Integral Representation of a Series Which One Includes the Mathieu A-Series

INTEGRAL REPRESENTATION OF A SERIES WHICH ONE INCLUDES THE MATHIEU A-SERIES

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Abstract. Integral expression is deduced for the series

$$\mathfrak{S}(r,\mu,\nu,\mathbf{a}) = \sum_{n=1}^{\infty} \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}\,,\frac{\nu-\mu}{2}+1\,;\nu+1\,;-\frac{r^2}{a(n)^2}\right)}{a(n)^{\nu-\mu+1}(a(n)^2+r^2)^{\mu-1/2}},$$

where $\mu > 1/2,\ \nu > -1/2,\ \mathbf{a}:\ 0 < a(1) < a(2) < \cdots < a(n) \uparrow \infty,\ r \in (0, a(1)]$, and ${}_2F_1$ is the Gauß hypergeometric function. The result precizes the integral expression for the generalized Qi type Mathieu $\mathbf{a}$-series $S(r,p,\mathbf{a}) = \sum_{n=0}^{\infty} a(n)(a(n)+r^2)^{-p-1}$ given in [1, (4.5)] generalizing some other results by Cerone and Lenard, Tomovski and Qi as well. Bounding inequalities are given for $\mathfrak{S}(r,\mu,\nu,\mathbf{a})$ using the derived integral expression.

1. Introduction

In this article we consider the series

$$\mathfrak{S}(r,\mu,\nu,\mathbf{a}) = \sum_{n=1}^{\infty} \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}\,,\frac{\nu-\mu}{2}+1\,;\nu+1\,;-\frac{r^2}{a(n)^2}\right)}{a(n)^{\nu-\mu+1}(a(n)^2+r^2)^{\mu-1/2}}, \tag{1}$$

where ${}_2F_1$ denotes the Gauß hypergeometric function and $(\mu-1/2, \nu+1/2) \in \mathbb{R}_+^2$, while the real sequence $\mathbf{a}:\ 0 < a(1) < a(2) < \cdots < a(n) \uparrow \infty$ and $r \in (0, a(1)]$. The special case of (1) is the Qi type Mathieu $\mathbf{a}$ - series

$$S(r,p,\mathbf{a}) = \sum_{n=1}^{\infty} \frac{a(n)}{(a(n)+r^2)^{p+1}}, \qquad r, p+1 > 0 \tag{2}$$

treated differently by the author in [6]. Indeed, it is $\mathfrak{S}(r, p+3/2, p-1/2, \mathbf{a}) = S(r,p,\mathbf{a})$, since ${}_2F_1(-1/2, 0; p+1/2; -(r/a(n))^2) \equiv 1$.

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In this article we give integral expression for $\mathfrak{S}(r,\mu,\nu,\mathbf{a})$. Our main tools are the closed (integral) form expression for a Dirichlet's series $\sum_{n=1}^\infty \exp\{-a(n)x\},\ x>0$ and $\mathbf{a}$ is positive, monotonous increasing divergent with $\lim_{n\to\infty} a(n) = \infty$. The sources for this are the suitable chapters of classical books [3, 4].

In the sequel $[u]$ denotes the integral part of u; $\mathcal{L}_s\{f\}, \mathcal{M}_z\{f\}$ stand for Laplace and Mellin transform of f and $J_\nu(x)$ is the $\nu^{\rm th}$ order Bessel function of first kind. Finally, $\mathbb{I}_B(t)$ denotes the characteristic function of the set B.

2. Integral representation of $\mathfrak{S}(r,\mu,\nu,\mathbf{a})$

Consider the formula

$$\int_0^\infty e^{-\alpha x}x^{\mu-1}J_\nu(\beta x)dx = \frac{\beta^\nu\Gamma(\nu+\mu)\,{}_2F_1\left(\frac{\nu-\mu+1}{2}\,,\frac{\nu-\mu}{2}+1\,;\nu+1\,;-\frac{\beta^2}{\alpha^2}\right)}{2^\nu\alpha^{\nu-\mu+1}\Gamma(\nu+1)(\alpha^2+\beta^2)^{\mu-1/2}}\,, \tag{3}$$

valid for $|\alpha|\ge|\beta|, \nu+\mu>0, \nu+1>0$, such that is listed (in equivalent form) as [2, **6.621 W421(3)**], then specify $\alpha\equiv a(n), \beta\equiv r$. Finally, the summation of (3) with respect to $n\in\mathbb{N}$ results in

$$\int_0^\infty \left(\sum_{n=1}^\infty e^{-a(n)x}\right)x^{\mu-1}J_\nu(rx)dx$$
$$= \left(\frac{r}{2}\right)^\nu\frac{\Gamma(\nu+\mu)}{\Gamma(\nu+1)}\sum_{n=1}^\infty \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}\,,\frac{\nu-\mu}{2}+1\,;\nu+1\,;-\frac{r^2}{a(n)^2}\right)}{a(n)^{\nu-\mu+1}(a(n)^2+r^2)^{\mu-1/2}}\,. \tag{4}$$

