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NORM INEQUALITIES FOR SEQUENCES OF OPERATORS RELATED TO THE SCHWARZ INEQUALITY

SEVER S. DRAGOMIR

Abstract. Some norm inequalities for sequences of linear operators defined on Hilbert spaces that are related to the classical Schwarz inequality are given. Applications for vector inequalities are also provided.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators that map H into H.

In many estimates one needs to use upper bounds for the norm of the linear combination of bounded linear operators $A_1,\dots,A_n$ with the scalars $\alpha_1,\dots,\alpha_n$, where separate information for scalars and operators are provided. In this situation, the classical approach is to use a Hölder type inequality as stated below

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|\left(\leq\sum_{i=1}^{n}|\alpha_i|\,\|A_i\|\right)\leq\begin{cases}\max\limits_{1\leq i\leq n}\{|\alpha_i|\}\sum\limits_{i=1}^{n}\|A_i\|;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^p\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}\|A_i\|^q\right)^{\frac{1}{q}}\\ \qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \max\limits_{1\leq i\leq n}\{\|A_i\|\}\sum\limits_{i=1}^{n}|\alpha_i|.\end{cases}$$

Notice that, the case when $p=q=2$, which provides the Cauchy-Bunyakovsky-Schwarz inequality

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|\leq\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}\|A_i\|^2\right)^{1/2}\tag{1.1}$$

is of special interest and of larger utility.

In the previous paper [1], in order to improve (1.1), we have established the following norm inequality for the operators $A_1,\dots,A_n\in B(H)$ and scalars $\alpha_1,\dots,\alpha_n\in$

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$\mathbb{K}$:

$$
(1.2)\quad \left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \le \begin{cases} \max\limits_{1\le i\le n}|\alpha_i|^2 \sum\limits_{i=1}^{n}\|A_i\|^2; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2 \max\limits_{1\le i\le n}\|A_i\|^2 \end{cases}
$$

$$
+\begin{cases} \max\limits_{1\le i\ne j\le n}\{|\alpha_i|\,|\alpha_j|\}\sum\limits_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^{2r}\right]^{\frac{1}{r}}\left(\sum\limits_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|^s\right)^{\frac{1}{s}} \\ \qquad \text{if } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^2\right]\max\limits_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|, \end{cases}
$$

where (1.2) should be seen as all the 9 possible configurations.

Some particular inequalities of interest that can be obtained from (1.2) and provide alternative bounds for the classical Cauchy-Bunyakovsky-Schwarz (CBS) inequality are the following [1]:

$$
(1.3)\quad \left\|\sum_{i=1}^{n}\alpha_i A_i\right\| \le \max_{1\le i\le n}|\alpha_i|\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|\right)^{\frac{1}{2}},
$$

$$
(1.4)\quad \left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \le \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{1\le i\le n}\|A_i\|^2+(n-1)\max_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|\right],
$$

$$
(1.5)\quad \left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \le \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{1\le i\le n}\|A_i\|^2+\left(\sum_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|^2\right)^{\frac{1}{2}}\right]
$$

and

$$
(1.6)\quad \left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \le \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}}\left(\sum_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}\right],
$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. In particular, for $p=q=2$, we have from (1.6)

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \le \left(\sum_{i=1}^{n}|\alpha_i|^4\right)^{\frac{1}{2}}\left[\left(\sum_{i=1}^{n}\|A_i\|^4\right)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}\left(\sum_{1\le i\ne j\le n}\left\|A_iA_j^*\right\|^2\right)^{\frac{1}{2}}\right]. \tag{1.7}$$

The aim of the present paper is to establish other upper bounds of interest for the quantity $\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|$, where, as above, $\alpha_1,\dots,\alpha_n$ are real or complex numbers, while $A_1,\dots,A_n$ are bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. These are compared with the (CBS) inequality (1.1) and shown that some times they are better. Applications for vector inequalities are also given.

