Integral Characterizations for Exponential Stability of Semigroups and Evolution Families on Banach Spaces

INTEGRAL CHARACTERIZATIONS FOR EXPONENTIAL STABILITY OF SEMIGROUPS AND EVOLUTION FAMILIES ON BANACH SPACES

C. BUŞE, N.S. BARNETT, P. CERONE, AND S.S. DRAGOMIR

Abstract. Let X be a real or complex Banach space and $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous and exponentially bounded evolution family on X. Let J be a non-negative functional on the positive cone of the space of all real-valued locally bounded functions on $\mathbb{R}_+ := [0,\infty)$. We suppose that J satisfies some extra-assumptions. Then the family $\mathcal{U}$ is uniformly exponentially stable provided that for every $x \in X$ we have:

$$\sup_{s\geq 0} J(||U(s+\cdot,s)x||) < \infty.$$

This result is connected to the uniform asymptotic stability of the well-posed linear and non-autonomous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= A(t)u(t), \quad t \geq s \geq 0, \\ u(s) &= x \quad x \in X. \end{cases}$$

In the autonomous case, i.e. when $U(t,s) = T(t-s)$ for some strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ we obtain the well-known theorems of Datko, Littman, Neerven, Pazy and Rolewicz.

1. Introduction

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norm of vectors in X and operators in $\mathcal{L}(X)$ will be denoted by $||\cdot||$. Let $\mathbf{T} := \{T(t)\}_{t\geq 0}$ be a semigroup of operators acting on X, that is, $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$, $T(0) = I$ the identity operator in $\mathcal{L}(X)$ and $T(t+s) = T(t) \circ T(s)$ for every $t \geq 0$ and $s \geq 0$. The semigroup $\mathbf{T}$ is called strongly continuous if for each $x \in X$ the map $t \mapsto T(t)x : [0,\infty) \to X$ is continuous. Every strongly continuous semigroup is locally bounded, that is, there exist $h > 0$ and $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \in [0,h]$. It is easy to see that every locally bounded semigroup is exponentially bounded, that is, there exist $\omega \in \mathbb{R}_+$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

It is well-known that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X and there exists $p \in [1,\infty)$ such that for each $x \in X$ one has

$$\int_0^\infty ||T(t)x||^p dt = M(p,x) < \infty, \tag{1.1}$$

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then $\mathbf{T}$ is exponentially stable, that is, its uniform growth bound

$$\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln ||T(t)||}{t},$$

is negative. This result is usually referred to as the Datko-Pazy theorem, see [6, 12]. An important application of the Datko-Pazy theorem can be found in [16]. A quantitative version of this theorem states that if $M(p,x)$ from (1.1) is equal to $C||x||^p$, where C is some positive constant, then $\omega_0(\mathbf{T}) < -\frac{1}{pC}$. See [10] Theorem 3.1.8 for details. An important generalization of the Datko-Pazy theorem was given by S. Rolewicz, [13]. In the autonomous case the Rolewicz theorem reads as follows. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If there exists a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and if*

$$\int_0^\infty \phi(||T(t)x||)dt := M_\phi(x) < \infty \text{ for each } x \in X, \tag{1.2}$$

then the semigroup $\mathbf{T}$ is exponentially stable. The same result was obtained independently by Littman [8]. In particular, from Rolewicz's theorem it follows that the Datko-Pazy theorem remains valid for $p \in (0,1)$. The condition (1.1) indicates that for each $x \in X$ the map $t \mapsto T(t)x$ belongs to $L^p(\mathbb{R}_+)$. Jan van Neerven has shown in [9] that a strongly continuous semigroup $\mathbf{T}$ on X is uniformly exponentially stable if there exists a Banach function space over $\mathbb{R}_+ := [0,\infty)$ with the property that

$$\lim_{t\to\infty} ||1_{[0,t]}||_E = \infty, \tag{1.3}$$

such that

$$||T(\cdot)x|| \in E \text{ for every } x \in X. \tag{1.4}$$

He has also shown that the autonomous variant of the Rolewicz theorem can be derived from his result by taking for E a suitable Orlicz space over $\mathbb{R}_+$. In another paper, [11], Jan van Neerven has come to the same conclusion by replacing either (1.1), (1.2) or (1.4) by the hypothesis that the set of all $x \in X$ for which the following inequality holds

$$J(||T(\cdot)x||) < \infty,$$

is of the second category in X. Here J is a certain lower semi-continuous functional as defined in Theorem 2 from [11]. The proof of this latter result is based on a non-trivial result from operator theory given by V. Müler, see Lemma 1 from [11], for further details. We give here a surprisingly simple proof for a result of the same type, moreover, we do not require the lower semi-continuity of J.

In order to introduce some non-autonomous results of this type we recall the notion of an evolution family.

A family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators on a Banach space X is a strongly continuous evolution family if

1. $U(t,t) = I$ and $U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$.
2. The map $t \mapsto U(t,s)x : [s,\infty) \to X$ is continuous for every $s \geq 0$ and every $x \in X$.

