A Complete Monotonicity of the Gamma Function

A COMPLETE MONOTONICITY OF THE GAMMA FUNCTION

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Abstract. The function $\frac{1}{x}\ln\Gamma(x+1)-\ln x+1$ is strictly completely monotonic on $(0,\infty)$.

The classical gamma function is usually defined for $\operatorname{Re} z>0$ by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{1}$$

The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [6, p. 16] as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{2}$$

$$\psi^{(m)}(x)=(-1)^{m+1}\int_0^\infty \frac{t^m}{1-e^{-t}}e^{-xt}\,\mathrm{d}t \tag{3}$$

for $x>0$ and $m\in\mathbb{N}$, where $\gamma=0.57721566490153286\cdots$ is the Euler-Mascheroni constant.

In 1985, D. Kershaw and A. Laforgia [5] showed that the function $x[\Gamma(1+\frac{1}{x})]^x$ is strictly increasing on $(0,\infty)$, which is equivalent to the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ being strictly decreasing on $(0,\infty)$. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0<x<1$, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [2, 10], it is proved that the function $f(x)=\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0,\infty)$ and the function $g(x)=\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0,\infty)$. Some new proofs for the monotonicity of the funtion $x^r[\Gamma(x+1)]^{1/x}$ on $(0,\infty)$ are given for $r\notin(0,1)$. In addition, if s is a positive real number, then for all real numbers $x>0$,

$$\frac{e^{-\gamma}}{[\Gamma(s+1)]^{1/s}}<\frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(x+s+1)]^{1/(x+s)}}<1, \tag{4}$$

$\lim_{x\to0}f(x)=e^{-\gamma}$ and $\lim_{x\to\infty}f(x)=e^{-1}$.

Using monotonicity and inequalities of the generalized weighted mean values [1, 7, 8, 12], the first author proved [9] that the functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$

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and $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ are increasing in $r>0$, $s>0$ and $x>0$. For any given $x>0$, the function $\frac{s\gamma(s,x)}{x^s}$ is decreasing in $s>0$.

In [3], N. Elezović, C. Giordana and J. Pečarić, among others, verified the convexity with respect to variable x of the function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$ for $|t-s|<1$.

Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^n(x) \geq 0 \tag{5}$$

for $x \in I$ and $n \geq 0$. If inequality (5) is strict for all $x \in I$ and for all $n \geq 0$, then f is said to be strictly completely monotonic. See [11] and references therein.

In this short note, we are about to prove a complete monotonicity result of a function involving the gamma function.

Theorem 1. *The function $\frac{1}{x}\ln\Gamma(x+1)-\ln x+1$ is strictly completely monotonic on $(0,\infty)$ and tends to ∞ as $x\to 0$ and to 0 as $x\to\infty$.*

Proof. It has been shown in [5] that the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ is strictly decreasing on $(0,\infty)$, then $f(x)=\frac{1}{x}\ln\Gamma(x+1)-\ln x+1$ is strictly decreasing on $(0,\infty)$. From the asymptotic expansion in [4]:

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi}+\frac{1}{12x}+O(x^{-3}) \quad \text{as } x\to\infty,$$

we conclude that $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to 0} f(x)=\infty$. This implies $f(x)>0$ for $x>0$.

Using Leibniz' rule

$$[u(x)v(x)]^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)v^{(n-k)}(x) \tag{6}$$

we obtain

$$\begin{aligned}
f^{(n)}(x)&=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}[\ln\Gamma(x+1)]^{(k)}-\frac{(-1)^{n-1}(n-1)!}{x^n}\\
&=\left(\frac{1}{x}\right)^{(n)}\ln\Gamma(x+1)+\sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}\psi^{(k-1)}(x+1)+\frac{(-1)^n n!}{nx^n}\\
&=\frac{(-1)^n n!}{x^{n+1}}\ln\Gamma(x+1)+\sum_{k=1}^{n}\frac{n!}{k!}\frac{(-1)^{n-k}}{x^{n-k+1}}\psi^{(k-1)}(x+1)+\frac{(-1)^n n!}{nx^n}\\
&\triangleq (-1)^n\frac{n!}{x^{n+1}}g(x),
\end{aligned} \tag{7}$$

and

$$g'(x)=\frac{(-1)^n}{n!}x^n\psi^{(n)}(x+1)+\frac{1}{n}. \tag{8}$$

Using (3) and $\frac{(n-1)!}{x^n}=\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t$ for $x>0$ and $n\in\mathbb{N}$, we conclude

$$\frac{1}{x^n}g'(x)=\frac{1}{n!}\int_0^\infty\left(1-\frac{t}{e^t-1}\right)t^{n-1}e^{-xt}\,\mathrm{d}t>0, \tag{9}$$

since $0 < \frac{t}{e^t-1} < 1$ for $x > 0$. Thus, the function g is strictly increasing and $g(x) > g(0) = 0$ on $(0, \infty)$, which implies $(-1)^n f^{(n)}(x) > 0$ for $x > 0$ and $n = 0, 1, 2, \dots$. The proof is complete. □

Remark 1. After this paper was finalized, a similar result in [13] was found: The function $1 + \frac{1}{x} \ln \Gamma(x + 1) - \ln(x + 1)$ is strictly completely monotone in $(-1, \infty)$ and tends to 1 as $x \to -1$ and to 0 as $x \to \infty$. This property is derived from the following integral representation:

$$\ln \Gamma(x + 1) = x \ln(x + 1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-t} \frac{1 - e^{-xt}}{t} \,\mathrm{d}t. \tag{10}$$

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