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# INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS OF COMPOSITE OPERATORS IN HILBERT SPACES

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ABSTRACT. Some new inequalities for the norm and the numerical radius of composite operators generated by a pair of operators are given.

## 1. INTRODUCTION

Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space. The *numerical range* of an operator  $T$  is the subset of the complex numbers  $\mathbb{C}$  given by [4, p. 1]:

$$(1.1) \quad W(T) = \{ \langle Tx, x \rangle, x \in H, \|x\| = 1 \}.$$

It is well known that (see [4]):

- (i) The numerical range of an operator is convex;
- (ii) The spectrum of an operator is contained in the closure of its numerical range;
- (iii)  $T$  is self-adjoint if and only if  $W(T)$  is real.

The *numerical radius*  $w(T)$  of an operator  $T$  on  $H$  is defined by [4, p. 8]

$$(1.2) \quad w(T) := \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

It is well known that  $w(\cdot)$  is a norm on the Banach algebra  $B(H)$  of all bounded linear operators acting on  $H$  and the following inequality holds true:

$$(1.3) \quad w(T) \leq \|T\| \leq 2w(T).$$

We recall some classical results involving the numerical radius of two linear operators  $A, B$ .

The following general result for the product of two operators holds [4, p. 37]:

**Theorem 1.** *If  $A, B$  are two bounded linear operators on the Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ , then*

$$(1.4) \quad w(AB) \leq 4w(A)w(B).$$

*In the case that  $AB = BA$ , then*

$$(1.5) \quad w(AB) \leq 2w(A)w(B).$$

The following results are also well known [4, p. 38].

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**Theorem 2.** *If  $A$  is a unitary operator that commutes with another operator  $B$ , then*

$$(1.6) \quad w(AB) \leq w(B).$$

*If  $A$  is an isometry and  $AB = BA$ , then (1.6) also holds true.*

We say that  $A$  and  $B$  *double commute* if  $AB = BA$  and  $AB^* = B^*A$ .

The following result holds [4, p. 38].

**Theorem 3** (Double commute). *If the operators  $A$  and  $B$  double commute, then*

$$(1.7) \quad w(AB) \leq w(B) \|A\|.$$

As a consequence of the above, we have [4, p. 39]:

**Corollary 1.** *Let  $A$  be a normal operator commuting with  $B$ . Then*

$$(1.8) \quad w(AB) \leq w(A) w(B).$$

For other results and historical comments on the above see [4, p. 39–41]. For more results on the numerical radius, see [5].

The main aim of this paper is to establish some new inequalities for composite operators generated by a pair of operators  $(A, B)$  under various assumptions. Namely, in one side, several inequalities involving the norm

$$\left\| \frac{A^*A + B^*B}{2} \right\|$$

and the numerical radius  $w(B^*A)$  are established. On the other side, upper bounds for the nonnegative quantities

$$\|A\| \|B\| - w(B^*A) \text{ and } \|A\|^2 \|B\|^2 - w^2(B^*A)$$

under special conditions for the operators involved are also given.

## 2. THE RESULTS

The following result may be stated:

**Theorem 4.** *Let  $A, B : H \rightarrow H$  be two bounded linear operators on the Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ . If  $r > 0$  and*

$$(2.1) \quad \|A - B\| \leq r,$$

*then*

$$(2.2) \quad \left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2}r^2.$$

*Proof.* For any  $x \in H$ ,  $\|x\| = 1$ , we have from (2.1) that

$$(2.3) \quad \|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2.$$

However

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \langle (A^*A)x, x \rangle + \langle (B^*B)x, x \rangle \\ &= \langle (A^*A + B^*B)x, x \rangle \end{aligned}$$

and by (2.3) we obtain

$$(2.4) \quad \langle (A^*A + B^*B)x, x \rangle \leq 2 |\langle (B^*A)x, x \rangle| + r^2$$

for any  $x \in H$ ,  $\|x\| = 1$ .

