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# COMPLETE MONOTONICITIES OF FUNCTIONS INVOLVING THE GAMMA AND DIGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

ABSTRACT. In the article, the completely monotonic results of the functions  $[\Gamma(x+1)]^{1/x}$ ,  $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ ,  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  and  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  in  $x \in (-1, \infty)$  for  $\alpha \in \mathbb{R}$  are obtained. In the final, three open problems are posed.

## 1. INTRODUCTION

The classical gamma function is usually defined for  $\operatorname{Re} z > 0$  by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \quad (1)$$

The psi or digamma function  $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ , the logarithmic derivative of the gamma function, and the polygamma functions can be expressed (See [1, 8] and [12, p. 16]) for  $x > 0$  and  $k \in \mathbb{N}$  as

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{1+n} - \frac{1}{x+n} \right), \quad (2)$$

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}}, \quad (3)$$

where  $\gamma = 0.57721566490153286 \dots$  is the Euler-Mascheroni constant.

A function  $f$  is said to be completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \quad (4)$$

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for  $x \in I$  and  $n \geq 0$ . If inequality (4) is strict for all  $x \in I$  and for all  $n \geq 0$ , then  $f$  is said to be strictly completely monotonic. For more information, please refer to [14, 15, 18, 23, 25] and references therein.

A function  $f$  is said to be logarithmically completely monotonic on an interval  $I$  if its logarithm  $\ln f$  satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \quad (5)$$

for  $k \in \mathbb{N}$  on  $I$ . If inequality (5) is strict for all  $x \in I$  and for all  $k \geq 1$ , then  $f$  is said to be strictly logarithmically completely monotonic.

In this article, using Leibnitz's formula and the formulas (2) and (3), the complete monotonicity properties of the functions  $[\Gamma(x+1)]^{1/x}$ ,  $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ ,  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  and  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  in  $x \in (-1, \infty)$  for  $\alpha \in \mathbb{R}$  are obtained. From these, some well known results are deduced, extended and generalized. The main results of this paper are as follows.

**Theorem 1.** *The function  $[\Gamma(x+1)]^{1/x}$  is strictly increasing in  $(-1, \infty)$ . The function  $\frac{\psi(x+1)}{x} - \frac{\ln \Gamma(x+1)}{x^2}$ , the logarithmic derivative of  $[\Gamma(x+1)]^{1/x}$ , is strictly completely monotonic in  $(-1, \infty)$ . The function  $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$  is logarithmically strictly completely monotonic with  $x \in (-1, \infty)$  for  $\alpha > 0$ .*

**Theorem 2.** *For  $\alpha \geq 1$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly decreasing and the function  $\frac{\ln \Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x+1}$ , the logarithmic derivative of  $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}$ , is strictly completely monotonic with  $x \in (-1, \infty)$ .*

Let  $\tau(s, t) = \frac{1}{s} [t - (t+s+1) \left(\frac{t}{t+1}\right)^{s+1}] > 0$  for  $(s, t) \in \mathbb{N} \times (0, \infty)$  and  $\tau_0 = \tau(s_0, t_0) > 0$  be the maximum of  $\tau(s, t)$  on the set  $\mathbb{N} \times (0, \infty)$ . For a given real number  $\alpha$  satisfying  $\alpha \leq \frac{1}{1+\tau_0} < 1$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly increasing and the function  $\frac{\psi(x+1)}{x} - \frac{\ln \Gamma(x+1)}{x^2} - \frac{\alpha}{x+1}$  is strictly completely monotonic in  $x \in (-1, \infty)$ .

**Theorem 3.** *For  $\alpha \leq 0$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly increasing and the function  $\frac{\psi(x+1)}{x} - \frac{\ln \Gamma(x+1)}{x^2} - \frac{\alpha}{x}$ , the logarithmic derivative of  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ , is strictly completely monotonic in  $(0, \infty)$ . For  $\alpha \geq 1$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly decreasing and the function  $\frac{\ln \Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x}$  is strictly completely monotonic in  $(0, \infty)$ .*

For  $\alpha \leq 0$  such that  $x^\alpha$  is real in  $(-1, 0)$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly decreasing and the function  $\frac{\ln \Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x}$  is strictly completely monotonic

in  $(-1, 0)$ . For  $\alpha \geq 1$  such that  $x^\alpha$  is real in  $(-1, 0)$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly increasing and the function  $\frac{\psi(x+1)}{x} - \frac{\ln \Gamma(x+1)}{x^2} - \frac{\alpha}{x}$  is strictly completely monotonic in  $(-1, 0)$ .

