\left|\overline{\beta} - \overline{\alpha}\right|,$$

hence the equivalence "$(ii) \Leftrightarrow (iii)$" is also proved. ∎

For two bounded linear operators $A, B \in B(H)$ and the vector $x \in H, \|x\| = 1$ define the functional

$$G(A, B; x) := \langle Ax, Bx\rangle - \langle Ax, x\rangle\langle x, Bx\rangle.$$

The following result concerning operator inequalities of Grüss' type may be stated:

THEOREM 1. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be so that the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then*

$$|G(A,B;x)| \tag{3.10}$$

$$\leq \frac{1}{4}|\beta-\alpha||\delta-\gamma| - \begin{cases} \left[\operatorname{Re}\langle C_{\alpha,\beta}(A)x,x\rangle \operatorname{Re}\langle C_{\gamma,\delta}(B)x,x\rangle\right]^{1/2} \\ \left|\left\langle\left(A-\frac{\alpha+\beta}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(B-\frac{\gamma+\delta}{2}I\right)x,x\right\rangle\right| \end{cases}$$

$$\left(\leq \frac{1}{4}|\beta-\alpha||\delta-\gamma|\right),$$

for any $x \in H, \|x\| = 1$.

If $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ then

$$|G(A,B;x)| \tag{3.11}$$

$$\leq \begin{cases} \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}}|\langle Ax,x\rangle\langle Bx,x\rangle|, \\ \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2} \\ \quad\times[|\langle Ax,x\rangle\langle Bx,x\rangle|]^{1/2}, \end{cases}$$

for any $x \in H, \|x\| = 1$.

If $\beta \neq -\alpha$, $\delta \neq -\gamma$ then

$$|G(A,B;x)| \tag{3.12}$$

$$\leq \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[|\beta+\alpha||\delta+\gamma|]^{1/2}}\left[(\|Ax\|+|\langle Ax,x\rangle|)(\|Bx\|+|\langle Bx,x\rangle|)\right]^{1/2},$$

for any $x \in H, \|x\| = 1$.

The proof follows by Lemmas 1, 2 and 3 on choosing $u = Ax, v = Bx$ and $e = x, x \in H, \|x\| = 1$.

REMARK 1. *We observe that*

$$G(A,B^*;x) = \langle BAx,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle, x \in H, \|x\| = 1$$

and since, by Lemma 4 the transform $C_{\alpha,\beta}(A)$ is accretive if and only if $C_{\overline{\alpha},\overline{\beta}}(A^*)$ is accretive, hence in all the inequalities (3.10)-(3.12) we can replace $G(A,B;x)$ by $G(A,B^*;x)$ getting other Grüss' type inequalities that will be used in the sequel.

In some applications, the case $B = A$ in both quantities $G(A,B;x)$ and $G(A,B^*;x)$ may be of interest. For the sake of simplicity, we denote

$$G_1(A;x) := G(A,A;x) = \|Ax\|^2 - |\langle Ax,x\rangle|^2 \geq 0$$

and

$$G_2(A;x) := G(A,A^*;x) = \left\langle A^2x,x\right\rangle - [\langle Ax,x\rangle]^2,$$

for $x \in H, \|x\| = 1$. For these quantities, that are related to the Schwarz's inequality, we can state the following result which is of interest:

COROLLARY 2. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be so that the transforms $C_{\alpha,\beta}(A)$ is accretive, then*

$$\max\left\{G_1(A;x), |G_2(A;x)|\right\} \leq \frac{1}{4}|\beta-\alpha|^2 - \left\{\begin{array}{c} \operatorname{Re}\langle C_{\alpha,\beta}(A)x, x\rangle \\ \\ \left|\left\langle\left(A-\frac{\alpha+\beta}{2}I\right)x, x\right\rangle\right|^2 \end{array}\right. \left(\leq \frac{1}{4}|\beta-\alpha|^2\right), \tag{3.13}$$

for any $x \in H, \|x\| = 1$.

If $\operatorname{Re}(\beta\overline{\alpha}) > 0$ then

$$\max\left\{G_1(A;x), |G_2(A;x)|\right\} \leq \left\{\begin{array}{c} \frac{1}{4}\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})}|\langle Ax, x\rangle|^2, \\ \\ \left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right)|\langle Ax, x\rangle| \end{array}\right., \tag{3.14}$$

for any $x \in H, \|x\| = 1$.