2. Some General Results

The following result containing 9 different inequalities may be stated:

Theorem 1. *Let $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ and $A_1,\dots,A_n\in B(H)$. Then*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \le \begin{cases} A \\ B \\ C \end{cases} \tag{2.1}$$

where

$$A:=\begin{cases} \max\limits_{1\le k\le n}|\alpha_k|^2\sum\limits_{i,j=1}^{n}\left\|A_iA_j^*\right\|, \\[2ex] \max\limits_{1\le k\le n}|\alpha_k|\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|\right)^s\right)^{\frac{1}{s}}, \\ \qquad\qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\[2ex] \max\limits_{1\le k\le n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|\right), \end{cases}$$

$$B:=\begin{cases} \max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}} \\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^t\right)^{\frac{1}{t}}\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \\ \qquad\qquad \text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\[2ex] \sum\limits_{i=1}^{n}|\alpha_i|\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\max\limits_{1\le i\le n}\left\{\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}\right\}, \end{cases} \tag{2.2}$$

for $p>1,\ \frac{1}{p}+\frac{1}{q}=1$ *and*

$$C:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\sum\limits_{k=1}^{n}|\alpha_k|\sum\limits_{i=1}^{n}\max\limits_{1\le j\le n}\left\{\left\|A_iA_j^*\right\|\right\},\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^m\right)^{\frac{1}{m}}\sum\limits_{k=1}^{n}|\alpha_k|\left[\sum\limits_{i=1}^{n}\left(\max\limits_{1\le j\le n}\left\{\left\|A_iA_j^*\right\|\right\}\right)^l\right]^{\frac{1}{l}},\\ \qquad\qquad\text{where } m,l>1,\ \frac{1}{m}+\frac{1}{l}=1;\\[2ex] \left(\sum\limits_{k=1}^{n}|\alpha_k|\right)^2\max\limits_{1\le i,j\le n}\left\{\left\|A_iA_j^*\right\|\right\}.\end{cases}$$

Proof. We observe, in the operator partial order of $B(H)$, we have that

$$0\le\left(\sum_{i=1}^{n}\alpha_iA_i\right)\left(\sum_{i=1}^{n}\alpha_iA_i\right)^*=\sum_{i=1}^{n}\alpha_iA_i\sum_{j=1}^{n}\overline{\alpha_j}A_j^*=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}A_iA_j^*. \tag{2.3}$$

Taking the norm in (2.3) and noticing that $\|UU^*\|=\|U\|^2$ for any $U\in B(H)$, we have:

$$\begin{aligned}\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2&=\left\|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}A_iA_j^*\right\|\\&\le\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i|\,|\alpha_j|\left\|A_iA_j^*\right\|\\&=\sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|\alpha_j|\left\|A_iA_j^*\right\|\right)=:M.\end{aligned}$$

Utilising Hölder's discrete inequality we have that

$$\sum_{j=1}^{n}|\alpha_j|\left\|A_iA_j^*\right\|\le\begin{cases}\max\limits_{1\le k\le n}|\alpha_k|\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|,\\[2ex] \left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}\\ \qquad\text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \sum\limits_{k=1}^{n}|\alpha_k|\max\limits_{1\le j\le n}\left\|A_iA_j^*\right\|,\end{cases}$$

for any $i\in\{1,\dots,n\}$.

This provides the following inequalities:

$$M \leq \begin{cases} \max\limits_{1\leq k\leq n} |\alpha_k| \sum\limits_{i=1}^{n} |\alpha_i| \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\| \right) =: M_1 \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |\alpha_i| \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\|^q \right)^{\frac{1}{q}} := M_p \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n} |\alpha_k| \sum\limits_{i=1}^{n} |\alpha_i| \left(\max\limits_{1\leq j\leq n} \left\| A_i A_j^* \right\| \right) := M_\infty . \end{cases}$$

Utilising Hölder's inequality for $r, s > 1$, $\frac{1}{r}+\frac{1}{s}=1$, we have:

$$\sum_{i=1}^{n} |\alpha_i| \left(\sum_{j=1}^{n} \left\| A_i A_j^* \right\| \right) \leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i,j=1}^{n} \left\| A_i A_j^* \right\| \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^r \right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\| \right)^s \right]^{\frac{1}{s}} \\ \qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\| \right), \end{cases}$$

and thus we can state that

$$M_1 \leq \begin{cases} \max\limits_{1\leq k\leq n} |\alpha_k|^2 \sum\limits_{i,j=1}^{n} \left\| A_i A_j^* \right\|; \\ \max\limits_{1\leq k\leq n} |\alpha_k| \left(\sum\limits_{i=1}^{n} |\alpha_i|^r \right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\| \right)^s \right)^{\frac{1}{s}}, \\ \qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \max\limits_{1\leq k\leq n} |\alpha_k| \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} \left\| A_i A_j^* \right\| \right), \end{cases}$$

and the first part of the theorem is proved.

By Hölder's inequality we can also have that (for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$)

$$M_p \leq \left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^t\right)^{\frac{1}{t}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \\ \qquad \text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\leq i\leq n}\left\{\left(\sum\limits_{j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}\right\}, \end{cases}$$

and the second part of (2.1) is proved.