**Theorem 4.** *A (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.*

As a direct consequence of combining Theorem 1 with Theorem 4, we have the following corollary.

**Corollary 1.** *The function  $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$  is strictly completely monotonic with  $x \in (-1, \infty)$  for  $\alpha > 0$ .*

In [3] and [4, p. 83], the following result was given: Let  $f$  and  $g$  be functions such that  $f \circ g$  is defined. If  $f$  and  $g'$  are completely monotonic, then  $f \circ g$  is also completely monotonic. Thus, from Theorem 1 and Theorem 2 and the fact that the exponential function  $e^{-x}$  is strictly completely monotonic in  $(-\infty, \infty)$ , the following corollary can be deduced.

**Corollary 2.** *The following complete monotonicity properties holds:*

- (1) *The function  $\frac{1}{[\Gamma(x+1)]^{1/x}}$  is strictly completely monotonic in  $(-1, \infty)$ .*
- (2) *For  $\alpha \geq 1$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly completely monotonic in  $(-1, \infty)$ . For a given real number  $\alpha$  with  $\alpha \leq \frac{1}{1+\tau_0} < 1$ , the function  $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}$  is strictly completely monotonic in  $(-1, \infty)$ .*
- (3) *For  $\alpha \leq 0$ , the function  $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$  is strictly completely monotonic in  $(0, \infty)$ . For  $\alpha \geq 1$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly completely monotonic in  $(0, \infty)$ . For  $\alpha \leq 0$  such that  $x^\alpha$  is real in  $(-1, 0)$ , the function  $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$  is strictly completely monotonic in  $(-1, 0)$ . For  $\alpha \geq 1$  such that  $x^\alpha$  is real in  $(-1, 0)$ , the function  $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$  is strictly completely monotonic in  $(-1, 0)$ .*

## 2. PROOFS OF THEOREMS

*Proof of Theorem 1.* For  $\alpha > 0$ , let

$$f_\alpha(x) = \frac{[\Gamma(x + \alpha + 1)]^{1/(x+\alpha)}}{[\Gamma(x + 1)]^{1/x}} \quad (6)$$

for  $x > -1$ .

By direct calculation and using Leibnitz's formula and formulas (2) and (3), we obtain for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \ln f_\alpha(x) &= \frac{\ln \Gamma(x + \alpha + 1)}{x + \alpha} - \frac{\ln \Gamma(x + 1)}{x} \triangleq g(x + \alpha) - g(x), \\ g^{(n)}(x) &= \frac{1}{x^{n+1}} \sum_{k=0}^n \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x + 1)}{k!} \triangleq \frac{h_n(x)}{x^{n+1}}, \end{aligned} \quad (7)$$

$$\begin{aligned} h'_n(x) &= x^n \psi^{(n)}(x + 1) \\ &\begin{cases} > 0, & \text{if } n \text{ is odd and } x \in (0, \infty), \\ \leq 0, & \text{if } n \text{ is odd and } x \in (-1, 0] \text{ or } n \text{ is even and } x \in (-1, \infty), \end{cases} \end{aligned} \quad (8)$$

where  $\psi^{(-1)}(x + 1) = \ln \Gamma(x + 1)$  and  $\psi^{(0)}(x + 1) = \psi(x + 1)$ . Hence, the function  $h_n(x)$  increases if  $n$  is odd and  $x \in (0, \infty)$  and decreases if  $n$  is odd and  $x \in (-1, 0)$  or  $n$  is even and  $x \in (-1, \infty)$ . Since  $h_n(0) = 0$ , it is easy to see that  $h_n(x) \geq 0$  if  $n$  is odd and  $x \in (-1, \infty)$  or  $n$  is even and  $x \in (-1, 0)$  and  $h_n(x) \leq 0$  if  $n$  is even and  $x \in (0, \infty)$ . Then, for  $x \in (-1, \infty)$ , we have  $g^{(n)}(x) \geq 0$  if  $n$  is odd and  $g^{(n)}(x) \leq 0$  if  $n$  is even. Since  $\lim_{x \rightarrow \infty} \frac{\psi^{(k)}(x+1)}{x^{n+1}} = 0$  for  $-1 \leq k \leq n$ , it is easy to see that  $\lim_{x \rightarrow \infty} g^{(n)}(x) = \lim_{x \rightarrow \infty} \frac{h_n(x)}{x^{n+1}} = 0$ . Therefore  $(-1)^{n+1} g^{(n)}(x) > 0$  with  $x \in (-1, \infty)$  for  $n \in \mathbb{N}$ . Then the function  $g'(x)$  is strictly completely monotonic and  $[\Gamma(x + 1)]^{1/x} = \exp(g(x))$  is strictly increasing in  $(-1, \infty)$ .