Taking the supremum over  $x \in H$ ,  $\|x\| = 1$  in (2.4) we get

$$(2.5) \quad w(A^*A + B^*B) \leq 2w(B^*A) + r^2$$

and since the operator  $A^*A + B^*B$  is self-adjoint, hence

$$w(A^*A + B^*B) = \|A^*A + B^*B\|$$

and by (2.5) we deduce the desired inequality (2.2). ■

**Remark 1.** We observe that, from the proof of the above theorem, we have the inequalities

$$(2.6) \quad 0 \leq \left\| \frac{A^*A + B^*B}{2} \right\| - w(B^*A) \leq \frac{1}{2} \|A - B\|^2,$$

provided that  $A, B$  are bounded linear operators in  $H$ .

The second inequality in (2.6) is obvious while the first inequality follows by the fact that

$$\begin{aligned} \langle (A^*A + B^*B)x, x \rangle &= \|Ax\|^2 + \|Bx\|^2 \\ &\geq 2\|Ax\|\|Bx\| \geq 2|\langle (B^*A)x, x \rangle| \end{aligned}$$

for any  $x \in H$ .

The inequality (2.2) is obviously a reach source of particular inequalities of interest.

Indeed, if we assume, for  $\lambda \in \mathbb{C}$  and a bounded linear operator  $T$ , that we have

$$(2.7) \quad \|T - \lambda T^*\| \leq r,$$

for a given positive number  $r$ , then by (2.6) we deduce the inequality

$$(2.8) \quad 0 \leq \left\| \frac{T^*T + |\lambda|^2 TT^*}{2} \right\| - |\lambda| w(T^2) \leq \frac{1}{2} r^2.$$

Now, if we assume that for  $\lambda \in \mathbb{C}$  and a bounded linear operator  $V$  we have that

$$(2.9) \quad \|V - \lambda I\| \leq r,$$

where  $I$  is the identity operator on  $H$ , then by (2.2) we deduce the inequality

$$0 \leq \left\| \frac{V^*V + |\lambda|^2 I}{2} \right\| - |\lambda| w(V) \leq \frac{1}{2} r^2.$$

As a dual approach, the following result may be noted as well:

**Theorem 5.** Let  $A, B : H \rightarrow H$  be two bounded linear operators on the Hilbert space  $H$ . Then

$$(2.10) \quad \left\| \frac{A + B}{2} \right\|^2 \leq \frac{1}{2} \left[ \left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A) \right].$$

*Proof.* We obviously have

$$\begin{aligned} \|Ax + Bx\|^2 &= \|Ax\|^2 + 2\operatorname{Re} \langle Ax, Bx \rangle + \|Bx\|^2 \\ &\leq \langle (A^*A + B^*B)x, x \rangle + 2|\langle (B^*A)x, x \rangle| \end{aligned}$$

for any  $x \in H$ .

Taking the supremum over  $x \in H$ ,  $\|x\| = 1$ , we get

$$\begin{aligned}\|A + B\|^2 &\leq w(A^*A + B^*B) + 2w(B^*A) \\ &= \|A^*A + B^*B\| + 2w(B^*A),\end{aligned}$$

from where we get the desired inequality (2.10). ■

**Remark 2.** The inequality (2.10) can generate some interesting particular results such as the following inequality

$$(2.11) \quad \left\| \frac{T + T^*}{2} \right\|^2 \leq \frac{1}{2} \left[ \left\| \frac{T^*T + TT^*}{2} \right\| + w(T^2) \right],$$

holding for each bounded linear operator  $T : H \rightarrow H$ .

The following result may be stated as well.

**Theorem 6.** Let  $A, B : H \rightarrow H$  be two bounded linear operators on the Hilbert space  $H$  and  $p \geq 2$ . Then

$$(2.12) \quad \left\| \frac{A^*A + B^*B}{2} \right\|^{\frac{p}{2}} \leq \frac{1}{4} [\|A - B\|^p + \|A + B\|^p].$$

*Proof.* We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [2]:

$$(2.13) \quad 2(\|a\|^p + \|b\|^p) \leq \|a + b\|^p + \|a - b\|^p$$

for any  $a, b \in H$  and  $p \geq 2$ .