If $\beta \neq -\alpha$ then

$$\max\left\{G_1(A;x), |G_2(A;x)|\right\} \leq \frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta+\alpha|}\left(\|Ax\| + |\langle Ax, x\rangle|\right), \tag{3.15}$$

for any $x \in H, \|x\| = 1$.

4. Reverse Inequalities for the Numerical Range

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator A is the subset of the complex numbers $\mathbb{C}$ given by [**21**, p. 1]:

$$W(A) = \left\{\langle Ax, x\rangle,\ x \in H,\ \|x\| = 1\right\}.$$

The *numerical radius* $w(A)$ of an operator A on H is given by [**21**, p. 8]:

$$w(A) = \sup\left\{|\lambda|, \lambda \in W(A)\right\} = \sup\left\{|\langle Ax, x\rangle|, \|x\| = 1\right\}. \tag{4.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [**21**, p. 9]:

THEOREM 2 (Equivalent norm). *For any $A \in B(H)$ one has*

$$w(A) \leq \|A\| \leq 2w(A). \tag{4.2}$$

The following reverses of the first inequality in (4.2), i.e., upper bounds under appropriate conditions for the bounded linear operator A for the nonnegative difference $\|A\|^2 - w^2(A)$ can be obtained:

THEOREM 3. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be so that the transforms $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \leq)\ \|A\|^2 - w^2(A) \leq \frac{1}{4}|\beta-\alpha|^2 - \left\{\begin{array}{c} \vartheta_i\left(C_{\alpha,\beta}(A)\right) \\ \\ w_i^2\left(A-\frac{\alpha+\beta}{2}I\right) \end{array}\right. \left(\leq \frac{1}{4}|\beta-\alpha|^2\right), \tag{4.3}$$

where, for a given operator B we have denoted $\vartheta_i(B) := \inf_{\|x\|=1} \operatorname{Re}\langle Ax, x\rangle$ and $w_i(B) := \inf_{\|x\|=1} |\langle Ax, x\rangle|$.

If $\operatorname{Re}(\beta\overline{\alpha}) > 0$ then

$$(0 \leq) \|A\|^2 - w^2(A) \leq \begin{cases} \frac{1}{4}\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} w^2(A), \\ \left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right) w(A). \end{cases} \tag{4.4}$$

If $\beta \neq -\alpha$ then

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta+\alpha|}\left(\|A\| + w(A)\right). \tag{4.5}$$

Proof. We give a short proof for the first inequality. The other results follow in a similar manner.

Utilising the inequality (3.13) we can write that

$$\|Ax\|^2 \leq |\langle Ax, x\rangle|^2 + \frac{1}{4}|\beta-\alpha|^2 - \operatorname{Re}\langle C_{\alpha,\beta}(A)x, x\rangle, \tag{4.6}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (4.6) we deduce the first inequality in (4.3). ∎

Corollary 3. *If $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive, then*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \begin{cases} \frac{1}{4}(M-m)^2 - \vartheta_i\left(C_{m,M}(A)\right), \\ \frac{1}{4}(M-m)^2 - w_i^2\left(A - \frac{m+M}{2}I\right), \\ \frac{1}{4}\frac{(M-m)^2}{mM} w^2(A), \\ \left(\sqrt{M} - \sqrt{m}\right)^2 w(A), \\ \frac{1}{4}\frac{(M-m)^2}{M+m}\left(\|A\| + w(A)\right). \end{cases} \tag{4.7}$$

Remark 2. *The inequalities in (4.4) and their consequences for positive M and m were obtained previously in* [**9**].

The following result is well known in the literature (see for instance [**34**]):

$$w(A^n) \leq w^n(A),$$

for each positive integer n and any operator $A \in B(H)$.

The following reverse inequalities for $n = 2$, can be stated:

Theorem 4. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be so that the transforms $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \leq) w^2(A) - w\left(A^2\right) \leq \frac{1}{4}|\beta-\alpha|^2 - \begin{cases} \vartheta_i\left(C_{\alpha,\beta}(A)\right) \\ w_i^2\left(A - \frac{\alpha+\beta}{2}I\right) \end{cases} \left(\leq \frac{1}{4}|\beta-\alpha|^2\right). \tag{4.8}$$

If $\operatorname{Re}(\beta\overline{\alpha})>0$ *then*

$$(0\leq)\, w^2(A)-w\left(A^2\right)\leq\begin{cases}\frac{1}{4}\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})}w^2(A),\\ \left(|\alpha+\beta|-2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right)w(A).\end{cases}\tag{4.9}$$

If $\beta\neq-\alpha$ *then*

$$(0\leq)\, w^2(A)-w\left(A^2\right)\leq\frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta+\alpha|}\left(\|A\|+w(A)\right).\tag{4.10}$$

Proof. We give a short proof for the fist inequality only. The other inequalities can be proved in a similar manner.