From  $(-1)^{n+1} g^{(n)}(x) \geq 0$  with  $x \in (-1, \infty)$  for  $n \in \mathbb{N}$ , it follows that  $g^{2k}(x)$  increases and  $g^{2k-1}(x)$  decreases with  $x \in (-1, \infty)$  for all  $k \in \mathbb{N}$ . This implies that  $(-1)^n [\ln f_\alpha(x)]^{(n)} \geq 0$ , and then the function  $\frac{[\Gamma(x + \alpha + 1)]^{1/(x + \alpha)}}{[\Gamma(x + 1)]^{1/x}}$  is logarithmically completely monotonic with  $x \in (-1, \infty)$ .  $\square$

*Proof of Theorem 2.* Let

$$\nu_\alpha(x) = \frac{[\Gamma(x + 1)]^{1/x}}{(x + 1)^\alpha} \quad (9)$$

for  $x \in (-1, \infty)$ . Then for  $n \in \mathbb{N}$ ,

$$\ln \nu_\alpha(x) = \frac{\ln \Gamma(x + 1)}{x} - \alpha \ln(x + 1), \quad (10)$$

$$[\ln \nu_\alpha(x)]^{(n)} = \frac{1}{x^{n+1}} \left[ h_n(x) + \frac{(-1)^n (n - 1)! \alpha x^{n+1}}{(x + 1)^n} \right] \triangleq \frac{\mu_{\alpha, n}(x)}{x^{n+1}}, \quad (11)$$

$$\mu'_{\alpha, n}(x) = x^n \psi^{(n)}(x + 1) + \frac{(-1)^n (n - 1)! \alpha x^n (x + n + 1)}{(x + 1)^{n+1}}$$

$$\begin{aligned}
 &= x^n \left[ \psi^{(n)}(x+1) + \frac{(-1)^n (n-1)! \alpha}{(x+1)^n} + \frac{(-1)^n n! \alpha}{(x+1)^{n+1}} \right] \\
 &= x^n \left\{ (-1)^{n+1} n! \sum_{i=1}^{\infty} \frac{1}{(x+i)^{n+1}} \right. \\
 &\quad \left. + (-1)^n (n-1)! \alpha \sum_{i=1}^{\infty} \left[ \frac{1}{(x+i)^n} - \frac{1}{(x+i+1)^n} \right] \right. \\
 &\quad \left. + (-1)^n n! \alpha \sum_{i=1}^{\infty} \left[ \frac{1}{(x+i)^{n+1}} - \frac{1}{(x+i+1)^{n+1}} \right] \right\} \quad (12) \\
 &= (-1)^n (n-1)! x^n \sum_{i=1}^{\infty} \left[ \frac{\alpha}{(x+i)^n} - \frac{\alpha}{(x+i+1)^n} \right. \\
 &\quad \left. - \frac{n\alpha}{(x+i+1)^{n+1}} + \frac{n(\alpha-1)}{(x+i)^{n+1}} \right] \\
 &= (n-1)! (-x)^n \sum_{i=1}^{\infty} \frac{[\alpha y + n(\alpha-1)](y+1)^{n+1} - \alpha(y+n+1)y^{n+1}}{y^{n+1}(y+1)^{n+1}} \\
 &= (n-1)! (-x)^n \sum_{i=1}^{\infty} \frac{\alpha[(y+n)(y+1)^{n+1} - (y+n+1)y^{n+1}] - n(y+1)^{n+1}}{y^{n+1}(y+1)^{n+1}} \\
 &= n! (-x)^n \sum_{i=1}^{\infty} \frac{1}{y^{n+1}} \left\{ \alpha \left[ 1 + \frac{1}{n} \left\langle y - (y+n+1) \left( \frac{y}{y+1} \right)^{n+1} \right\rangle \right] - 1 \right\},
 \end{aligned}$$

where  $y = x + i > 0$ .