Utilising (2.13) we may write

$$(2.14) \quad 2(\|Ax\|^p + \|Bx\|^p) \leq \|Ax + Bx\|^p + \|Ax - Bx\|^p$$

for any  $x \in H$ .

Now, observe that

$$\|Ax\|^p + \|Bx\|^p = \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}}$$

and by the elementary inequality:

$$\frac{\alpha^q + \beta^q}{2} \geq \left(\frac{\alpha + \beta}{2}\right)^q, \quad \alpha, \beta \geq 0 \quad \text{and} \quad q \geq 1$$

we have

$$\begin{aligned}(2.15) \quad \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}} &\geq 2^{1-\frac{p}{2}} \left(\|Ax\|^2 + \|Bx\|^2\right)^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}} [\langle (A^*A + B^*B)x, x \rangle]^{\frac{p}{2}}.\end{aligned}$$

Combining (2.14) with (2.15) we get

$$(2.16) \quad \frac{1}{4} [\|Ax - Bx\|^p + \|Ax + Bx\|^p] \geq \left| \left\langle \left( \frac{A^*A + B^*B}{2} \right) x, x \right\rangle \right|^{\frac{p}{2}}$$

for any  $x \in H$ ,  $\|x\| = 1$ . Taking the supremum over  $x \in H$ ,  $\|x\| = 1$ , and taking into account that

$$w\left(\frac{A^*A + B^*B}{2}\right) = \left\| \frac{A^*A + B^*B}{2} \right\|,$$

we deduce the desired result (2.12). ■

**Remark 3.** If  $p = 2$ , then we have the inequality:

$$(2.17) \quad \left\| \frac{A^*A + B^*B}{2} \right\| \leq \left\| \frac{A - B}{2} \right\|^2 + \left\| \frac{A + B}{2} \right\|^2,$$

for any  $A, B$  bounded linear operators. This result can also be obtained directly on utilising the parallelogram identity.

We also should observe that for  $A = T$  and  $B = T^*$ ,  $T$  a normal operator, the inequality (2.12) becomes

$$\|T\|^p \leq \frac{1}{4} [\|T - T^*\|^p + \|T + T^*\|^p],$$

where  $p \geq 2$ .

The following result may be stated as well.

**Theorem 7.** Let  $A, B : H \rightarrow H$  be two bounded linear operators on the Hilbert space  $H$  and  $r \geq 1$ . If  $A^*A \geq B^*B$  in the operator order or, equivalently,  $\|Ax\| \geq \|Bx\|$  for any  $x \in H$ , then:

$$(2.18) \quad \left\| \frac{A^*A + B^*B}{2} \right\|^r \leq \|A\|^{r-1} \|B\|^{r-1} w(B^*A) + \frac{1}{2} r^2 \|A\|^{2r-2} \|A - B\|^2.$$

*Proof.* We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [3]:

$$(2.19) \quad \|a\|^{2r} + \|b\|^{2r} \leq 2 \|a\|^{r-1} \|b\|^{r-1} \operatorname{Re} \langle a, b \rangle + r^2 \|a\|^{2r-2} \|a - b\|^2,$$

where  $r \geq 1$ ,  $a, b \in H$  and  $\|a\| \geq \|b\|$ .

Utilising (2.19) we can state that:

$$(2.20) \quad \begin{aligned} \|Ax\|^{2r} + \|Bx\|^{2r} \\ \leq 2 \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx \rangle| + r^2 \|Ax\|^{2r-2} \|Ax - Bx\|^2, \end{aligned}$$

for any  $x \in H$ .

As in the proof of Theorem 6, we also have

$$(2.21) \quad 2^{1-r} [(\langle (A^*A + B^*B)x, x \rangle)]^r \leq \|Ax\|^{2r} + \|Bx\|^{2r},$$

for any  $x \in H$ .

Therefore, by (2.20) and (2.21) we deduce

$$(2.22) \quad \begin{aligned} \left[ \left\langle \left( \frac{A^*A + B^*B}{2} \right) x, x \right\rangle \right]^r \\ \leq \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx \rangle| + \frac{1}{2} r^2 \|A\|^{2r-2} \|Ax - Bx\|^2 \end{aligned}$$

for any  $x \in H$ .