Utilising the inequality (3.13) we can write that

$$|\langle Ax,x\rangle|^2-\left|\left\langle A^2x,x\right\rangle\right|\leq\left|\left\langle A^2x,x\right\rangle-\left[\langle Ax,x\rangle\right]^2\right|\leq\frac{1}{4}|\beta-\alpha|^2-\operatorname{Re}\langle C_{\alpha,\beta}(A)x,x\rangle,$$

for any $x\in H,\ \|x\|=1$, which implies that

$$|\langle Ax,x\rangle|^2\leq\left|\left\langle A^2x,x\right\rangle\right|+\frac{1}{4}|\beta-\alpha|^2-\operatorname{Re}\langle C_{\alpha,\beta}(A)x,x\rangle\tag{4.11}$$

for any $x\in H,\ \|x\|=1$. Taking the supremum over $x\in H,\ \|x\|=1$ in (4.11) we deduce the desired inequality in (4.8). ∎

Remark 3. *If* $A\in B(H)$ *and* $M>m>0$ *are such that the transform* $C_{m,M}(A)=(A^*-mI)(MI-A)$ *is accretive, then all the inequalities in (4.7) hold true with the left side replaced by the nonnegative quantity* $w^2(A)-w\left(A^2\right)$.

5. New Inequalities of the Kantorovich Type

The following result comprising some inequalities for the Kantorovich functional can be stated:

Theorem 5. *Let* $A\in B(H)$ *and* $\alpha,\beta\in\mathbb{K}$ *be so that the transforms* $C_{\alpha,\beta}(A)$ *is accretive. If* $\operatorname{Re}(\beta\overline{\alpha})>0$ *and the operator* $-i\operatorname{Im}(\beta\overline{\alpha})C_{\alpha,\beta}(A)$ *is accretive, then*

$$|K(A;x)-1|\tag{5.1}$$

$$\leq\begin{cases}\frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta\alpha|}-\left[\operatorname{Re}\langle C_{\alpha,\beta}(A)x,x\rangle\operatorname{Re}\left\langle C_{\frac{1}{\alpha},\frac{1}{\beta}}\left(A^{-1}\right)x,x\right\rangle\right]^{1/2},\\ \frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta\alpha|}-\left|\left\langle\left(A-\frac{\alpha+\beta}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1}-\frac{\alpha+\beta}{2\alpha\beta}I\right)x,x\right\rangle\right|,\\ \frac{1}{4}\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})}|\langle Ax,x\rangle|\left|\left\langle A^{-1}x,x\right\rangle\right|,\\ \frac{|\beta+\alpha|-2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}}{|\beta\alpha|^{1/2}}\left[|\langle Ax,x\rangle|\left|\left\langle A^{-1}x,x\right\rangle\right|\right]^{1/2},\\ \frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta\alpha|^{1/2}|\beta+\alpha|}\left[\left(\|Ax\|+|\langle Ax,x\rangle|\right)\left(\left\|A^{-1}x\right\|+\left|\left\langle A^{-1}x,x\right\rangle\right|\right)\right]^{1/2},\end{cases}$$

for any $x\in H,\ \|x\|=1$.

PROOF. Utilising Proposition 2, we have for each $x \in H,\ \|x\| = 1$ that

$$\begin{aligned}
& \operatorname{Re}\left\langle C_{\frac{1}{\alpha},\frac{1}{\beta}}\left(A^{-1}\right)x,x\right\rangle \\
= & \frac{1}{|\beta\alpha|^2}\operatorname{Re}\left[\overline{\beta\alpha}\left\langle \left(A^{-1}\right)^* C_{\alpha,\beta}\left(A\right)A^{-1}x,x\right\rangle\right] \\
= & \frac{1}{|\beta\alpha|^2}\left[\operatorname{Re}\left(\beta\overline{\alpha}\right)\cdot\operatorname{Re}\left\langle \left(A^{-1}\right)^* C_{\alpha,\beta}\left(A\right)A^{-1}x,x\right\rangle\right. \\
& \left.+\operatorname{Im}\left(\beta\overline{\alpha}\right)\cdot\operatorname{Im}\left\langle \left(A^{-1}\right)^* C_{\alpha,\beta}\left(A\right)A^{-1}x,x\right\rangle\right] \\
= & \frac{1}{|\beta\alpha|^2}\left[\operatorname{Re}\left(\beta\overline{\alpha}\right)\cdot\operatorname{Re}\left\langle \left(A^{-1}\right)^* C_{\alpha,\beta}\left(A\right)A^{-1}x,x\right\rangle\right. \\
& \left.+\operatorname{Re}\left\langle \left(A^{-1}\right)^*\left(-i\operatorname{Im}\left(\beta\overline{\alpha}\right)C_{\alpha,\beta}\left(A\right)\right)A^{-1}x,x\right\rangle\right] \\
\geq & 0,
\end{aligned}$$