In [5, p. 28] and [11, p. 154], the Bernoulli's inequality states that if  $x \geq -1$  and  $x \neq 0$  and if  $\alpha > 1$  or if  $\alpha < 0$  then  $(1+x)^\alpha > 1 + \alpha x$ . This means that  $1 + \frac{s+1}{t} < (1 + \frac{1}{t})^{s+1}$  for  $t > 0$ , which is equivalent to  $t - (t+s+1) \left( \frac{t}{t+1} \right)^{s+1} > 0$  for  $t > 0$ , and then  $\tau(s, t) > 0$  for  $s \geq 1$  and  $t > 0$  and  $\tau(s, 0) = 0$ .

From  $\tau(s, t) > 0$ , it is deduced that  $[\alpha y + n(\alpha-1)](y+1)^{n+1} - \alpha(y+n+1)y^{n+1} > 0$  for  $y = x + i > 0$  and  $n \in \mathbb{N}$  if  $\alpha \geq 1$ . Therefore, for  $\alpha \geq 1$ , we have

$$\mu'_{\alpha, n}(x) \begin{cases} > 0, & \text{if } n \text{ is even and } x \in (-1, 0) \cup (0, \infty) \text{ or } n \text{ is odd and } x \in (-1, 0), \\ < 0, & \text{if } n \text{ is odd and } x \in (0, \infty), \end{cases}$$

and then  $\mu_{\alpha, n}(x)$  is strictly increasing with  $x \in (-1, \infty)$  if  $n$  is even or with  $x \in (-1, 0)$  if  $n$  is odd and  $\mu_{\alpha, n}(x)$  is strictly decreasing with  $x \in (0, \infty)$  if  $n$  is odd. Since  $\mu_{\alpha, n}(0) = 0$ , thus  $\mu_{\alpha, n}(x) < 0$  with  $x > -1$  and  $x \neq 0$  if  $n$  is odd or with  $x \in (-1, 0)$  if  $n$  is even and  $\mu_{\alpha, n}(x) > 0$  with  $x \in (0, \infty)$  if  $n$  is even. Therefore, from  $\lim_{x \rightarrow \infty} [\ln \nu_\alpha(x)]^{(n)} = 0$ , it is deduced that  $[\ln \nu_\alpha(x)]^{(n)} > 0$  if  $n$  is even

and  $[\ln \nu_\alpha(x)]^{(n)} < 0$  if  $n$  is odd, which is equivalent to  $(-1)^n [\ln \nu_\alpha(x)]^{(n)} > 0$  in  $x \in (-1, \infty)$  for  $n \in \mathbb{N}$  and  $\alpha \geq -1$ . Hence, if  $\alpha \geq 1$ , then the function  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly decreasing and the function  $\frac{\ln \Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x+1}$  is strictly completely monotonic in  $x \in (-1, \infty)$ .

It is clear that  $\tau_0 > 0$ . When  $\alpha \leq \frac{1}{1+\tau_0} < 1$ , it follows that  $\mu'_{\alpha,n}(x) < 0$  and  $\mu_{\alpha,n}(x)$  is decreasing with  $x \in (-1, \infty)$  and  $x \neq 0$  for  $n$  an even integer or with  $x \in (-1, 0)$  for  $n$  an odd integer, and  $\mu'_{\alpha,n}(x) > 0$  and  $\mu_{\alpha,n}(x)$  is increasing with  $x \in (0, \infty)$  for  $n$  an odd integer. Since  $\mu_{\alpha,n}(0) = 0$  and  $\lim_{x \rightarrow \infty} [\ln \nu_\alpha(x)]^{(n)} = 0$ , we have  $[\ln \nu_\alpha(x)]^{(n)} < 0$  for  $n$  an even and  $[\ln \nu_\alpha(x)]^{(n)} > 0$  for  $n$  an odd in  $x \in (-1, \infty)$ , this implies that  $(-1)^{n+1} [\ln \nu_\alpha(x)]^{(n)} > 0$  in  $x \in (-1, \infty)$  for  $n \in \mathbb{N}$ . Therefore  $\nu_\alpha(x)$  is strictly increasing and  $(-1)^{n-1} \{[\ln \nu_\alpha(x)]'\}^{(n-1)} > 0$  in  $(-1, \infty)$  for  $n \in \mathbb{N}$ . Hence, if  $\alpha \leq \frac{1}{1+\tau_0}$ , then the function  $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly increasing and the function  $\frac{\psi(x+1)}{x} - \frac{\ln \Gamma(x+1)}{x^2} - \frac{\alpha}{x+1}$  is strictly completely monotonic in  $(-1, \infty)$ .  $\square$