Taking the supremum in (2.22) we obtain the desired result (2.18). ▀

**Remark 4.** Following [4, p. 156], we recall that the bounded linear operator  $V$  is hyponormal, if

$$\|V^*x\| \leq \|Vx\| \text{ for all } x \in H.$$

Now, if we choose in (2.18)  $A = V$  and  $B = V^*$ , then, on taking into account that for hyponormal operators  $w(V^2) = \|V\|^2$ , we get the inequality

$$(2.23) \quad \left\| \frac{V^*V + VV^*}{2} \right\|^r \leq \|V\|^{2r-2} \left[ \|V\|^2 + \frac{1}{2} r^2 \|V - V^*\|^2 \right],$$

holding for any hyponormal operator  $V$  and any  $r \geq 1$ .

### 3. FURTHER INEQUALITIES FOR AN INVERTIBLE OPERATOR

In this section we assume that  $B : H \rightarrow H$  is an invertible bounded linear operator and let  $B^{-1} : H \rightarrow H$  be its inverse. Then, obviously,

$$(3.1) \quad \|Bx\| \geq \frac{1}{\|B^{-1}\|} \|x\| \quad \text{for any } x \in H,$$

where  $\|B^{-1}\|$  denotes the norm of the inverse  $B^{-1}$ .

The following result holds true:

**Theorem 8.** *Let  $A, B : H \rightarrow H$  be two bounded linear operators on  $H$  and  $B$  is invertible such that, for a given  $r > 0$ ,*

$$(3.2) \quad \|A - B\| \leq r.$$

*Then:*

$$(3.3) \quad \|A\| \leq \|B^{-1}\| \left[ w(B^*A) + \frac{1}{2}r^2 \right].$$

*Proof.* The condition (3.2) is obviously equivalent to:

$$(3.4) \quad \|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle (B^*A)x, x \rangle + r^2$$

for any  $x \in H$ ,  $\|x\| = 1$ .

Since, by (3.1),

$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2, \quad x \in H$$

and  $\operatorname{Re} \langle (B^*A)x, x \rangle \leq |\langle (B^*A)x, x \rangle|$ , hence by (3.4) we get

$$(3.5) \quad \|Ax\|^2 + \frac{\|x\|^2}{\|B^{-1}\|^2} \leq 2 |\langle (B^*A)x, x \rangle| + r^2$$

for any  $x \in H$ ,  $\|x\| = 1$ .

Taking the supremum over  $x \in H$ ,  $\|x\| = 1$  in (3.5), we have

$$(3.6) \quad \|A\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2w(B^*A) + r^2.$$

By the elementary inequality

$$(3.7) \quad \frac{2\|A\|}{\|B^{-1}\|} \leq \|A\|^2 + \frac{1}{\|B^{-1}\|^2}$$

and by (3.6) we then deduce the desired result (3.3). ■

**Remark 5.** *If we choose above  $B = \lambda I$ ,  $\lambda \neq 0$ , then we get the inequality*

$$(3.8) \quad (0 \leq) \|A\| - w(A) \leq \frac{1}{2|\lambda|} r^2,$$

*provided  $\|A - \lambda I\| \leq r$ . This result has been obtained in the earlier paper [1].*

*Also, if we assume that  $B = \lambda A^*$ ,  $A$  is invertible, then we obtain*

$$(3.9) \quad \|A\| \leq \|A^{-1}\| \left[ w(A^2) + \frac{1}{2|\lambda|} r^2 \right],$$

*provided  $\|A - \lambda A^*\| \leq r$ ,  $\lambda \neq 0$ .*

The following result may be stated as well:

**Theorem 9.** *Let  $A, B : H \rightarrow H$  be two bounded linear operators on  $H$ . If  $B$  is invertible and for  $r > 0$ ,*

$$(3.10) \quad \|A - B\| \leq r,$$

then

$$(3.11) \quad (0 \leq) \|A\| \|B\| - w(B^* A) \leq \frac{1}{2} r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}.$$