Denoting the series in (4) by $\mathfrak{S}(r,\mu,\nu,\mathbf{a})$ we finish the first part of our exposition.

The Dirichlet series in the integrand of (4) possesses Laplace integral form

$$\sum_{n=1}^\infty e^{-a(n)x} = x\int_0^\infty e^{-xt}A(t)dt, \tag{5}$$

where the so-called *counting function* $A(t)$ is given as

$$A(t) = \sum_{a(n)\le t} 1 = [a^{-1}(t)], \tag{6}$$

see [3, **IV.**], [4, Part **C**, **I.1**, **V.5**] or [6, §1,2]. Here a^{-1} denotes the inverse of the function $a(x), x\in\mathbb{R}_+$, whose restriction $\mathbf{a}$ is mentioned in $\mathfrak{S}(r,\mu,\nu,\mathbf{a})$. Therefore, replacing (6) into (5) we deduce the following result.

Theorem 1. *Let $\mu > 1/2, \nu + 1/2 > 0$ and let $\mathbf{a}$ be the restriction of the monotonous increasing function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+,\ a(0) = 0$ to $\mathbb{N}_0$, such that*

$$0 < a(1) < a(2) < \cdots < a(n) \uparrow \infty \quad as \quad n \to \infty\ ,$$

Then it holds true

$$\mathfrak{S}(r, \mu, \nu, \mathbf{a}) = \left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} \int_0^\infty \int_{a(1)}^\infty e^{-xt} x^{\mu} J_\nu(rx)[a^{-1}(t)]dxdt. \tag{7}$$

for all $0 < r \le a(1)$.

Proof. We remark that a has the unique inverse being a^{-1} monotonous. So, by the same reason $[a^{-1}(t)] = 0$ for all $a(0) = 0 \le a^{-1}(t) < 1$, i.e. for $0 \le t < a(1)$. □

Bearing on mind the Laplace and Mellin transform notations, the integral representation (7) becomes

$$\mathfrak{S}(r, \mu, \nu, \mathbf{a}) = \left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} \mathcal{M}_{\mu+1}\{\mathcal{L}_x\{[a^{-1}(t)]\} J_\nu(rx)\}. \tag{8}$$

We will consider the case $\mathfrak{S}(r, p+3/2, p-1/2, \mathbf{a})$ separately beacuse it represents the Qi type Mathieu $\mathbf{a}$ - series $S(r, p, \mathbf{a})$. We remark that S is discussed from different points of view in [6, 7, 8].

Corollary 1.1. *For all $r > 0, p+1 > 0$ and $\mathbf{a}$ being the same as in* Theorem 1*, we have*

$$S(r, p, \mathbf{a}) = \frac{2\sqrt{\pi}}{(2r)^{p-1/2}\Gamma(p+1)} \int_0^\infty \int_{a(1)}^\infty e^{-xt} x^{p+3/2} J_{p-1/2}(rx)[a^{-1}(t)]dxdt. \tag{9}$$

Proof. The parametrization by p gives $\mu + 2 = \nu$, so (3) becomes

$$\int_0^\infty e^{-\alpha x} x^{\nu+1} J_\nu(\beta x) dx = \frac{2\alpha(2\beta)^\nu \Gamma(\nu+3/2)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+3/2}}\ ,$$

valid for all $\nu + 1 > 0, \Re(\alpha) > |\Im(\beta)|$, see [2, **6.623 W422(6)**]. Having on mind the connection between $\mathfrak{S}$ and S, we see that the restriction $r \in (0, a(1)]$ necessary to the convergence of ${}_2F_1$ becomes $r \in \mathbb{R}_+$. Easy repretition of the procedure given in hte proof of the Theorem1 gives us (9). □

It is not hard to see that (9) is the desired closed integral expression for the relation [1, (4.5)] by Cerone and Lenard. In the same time

$$F(\mathbf{a}) = \sum_{n=1}^{\infty} e^{-a(n)x} = x \int_{a(1)}^{\infty} e^{-xt}[a^{-1}(t)]dt, \qquad x > 0, \tag{5 \& 6}$$

is an answer to the question of closed form expression for the series $F(\mathbf{a})$, posed as open problem in the same article, for monotonous $\mathbf{a}$.