showing that the operator $C_{\frac{1}{\alpha},\frac{1}{\beta}}\left(A^{-1}\right)$ is also accretive.

Now, on applying Theorem 1 for the difference $\langle BAx,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle$ and for the choices $B = A^{-1}$ and $\delta = 1/\beta$, $\gamma = 1/\alpha$, we get the desired inequality (5.1). The details are omitted. ∎

REMARK 4. *A sufficient simple condition for the second assumption to hold in the above theorem is that $\beta\overline{\alpha}$ is a positive real number.*

However, for practical applications the following even more particular case is of interest:

COROLLARY 4. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then*

$$|K(A;x) - 1| \tag{5.2}$$

$$\leq \begin{cases}
\frac{1}{4}\frac{(M-m)^2}{mM} - \left[\operatorname{Re}\langle C_{m,M}(A)x,x\rangle \operatorname{Re}\left\langle C_{\frac{1}{m},\frac{1}{M}}\left(A^{-1}\right)x,x\right\rangle\right]^{1/2}, \\
\frac{1}{4}\frac{(M-m)^2}{mM} - \left|\left\langle\left(A - \frac{m+M}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1} - \frac{m+M}{2mM}I\right)x,x\right\rangle\right|, \\
\frac{1}{4}\frac{(M-m)^2}{mM}\left|\langle Ax,x\rangle\right|\left|\left\langle A^{-1}x,x\right\rangle\right|, \\
\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\left[\left|\langle Ax,x\rangle\right|\left|\left\langle A^{-1}x,x\right\rangle\right|\right]^{1/2}, \\
\frac{1}{4}\frac{(M-m)^2}{\sqrt{mM}(m+M)}\left[\left(\|Ax\| + \left|\langle Ax,x\rangle\right|\right)\left(\left\|A^{-1}x\right\| + \left|\left\langle A^{-1}x,x\right\rangle\right|\right)\right]^{1/2},
\end{cases}$$

for any $x \in H,\ \|x\| = 1$.

Finally, on returning to the original assumptions, we can state the following results which both provides refinements for the additive version of the operator Kantorovich inequality (1.4) as well as other similar results that apparently were not discovered before:

COROLLARY 5. *Let A be a selfadjoint operator on H and $M > m > 0$ such that $MI \geq A \geq mI$ in the partial operator order of $B(H)$. Then*

$$0 \leq K(A;x) - 1 \tag{5.3}$$

$$\leq \begin{cases} \frac{1}{4}\frac{(M-m)^2}{mM} - \left[\operatorname{Re}\langle C_{m,M}(A)x,x\rangle \operatorname{Re}\left\langle C_{\frac{1}{m},\frac{1}{M}}\left(A^{-1}\right)x,x\right\rangle\right]^{1/2}, \\ \frac{1}{4}\frac{(M-m)^2}{mM} - \left|\left\langle\left(A-\frac{m+M}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1}-\frac{m+M}{2mM}I\right)x,x\right\rangle\right|, \\ \frac{1}{4}\frac{(M-m)^2}{mM}\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle, \\ \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle\right]^{1/2}, \\ \frac{1}{4}\frac{(M-m)^2}{\sqrt{mM}(m+M)}\left[\left(\|Ax\|+\langle Ax,x\rangle\right)\left(\left\|A^{-1}x\right\|+\left\langle A^{-1}x,x\right\rangle\right)\right]^{1/2}, \end{cases}$$

for any $x \in H$, $\|x\| = 1$.

The proof is obvious by Corollary 5 on noticing that the fact $MI \geq A \geq mI$ for a selfadjoint operator A implies that $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. The reverse is not true in general.

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