*Proof.* The condition (3.10) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any  $x \in H$ , which is clearly equivalent to

$$(3.12) \quad \|Ax\|^2 + \|B\|^2 \leq 2 \operatorname{Re} \langle B^* Ax, x \rangle + r^2 + \|B\|^2 - \|Bx\|^2.$$

Since

$$\operatorname{Re} \langle B^* Ax, x \rangle \leq |\langle B^* Ax, x \rangle|, \quad \|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2$$

and

$$\|Ax\|^2 + \|B\|^2 \geq 2 \|B\| \|Ax\|$$

for any  $x \in H$ , hence by (3.12) we get

$$(3.13) \quad 2 \|B\| \|Ax\| \leq 2 |\langle B^* Ax, x \rangle| + r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}$$

for any  $x \in H$ ,  $\|x\| = 1$ .

Taking the supremum over  $x \in H$ ,  $\|x\| = 1$  we deduce the desired result (3.11). ■

**Remark 6.** *If we choose in Theorem 9,  $B = \lambda A^*$ ,  $\lambda \neq 0$ ,  $A$  is invertible, then we get the inequality:*

$$(3.14) \quad (0 \leq) \|A\|^2 - w(A^2) \leq \frac{1}{2|\lambda|} r^2 + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{\|A^{-1}\|^2}$$

provided  $\|A - \lambda A^*\| \leq r$ .

The following result may be stated as well.

**Theorem 10.** *Let  $A, B : H \rightarrow H$  be two bounded linear operators on  $H$ . If  $B$  is invertible and for  $r > 0$  we have*

$$(3.15) \quad \|A - B\| \leq r < \|B\|,$$

then

$$(3.16) \quad \|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left( w(B^* A) + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{2 \|B^{-1}\|^2} \right).$$

*Proof.* The first part of condition (3.15) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any  $x \in H$ , which is clearly equivalent to

$$(3.17) \quad \|Ax\|^2 + \|B\|^2 - r^2 \leq 2 \operatorname{Re} \langle B^* Ax, x \rangle + \|B\|^2 - \|Bx\|^2.$$



Since

$$\begin{aligned} \operatorname{Re} \langle B^* A x, x \rangle &\leq |\langle B^* A x, x \rangle|, \\ \|B x\|^2 &\geq \frac{1}{\|B^{-1}\|^2} \|x\|^2 \end{aligned}$$

and, by the second part of (3.15),

$$\|A x\|^2 + \|B\|^2 - r^2 \geq 2\sqrt{\|B\|^2 - r^2} \|A x\|,$$

for any  $x \in H$ , hence by (3.17) we get

$$(3.18) \quad 2\|A x\| \sqrt{\|B\|^2 - r^2} \leq 2|\langle B^* A x, x \rangle| + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}$$

for any  $x \in H$ ,  $\|x\| = 1$ .

Taking the supremum over  $x \in H$ ,  $\|x\| = 1$  in (3.18), we deduce the desired inequality (3.16). ■

**Remark 7.** *The above Theorem 10 has some particular cases of interest. For instance, if we choose  $B = \lambda I$ , with  $|\lambda| > r$ , then (3.15) is obviously fulfilled and by (3.16) we get*

$$(3.19) \quad \|A\| \leq \frac{w(A)}{\sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}},$$

provided  $\|A - \lambda I\| \leq r$ . This result has been obtained in the earlier paper [1].

On the other hand, if in the above we choose  $B = \lambda A^*$  with  $\|A\| \geq \frac{r}{|\lambda|}$  ( $\lambda \neq 0$ ), then by (3.16) we get

$$(3.20) \quad \|A\| \leq \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\lambda|}\right)^2}} \left[ w(A^2) + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{2\|A^{-1}\|^2} \right],$$

provided  $\|A - \lambda A^*\| \leq r$ .

The following result may be stated as well.