Specified values of the parameters r, μ, ν and assumptions upon the behaviour of the sequence $\mathbf{a}$ cover certain questions, posed as open problems by Feng Qi, and by Tomovski, compare [5, 7, 8].

Theorem 2. *Let the situation be the same as in* Theorem 1. *Then we have the sharp inequality*

$$0 \le \left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} M(r,\mu,\mathbf{a}) - \mathfrak{S}(r,\mu,\nu,\mathbf{a}) < \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}\,, \frac{\nu-\mu}{2}+1\,; \nu+1\,; -\frac{r^2}{a(1)^2}\right)}{a(1)^{\nu-\mu+1}(a(1)^2+r^2)^{\mu-1/2}}, \tag{10}$$

where

$$M(r,\mu,\mathbf{a}) = \mathcal{M}_{\mu+1}\{\mathcal{L}_x\{a^{-1}(t)\mathbb{I}_{[a(1),\infty)}(t)\}J_\nu(rx)\}.$$

Proof. Applying the well-known (sharp) bilateral inequality

$$a^{-1}(t) - 1 < [a^{-1}(t)] \le a^{-1}(t), \qquad t \in \mathbb{R}_+$$

to the integrand in (7) we clearly deduce the sharp bilateral bounding inequality which lower bound is equal to

$$\begin{aligned} &\left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} \int_0^{\infty} x^{\mu} J_\nu(rx)dx \int_0^{\infty} e^{-xt}(a^{-1}(t)-1)\mathbb{I}_{[a(1),\infty)}(t)dt \\ &\qquad = \left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} \left(M(r,\mu,\mathbf{a}) - \int_0^{\infty} e^{-a(1)x}x^{\mu-1}J_\nu(rx)dx\right). \end{aligned} \tag{11}$$

The integral in (11) we calculate putting $\alpha = a(1), \beta = r$ into (3). So, (11) takes the value

$$\left(\frac{2}{r}\right)^{\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu)} M(r,\mu,\mathbf{a}) - \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}\,, \frac{\nu-\mu}{2}+1\,; \nu+1\,; -\frac{r^2}{a(1)^2}\right)}{a(1)^{\nu-\mu+1}(a(1)^2+r^2)^{\mu-1/2}}\,. \tag{12}$$

The upper bound we get by the same procedure, and it is equal to the first addend in (12). Now, collecting these estimates easy calculations lead to the asserted bilateral bounding inequality (10). □

Finally, it could be mentioned that this results enable to give bilateral bounding inequalities for Qi type Mathieu **a** - series by using suitable values for parameters μ and ν in (10). However, this, and the caused calculations we leave to the interested reader.

References

[1] P. Cerone and C. T. Lenard, On integral forms of generalized Mathieu series, *J. Inequal. Pure Appl. Math.* **4**(5)(2003), Art. 100. Available online at `http://jipam.vu.edu.au/v4n5/100_01.html`.

[2] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products*, 4.Ed. supervised by Yu. V. Geronimus and M. Yu. Tseitlin, Fizmatgiz, Moscow, 1963. (In Russian)

[3] G. H. Hardy and M. Riesz, "The General Theory of Dirichlet's Series", University Press, Cambridge, 1915.

[4] J. Karamata, "Theory and Praxis of Stieltjes Integral", Srpska Akademija Nauka, Posebna izdanja **CLIV**, Matematički institut, Knjiga **I**, Beograd, 1949. (in Serbian)

[5] Ž. Tomovski, New double inequalities for Mathieu type series, *RGMIA Research Report Collection* **6**(2), Art. 17. Available online at `http://rgmia.vu.edu.au/v6n2.html`.

[6] T. K. Pogány, Integral representation of Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series, *RGMIA Research Report Collection* **7**(1) (2004), Art. 9. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[7] F. Qi, Inequalities for Mathieu series, *RGMIA Research Report Collection* **4**(2) (2001), Art. 3, 187-193. Available online at `http://rgmia.vu.edu.au/v4n2.html`.

[8] F. Qi, Integral expression and inequalities of Mathieu type series, *RGMIA Research Report Collection* **6**(2) (2003), Art. 10. Available online at `http://rgmia.vu.edu.au/v6n2.html`.

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