Finally, we may state that

$$M_\infty \leq \sum_{k=1}^{n}|\alpha_k| \times \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|\sum\limits_{i=1}^{n}\max\limits_{1\leq j\leq n}\left\{\left\|A_iA_j^*\right\|\right\} \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^m\right)^{\frac{1}{m}}\left[\sum\limits_{i=1}^{n}\left(\max\limits_{1\leq j\leq n}\left\{\left\|A_iA_j^*\right\|\right\}\right)^l\right]^{\frac{1}{l}} \\ \qquad \text{where } m,l>1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\leq i,j\leq n}\left\{\left\|A_iA_j^*\right\|\right\}, \end{cases}$$

giving the last part of (2.1). ∎

Remark 1. *It is obvious that out of (2.1) one can obtain various particular inequalities. For instance, the choice $t=2$, $p=2$ (therefore $u=q=2$) in the $B-$branch of (2.2) gives:*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^2\right)^{\frac{1}{2}} = \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{i=1}^{n}\|A_i\|^4+\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|\right)^{\frac{1}{2}}. \tag{2.4}$$

If we consider now the usual Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}\|A_i\|^2, \tag{2.5}$$

and observe that

$$\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i,j=1}^{n}\|A_i\|^2\left\|A_j^*\right\|^2\right)^{\frac{1}{2}} = \sum_{i=1}^{n}\|A_i\|^2,$$

then we can conclude that (2.4) is a refinement of the (CBS) inequality (2.5).

Corollary 1. *Let* $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ *and* $A_1, \dots, A_n \in B(H)$ *so that* $A_i A_j^* = 0$ *with* $i \neq j$. *Then*

$$\left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \leq \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases} \tag{2.6}$$

where

$$\tilde{A} := \begin{cases} \max\limits_{1 \leq k \leq n} |\alpha_k|^2 \sum\limits_{i=1}^{n} \|A_i\|^2 ; \\ \\ \max\limits_{1 \leq k \leq n} |\alpha_k| \left(\sum\limits_{i=1}^{n} |\alpha_i|^r \right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \|A_i\|^{2s} \right)^{\frac{1}{s}}, \\ \qquad\qquad \text{where } r > 1, \ \frac{1}{r} + \frac{1}{s} = 1; \\ \\ \max\limits_{1 \leq k \leq n} |\alpha_k| \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1 \leq i \leq n} \left\{ \|A_i\|^2 \right\}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max\limits_{1 \leq i \leq n} |\alpha_i| \left(\sum\limits_{k=1}^{n} |\alpha_k|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} \|A_i\|^2 ; \\ \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^t \right)^{\frac{1}{t}} \left(\sum\limits_{k=1}^{n} |\alpha_k|^p \right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n} \|A_i\|^{2u} \right]^{\frac{1}{u}}, \\ \qquad\qquad \text{where } t > 1, \ \frac{1}{t} + \frac{1}{u} = 1; \\ \\ \sum\limits_{i=1}^{n} |\alpha_i| \left(\sum\limits_{k=1}^{n} |\alpha_k|^p \right)^{\frac{1}{p}} \max\limits_{1 \leq i \leq n} \left\{ \|A_i\|^2 \right\}, \end{cases}$$

where $p > 1$ *and*

$$\tilde{C} := \begin{cases} \max\limits_{1 \leq i \leq n} |\alpha_i| \sum\limits_{k=1}^{n} |\alpha_k| \sum\limits_{i=1}^{n} \|A_i\|^2 ; \\ \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^m \right)^{\frac{1}{m}} \sum\limits_{k=1}^{n} |\alpha_k| \left(\sum\limits_{i=1}^{n} \|A_i\|^{2l} \right)^{\frac{1}{l}}, \\ \qquad\qquad \text{where } m, l > 1, \ \frac{1}{m} + \frac{1}{l} = 1; \\ \\ \left(\sum\limits_{k=1}^{n} |\alpha_k| \right)^2 \max\limits_{1 \leq i,j \leq n} \left\{ \|A_i\|^2 \right\}. \end{cases}$$

3. Other Results

A different approach is embodied in the following theorem:

Theorem 2. *If* $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ *and* $A_1, \dots, A_n \in B(H)$, *then*

$$\left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \sum_{j=1}^{n} \left\| A_i A_j^* \right\| \tag{3.1}$$
$$\leq \begin{cases} \sum_{i=1}^{n} |\alpha_i|^2 \max_{1\leq i\leq n} \left[\sum_{j=1}^{n} \left\| A_i A_j^* \right\| \right]; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left\| A_i A_j^* \right\| \right)^q \right]^{\frac{1}{q}} \\ \qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1\leq i\leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} \left\| A_i A_j^* \right\|. \end{cases}$$