**Theorem 11.** *Let  $A, B$  and  $r$  be as in Theorem 8. Moreover, if*

$$(3.21) \quad \|B^{-1}\| < \frac{1}{r},$$

then

$$(3.22) \quad \|A\| \leq \frac{\|B^{-1}\|}{\sqrt{1 - r^2 \|B^{-1}\|^2}} w(B^* A).$$

*Proof.* Observe that, by (3.6) we have

$$(3.23) \quad \|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2} \leq 2w(B^* A).$$

Utilising the elementary inequality

$$(3.24) \quad 2 \frac{\|A\|}{\|B^{-1}\|} \sqrt{1 - r^2 \|B^{-1}\|^2} \leq \|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2},$$

which can be stated since (3.21) is assumed to be true, hence by (3.23) and (3.24) we deduce the desired result (3.22). ■

**Remark 8.** If we assume that  $B = \lambda A^*$  with  $\lambda \neq 0$  and  $A$  an invertible operator, then, by applying Theorem 11, we get the inequality:

$$(3.25) \quad \|A\| \leq \frac{\|A^{-1}\| w(A^2)}{\sqrt{|\lambda|^2 - r^2 \|A^{-1}\|^2}},$$

provided  $\|A - \lambda A^*\| \leq r$  and  $\|A^{-1}\| \leq \frac{|\lambda|}{r}$ .

The following result may be stated as well.

**Theorem 12.** Let  $A, B : H \rightarrow H$  be two bounded linear operators. If  $r > 0$  and  $B$  is invertible with the property that  $\|A - B\| \leq r$  and

$$(3.26) \quad \frac{1}{\sqrt{r^2 + 1}} \leq \|B^{-1}\| < \frac{1}{r},$$

then

$$(3.27) \quad \|A\|^2 \leq w^2(B^*A) + 2w(B^*A) \cdot \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}.$$

*Proof.* Let  $x \in H$ ,  $\|x\| = 1$ . Then by (3.5) we have

$$(3.28) \quad \|Ax\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2|\langle B^*Ax, x \rangle| + r^2,$$

and since

$$\frac{1}{\|B^{-1}\|^2} - r^2 > 0,$$

we can conclude that  $|\langle B^*Ax, x \rangle| > 0$  for any  $x \in H$ ,  $\|x\| = 1$ .

Dividing in (3.28) with  $|\langle B^*Ax, x \rangle| > 0$ , we obtain

$$(3.29) \quad \frac{\|Ax\|^2}{|\langle B^*Ax, x \rangle|} \leq 2 + \frac{r^2}{|\langle B^*Ax, x \rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x \rangle|}.$$

Subtracting  $|\langle B^*Ax, x \rangle|$  from both sides of (3.29), we get

$$(3.30) \quad \begin{aligned} & \frac{\|Ax\|^2}{|\langle B^*Ax, x \rangle|} - |\langle B^*Ax, x \rangle| \\ & \leq 2 - |\langle B^*Ax, x \rangle| - \frac{1 - r^2 \|B^{-1}\|^2}{|\langle B^*Ax, x \rangle| \|B^{-1}\|^2} \\ & = 2 - \frac{2\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} - \left( \sqrt{|\langle B^*Ax, x \rangle|} - \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\| \sqrt{|\langle B^*Ax, x \rangle|}} \right)^2 \\ & \leq 2 \left( \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} \right), \end{aligned}$$

which gives:

$$(3.31) \quad \|Ax\|^2 \leq |\langle B^*Ax, x \rangle|^2 + 2|\langle B^*Ax, x \rangle| \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}.$$

We also remark that, by (3.26) the quantity

$$\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \geq 0,$$

hence, on taking the supremum in (3.31) over  $x \in H$ ,  $\|x\| = 1$ , we deduce the desired inequality. ■

**Remark 9.** *It is interesting to remark that if we assume  $\lambda \in \mathbb{C}$  with  $0 < r \leq |\lambda| \leq \sqrt{r^2 + 1}$  and  $\|A - \lambda I\| \leq r$ , then by (3.2) we can state the following inequality:*

$$(3.32) \quad \|A\|^2 \leq |\lambda|^2 w(A^2) + 2|\lambda| \left(1 - \sqrt{|\lambda|^2 - r^2}\right) w(A).$$

Also, if  $\|A - A^*\| \leq r$ ,  $A$  is invertible and  $\frac{1}{\sqrt{r^2 + 1}} \leq \|A^{-1}\| \leq \frac{1}{r}$ , then, by (3.27) we also have

$$(3.33) \quad \|A\|^2 \leq w^2(A^2) + 2w(A^2) \cdot \frac{\|A^{-1}\| - \sqrt{1 - r^2 \|A^{-1}\|^2}}{\|A^{-1}\|}.$$

One can also prove the following result.