*Proof of Theorem 3.* The procedure is same as the one of Theorem 2. Hence, we leave it to the readers.  $\square$

*Proof of Theorem 4.* It is clear that  $\exp \phi(x) \geq 0$ . Further, it is easy to see that  $[\exp \phi(x)]' = \phi'(x) \exp \phi(x) \leq 0$  and  $[\exp \phi(x)]'' = \{\phi''(x) + [f'(x)]^2\} \exp \phi(x) \geq 0$ .

Suppose  $(-1)^k [\exp \phi(x)]^{(k)} \geq 0$  for all nonnegative integers  $k \leq n$ , where  $n \in \mathbb{N}$  is a given positive integer. By Leibnitz's formula, we have

$$\begin{aligned}
(-1)^{n+1} [\exp \phi(x)]^{(n+1)} &= (-1)^{n+1} \{[\exp \phi(x)]'\}^{(n)} \\
&= (-1)^{n+1} [\phi'(x) \exp \phi(x)]^{(n)} \\
&= (-1)^{n+1} \sum_{i=0}^n \binom{n}{i} \phi^{(i+1)}(x) [\exp \phi(x)]^{(n-i)} \\
&= \sum_{i=0}^n \binom{n}{i} [(-1)^{i+1} \phi^{(i+1)}(x)] \{(-1)^{n-i} [\exp \phi(x)]^{(n-i)}\} \\
&\geq 0.
\end{aligned} \tag{13}$$

By induction, it is proved that the function  $\exp \phi(x)$  is completely monotonic.  $\square$

## 3. REMARKS

*Remark 1.* In [10, 13], among other things, the following monotonicity results were obtained

$$\begin{aligned} [\Gamma(1+k)]^{1/k} &< [\Gamma(2+k)]^{1/(k+1)}, \quad k \in \mathbb{N}; \\ \left[ \Gamma \left( 1 + \frac{1}{x} \right) \right]^x &\text{ decreases with } x > 0. \end{aligned}$$

These are extended and generalized in [16]: The function  $[\Gamma(r)]^{1/(r-1)}$  is increasing in  $r > 0$ . Clearly, Theorem 1 generalizes and extends these results for the range of the argument.

*Remark 2.* It is proved in [19] that the function  $\frac{1}{x} \ln \Gamma(x+1) - \ln x + 1$  is strictly completely monotonic on  $(0, \infty)$  and tends to  $+\infty$  as  $x \rightarrow 0$  and to 0 as  $x \rightarrow \infty$ . A similar result was found in [24]: The function  $1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$  is strictly completely monotonic on  $(-1, \infty)$  and tends to 1 as  $x \rightarrow -1$  and to 0 as  $x \rightarrow \infty$ . Our main results generalize these ones.

*Remark 3.* From our main results, the following can be deduced: Let  $n$  be natural number. Then the sequence  $\frac{\sqrt[n]{n!}}{n+k/(n+k+1)!}$  are increasing with  $n \in \mathbb{N}$ .

*Remark 4.* A function  $f$  is logarithmic convex on an interval  $I$  if  $f$  is positive and  $\ln f$  is convex on  $I$ . Since  $f(x) = \exp[\ln f(x)]$ , it follows that a logarithmic convex function is convex.