Proof. From the proof of Theorem 1 we have that

$$\left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| \left\| A_i A_j^* \right\|.$$

Using the simple observation that

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} \left(|\alpha_i|^2 + |\alpha_j|^2 \right), \quad i, j \in \{1, \dots, n\},$$

we have

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| \left\| A_i A_j^* \right\| &\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[|\alpha_i|^2 + |\alpha_j|^2 \right] \left\| A_i A_j^* \right\| \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[|\alpha_i|^2 \left\| A_i A_j^* \right\| + |\alpha_j|^2 \left\| A_i A_j^* \right\| \right] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i|^2 \left\| A_i A_j^* \right\|, \end{aligned}$$

which proves the first inequality in (3.1).

The second part follows by Hölder's inequality and the details are omitted. ∎

Remark 2. *If in (3.1) we choose* $\alpha_1 = \cdots = \alpha_n = 1$, *then we get*

$$\left\| \sum_{i=1}^{n} A_i \right\| \leq \left(\sum_{i=1}^{n} \|A_i\|^2 + \sum_{1\leq i\neq j\leq n} \left\| A_i A_j^* \right\| \right)^{1/2} \leq \sum_{i=1}^{n} \|A_i\|,$$

which is a refinement for the generalised triangle inequality.

The following corollary may be stated:

Corollary 2. *If $A_1,\dots,A_n \in B(H)$ are such that $A_iA_j^* = 0$ for $i \neq j$, $i,j \in \{1,\dots,n\}$, then*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\|A_i\|^2 \tag{3.2}$$
$$\leq \left(\begin{cases}\sum_{i=1}^{n}|\alpha_i|^2\max_{1\leq i\leq n}\|A_i\|^2; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\sum_{j=1}^{n}\|A_i\|^{2q}\right]^{\frac{1}{q}} \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max_{1\leq i\leq n}|\alpha_i|^2\sum_{i=1}^{n}\|A_i\|^2\end{cases}\right).$$

Finally, the following result may be stated as well:

Theorem 3. *If $\alpha_1,\dots,\alpha_n \in \mathbb{K}$ and $A_1,\dots,A_n \in B(H)$, then*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \leq \begin{cases}\max_{1\leq i\leq n}|\alpha_i|^2\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|; \\ \left(\sum_{i=1}^{n}|\alpha_i|^p\right)^{\frac{2}{p}}\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}} \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\sum_{i=1}^{n}|\alpha_i|\right)^2\max_{1\leq i,j\leq n}\left\{\left\|A_iA_j^*\right\|\right\}.\end{cases} \tag{3.3}$$

Proof. We know that

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i||\alpha_j|\left\|A_iA_j^*\right\| =: P.$$

Firstly, we obviously have that

$$P \leq \max_{1\leq i,j\leq n}\{|\alpha_i||\alpha_j|\}\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\| = \max_{1\leq i\leq n}|\alpha_i|^2\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|.$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned} P &\leq \left[\sum_{i,j=1}^{n}\left(|\alpha_i|\,|\alpha_j|\right)^p\right]^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n}|\alpha_i|^p\sum_{j=1}^{n}|\alpha_j|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n}|\alpha_i|^p\right)^{\frac{2}{p}}\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Finally, we have

$$\begin{aligned} P &\leq \max_{1\leq i,j\leq n}\left\{\left\|A_iA_j^*\right\|\right\}\sum_{i,j=1}^{n}|\alpha_i|\,|\alpha_j| \\ &= \left(\sum_{i=1}^{n}|\alpha_i|\right)^2\max_{1\leq i,j\leq n}\left\{\left\|A_iA_j^*\right\|\right\} \end{aligned}$$

and the theorem is proved. ∎

Corollary 3. *If $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ and $A_1,\dots,A_n\in B(H)$ are such that $A_iA_j^*=0$ for $i,j\in\{1,\dots,n\}$ with $i\neq j$, then*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|^2\sum\limits_{i=1}^{n}\|A_i\|^2; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^p\right)^{\frac{2}{p}}\left(\sum\limits_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{q}}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2\max\limits_{1\leq i\leq n}\left\{\|A_i\|^2\right\}.\end{cases} \tag{3.4}$$

4. Vector Inequalities

As pointed out in our previous paper [1], the operator inequalities obtained above may provide various vector inequalities of interest.