**Theorem 13.** *Let  $A, B : H \rightarrow H$  be two bounded linear operators. If  $r > 0$  and  $B$  is invertible with the property that  $\|A - B\| \leq r$  and  $\|B^{-1}\| \leq \frac{1}{r}$ , then*

$$(3.34) \quad (0 \leq) \|A\|^2 \|B\|^2 - w^2(B^* A) \\ \leq 2w(B^* A) \cdot \frac{\|B\|}{\|B^{-1}\|} \left( \|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right).$$

*Proof.* We subtract the quantity  $\frac{|\langle B^* A x, x \rangle|}{\|B\|^2}$  from both sides of (3.29) to obtain

$$(3.35) \quad 0 \leq \frac{\|Ax\|^2}{|\langle B^* A x, x \rangle|} - \frac{|\langle B^* A x, x \rangle|}{\|B\|^2} \\ \leq 2 - \frac{|\langle B^* A x, x \rangle|}{\|B\|^2} - \frac{1 - r^2 \|B^{-1}\|^2}{|\langle B^* A x, x \rangle| \|B^{-1}\|^2} \\ = 2 - 2 \cdot \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B\| \|B^{-1}\|} - \left( \frac{\sqrt{|\langle B^* A x, x \rangle|}}{\|B\|} - \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\sqrt{|\langle B^* A x, x \rangle|} \|B^{-1}\|} \right)^2 \\ \leq 2 \cdot \frac{\left( \|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right)}{\|B\| \|B^{-1}\|},$$

which is equivalent with

$$(3.36) \quad (0 \leq) \|Ax\|^2 \|B\|^2 - |\langle B^* A x, x \rangle|^2 \\ \leq 2 \frac{\|B\|}{\|B^{-1}\|} |\langle B^* A x, x \rangle| \left( \|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right)$$

for any  $x \in H$ ,  $\|x\| = 1$ .

The inequality (3.36) also shows that  $\|B\| \|B^{-1}\| \geq \sqrt{1 - r^2 \|B^{-1}\|^2}$  and then, by (3.36), we get

$$(3.37) \quad \|Ax\|^2 \|B\|^2 \leq |\langle B^* Ax, x \rangle|^2 + 2 \frac{\|B\|}{\|B^{-1}\|} |\langle B^* Ax, x \rangle| \left( \|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right)$$

for any  $x \in X$ ,  $\|x\| = 1$ .

Taking the supremum in (3.37) we deduce the desired inequality (3.34). ■

**Remark 10.** *The above Theorem 13 has some particular instances of interest as follows. If, for instance, we choose  $B = \lambda I$  with  $|\lambda| \geq r > 0$  and  $\|A - \lambda I\| \leq r$ , then by (3.34) we obtain the inequality*

$$(3.38) \quad (0 \leq) \|A\|^2 - w^2(A) \leq 2 |\lambda| w(A) \left( 1 - \sqrt{1 - \frac{r^2}{|\lambda|^2}} \right).$$

Also, if  $A$  is invertible,  $\|A - \lambda A^*\| \leq r$  and  $\|A^{-1}\| \leq \frac{|\lambda|}{r}$ , then by (3.34) we can state:

$$(3.39) \quad (0 \leq) \|A\|^4 - w^2(A^2) \leq 2 |\lambda| w(A^2) \cdot \frac{\|A\|}{\|A^{-1}\|} \left( \|A\| \|A^{-1}\| - \sqrt{1 - \frac{r^2}{|\lambda|^2} \|A^{-1}\|^2} \right).$$

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