*Remark 5.* Straightforward computation shows that the maximum  $\tau_2$  of  $\tau(2, t)$  in  $(0, \infty)$  is

$$\tau \left( 2, \frac{2 + \sqrt{7}}{3} \right) = \frac{1}{2} \left[ \frac{2 + \sqrt{7}}{3} - \frac{(2 + \sqrt{7})^3 \left( 3 + \frac{2 + \sqrt{7}}{3} \right)}{27 \left( 1 + \frac{2 + \sqrt{7}}{3} \right)^3} \right] = 0.264076 \dots \quad (14)$$

and the maximum  $\tau_3$  of  $\tau(3, t)$  in  $(0, \infty)$  is

$$\tau \left( 3, \frac{5}{9} + \frac{\sqrt[3]{2836 - 54\sqrt{406}}}{18} + \frac{\sqrt[3]{1418 + 27\sqrt{406}}}{9\sqrt[3]{4}} \right) = 0.271807 \dots \quad (15)$$

If  $\alpha \leq \frac{1}{1+\tau_2} = 0.791091378310519808 \dots$ , then  $\mu'_{\alpha,2}(x) \leq 0$  and  $\mu_{\alpha,2}(x)$  decreases in  $(-1, \infty)$ . Since  $\mu_{\alpha,2}(0) = 0$  and  $\lim_{x \rightarrow \infty} [\ln \nu_{\alpha}(x)]^{(2)} = 0$ , it is obtained that



$[\ln \nu_\alpha(x)]^{(2)} < 0$ . Therefore the function  $\nu_\alpha(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$  is strictly increasing and strictly logarithmically concave for  $\alpha \leq \frac{1}{1+\tau_2}$  in  $(-1, \infty)$ .

If  $\alpha \leq \frac{1}{1+\tau_3} = 0.7862824583608 \dots$ , then  $\mu'_{\alpha,3}(x) < 0$  and  $\mu_{\alpha,3}(x)$  decreases in  $(-1, 0)$  and  $\mu'_{\alpha,3}(x) > 0$  and  $\mu_{\alpha,3}(x)$  increases in  $(0, \infty)$ . Thus  $\mu_{\alpha,3}(x) \geq 0$  and then  $[\ln \nu_\alpha(x)]^{(3)} > 0$  in  $(-1, \infty)$ . Hence  $[\ln \nu_\alpha(x)]^{(2)}$  is strictly increasing in  $(-1, \infty)$  if  $\alpha \leq \frac{1}{1+\tau_3}$ .

MATHEMATICA shows that  $\tau_0 > 0.2980 \dots$ .

*Remark 6.* The motivation of this paper has been exposted in detail in [21] and a lot of literature is listed therein. Please also refer to [2, 6, 7, 9, 17, 20, 22].

#### 4. OPEN PROBLEMS

A function  $f(t)$  is said to be absolutely monotonic on an interval  $I$  if it has derivatives of all orders and  $f^{(k)}(t) \geq 0$  for  $t \in I$  and  $k \in \mathbb{N}$ . A function  $f(t)$  is said to be regularly monotonic if it and its derivatives of all orders have constant sign (+ or -; not all the same) on  $(a, b)$ . A function  $f(t)$  is said to be absolutely convex on  $(a, b)$  if it has derivatives of all orders and  $f^{(2k)}(t) \geq 0$  for  $t \in (a, b)$  and  $k \in \mathbb{N}$ .

The function  $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$  can be expressed as

$$\frac{x+\alpha \sqrt{\int_0^\infty t^{x+\alpha} e^{-t} dt}}{x \sqrt{\int_0^\infty t^x e^{-t} dt}}, \quad (16)$$

where  $\int_0^\infty e^{-t} dt = 1$ . Then we propose the following

**Open Problem 1.** Let  $w(x) \geq 0$  be a nonnegative weight defined on a domain  $\Omega$  with  $\int_\Omega w(x) dx = 1$ . Find conditions about  $w(x)$  and  $f(x) \geq 0$  such that the ratio between two power means

$$\mathcal{Q}(t) = \frac{[\int_\Omega w(x) f^{t+\alpha}(x) dx]^{1/(t+\alpha)}}{[\int_\Omega w(x) f^t(x) dx]^{1/t}} \quad (17)$$

is completely (absolutely, regularly) monotonic (convex) with  $t \in \mathbb{R}$  for a given number  $\alpha > 0$ .

**Open Problem 2.** Find conditions about  $\alpha$  and  $\beta$  such that the ratio

$$\mathcal{F}(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+\beta)^\alpha} \quad (18)$$

is completely (absolutely, regularly) monotonic (convex) with  $x > -1$ .

**Open Problem 3.** For  $(s, t) \in \mathbb{N} \times (0, \infty)$ , find the maximum of the following

$$\tau(s, t) = \frac{1}{s} \left[ t - (t + s + 1) \left( \frac{t}{t + 1} \right)^{s+1} \right]. \quad (19)$$

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