If by $M(\boldsymbol{\alpha},\mathbf{A})$ we denote any of the bounds provided by (2.1), (2.4), (3.1) or (3.3) for the quantity $\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2$, then we may state the following general fact:

Under the assumptions of Theorem 1, we have:

$$\left\|\sum_{i=1}^{n}\alpha_iA_ix\right\|^2\leq\|x\|^2M(\boldsymbol{\alpha},\mathbf{A}). \tag{4.1}$$

for any $x \in H$ *and*

$$\left|\sum_{i=1}^{n} \alpha_i \langle A_i x, y\rangle\right|^2 \leq \|x\|^2 \|y\|^2 M(\boldsymbol{\alpha}, \mathbf{A}). \tag{4.2}$$

for any $x, y \in H$, *respectively.*

The proof follows by the Schwarz inequality in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, see for instance [1], and the details are omitted.

Now, we consider the non zero vectors $y_1, \dots, y_n \in H$. Define the operators [1]

$$A_i : H \to H, \quad A_i x = \frac{(x, y_i)}{\|y_i\|} \cdot y_i, \quad i \in \{1, \dots, n\}.$$

Since

$$\|A_i\| = \|y_i\|, \quad i \in \{1, \dots, n\}$$

then A_i are bounded linear operators in H. Also, since

$$(A_i x, x) = \frac{|(x, y_i)|^2}{\|y_i\|} \geq 0, \quad x \in H, \quad i \in \{1, \dots, n\}$$

and

$$(A_i x, z) = \frac{(x, y_i)(y_i, z)}{\|y_i\|},$$
$$(x, A_i z) = \frac{(x, y_i)(y_i, z)}{\|y_i\|},$$

giving

$$(A_i x, z) = (x, A_i z), \quad x, z \in H, \quad i \in \{1, \dots, n\},$$

we may conclude that A_i $(i = 1, \dots, n)$ are positive self-adjoint operators on H.

Since, for any $x \in H$, one has

$$\|(A_i A_j)(x)\| = \frac{|(x, y_j)| |(y_j, y_i)|}{\|y_j\|}, \quad i, j \in \{1, \dots, n\},$$

then we deduce that

$$\|A_i A_j\| = |(y_i, y_j)|; \quad i, j \in \{1, \dots, n\}.$$

If $(y_i)_{i=\overline{1,n}}$ is an orthogonal family on H, then $\|A_i\| = 1$ and $A_i A_j = 0$ for $i, j \in \{1, \dots, n\}$, $i \neq j$.

Now, utilising, for instance, the inequalities in Theorem 2 we may state that:

$$\left\|\sum_{i=1}^{n}\alpha_i\frac{(x,y_i)}{\|y_i\|}y_i\right\|^2 \le \|x\|^2\sum_{i=1}^{n}|\alpha_i|^2\sum_{j=1}^{n}|(y_i,y_j)| \tag{4.3}$$

$$\le \|x\|^2\times\begin{cases}\sum_{i=1}^{n}|\alpha_i|^2\max_{1\le i\le n}\left[\sum_{j=1}^{n}|(y_i,y_j)|\right]; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)^q\right]^{\frac{1}{q}} \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max_{1\le i\le n}|\alpha_i|^2\sum_{i,j=1}^{n}|(y_i,y_j)|.\end{cases}$$

for any $x,y_1,\dots,y_n\in H$ and $\alpha_n,\dots,\alpha_n\in\mathbb{K}$.

The proof follows on choosing $A_i=\frac{(\cdot,y_i)}{\|y_i\|}y_i$ in Theorem 2 and taking into account that $\|A_i\|=\|y_i\|$,

$$\left\|A_iA_j^*\right\|=|(y_i,y_j)|,\qquad i,j\in\{1,\dots,n\}.$$

We omit the details.

The choice $\alpha_i=\|y_i\|$ $(i=1,\dots,n)$ will produce some interesting bounds for the norm of the Fourier series

$$\left\|\sum_{i=1}^{n}(x,y_i)\,y_i\right\|.$$

Notice that the vectors y_i $(i=1,\dots,n)$ are not necessarily orthonormal.

Similar inequalities may be stated if one uses the other two main theorems. For the sake of brevity, they will not be stated here.

References

[1] S.S. DRAGOMIR, Some Schwarz type inequalities for sequences of operators in Hilbert spaces, *Bull. Austral. Math. Soc.*, **73**(2006), 17-26.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC, 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`