The family $\mathcal{U}$ is exponentially bounded if there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$||U(t,s)|| \leq M_\omega e^{\omega(t-s)} \text{ for } t \geq s \geq 0. \tag{1.5}$$

Then $\omega(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \text{there is } M_\omega \geq 0 \text{ such that (1.5) holds}\}$ is called the growth bound of $\mathcal{U}$. The family $\mathcal{U}$ is uniformly exponentially stable if its growth bound is negative.

In [1] it is proved that an exponentially bounded evolution family $\mathcal{U}$ is uniformly exponentially stable if there exists a solid space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $||U(s+\cdot, s)x||$ belongs to E and

$$\sup_{s\geq 0} |||U(s+\cdot, s)x|| := K(x) < \infty.$$

The non-autonomous Datko theorem, [7], follows from this by taking $E = L^p(\mathbb{R}_+)$. The theorem of Rolewicz, [14], can be derived as well by taking for E a suitable Orlicz space over $\mathbb{R}_+$, see Theorem 2.10 from [1]. New guidelines about the proof of the Datko theorem can be found in [5] and [15]. In this paper we propose a more natural generalization of the theorems of Datko and Rolewicz which can also be extended to the general non-autonomous case. For some recently obtained autonomous or periodic versions of the above; see [4], [11].

2. A Generalization of the Datko-Pazy Theorem

We begin by stating and proving two lemmas which are useful later.

Lemma 1. *Let $\mathbf{T} = \{T(t) : t \geq 0\}$ be a locally bounded semigroup on a Banach space X. If for each $x \in X$ there exists $t(x) > 0$ such that $T(t(x))x = 0$, then $\mathbf{T}$ is uniformly exponentially stable.*

Proof. It is easy to see that $\mathbf{T}$ is uniformly bounded. Indeed, if not, then there exists a sequence (t_n) of positive real numbers with $t_n \to \infty$ such that $||T(t_n)|| \to \infty$. By the Uniform Boundedness Theorem it follows that there exists $x \in X$ such that $||T(t_n)x|| \to \infty$. This is in contradiction to the hypothesis. Now let $\nu > 0$. The semigroup $\{e^{\nu t}T(t)\}$ verifies the hypothesis of the present Lemma and it is uniformly bounded. Finally, we deduce that $\mathbf{T}$ is uniformly exponentially stable. ∎

Lemma 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a locally bounded semigroup such that for each $x \in X$ the map $t \mapsto ||T(t)x||$ is continuous on $(0, \infty)$. If there exist a positive h and $0 < q < 1$ such that for all $x \in X$ there exists $t(x) \in (0, h]$ with*

$$||T(t(x))x|| \leq q||x||, \tag{2.1}$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Proof. Let $x \in X$ be fixed and $t_1 \in (0, h]$ such that $||T(t_1)x|| \leq q||x||$, then there exists $t_2 \in (0, h]$ such that

$$||T(t_2 + t_1)x|| \leq q||T(t_1)x|| \leq q^2||x||.$$

By mathematical induction it is easy to see that there exists a sequence (t_n), with $0 < t_n \leq h$ such that $||T(s_n)x|| \leq q^n||x||$, where $s_n := t_1 + t_2 + \cdots + t_n$.

If $s_n \to \infty$, then for each $t \in [s_n, s_{n+1}]$ we have that $t < (n + 1)h$ and

$$||T(t)x|| \leq Mq^n||x|| \leq Me^{-\ln(q)}e^{\frac{\ln(q)}{T}t}||x||,$$

that is, $\mathbf{T}$ is exponentially stable.

If the sequence (s_n) is bounded, let $t(x)$ be the limit of (s_n). By the assumption of continuity it follows that $T(t(x)) = 0$ and then application of Lemma 1 completes the proof. ∎

We can now state the main result of this section.

Theorem 1. *Let $\mathcal{M}_{loc}([0,\infty))$ be the space of all real valued locally bounded functions on $\mathbb{R}_+ = [0,\infty)$ endowed with the topology of uniform convergence on bounded sets and $\mathcal{M}_{loc}^{+}(\mathbb{R}_+)$ its positive cone.*

Let $J : \mathcal{M}_{loc}^{+}(\mathbb{R}_+) \to [0,\infty]$ be a map with the following properties:

1. *J is nondecreasing.*

2. *For each positive real number ρ,*

$$\lim_{t\to\infty} J(\rho \cdot 1_{[0,t]}) = \infty.$$

If $\mathbf{T}$ is a semigroup on a Banach space X as in Lemma 2 such that

$$\sup_{||x||\leq 1} J(||T(\cdot)x||) := K_J < \infty, \tag{2.2}$$

then $\mathbf{T}$ is exponentially stable.

Proof. Suppose that $\mathbf{T}$ is not exponentially stable. For all $h > 0$ and all $0 < q < 1$ then there exists $x_0 \in X$ of norm one such that

$$||T(t)x_0|| > q \text{ for every } t \in [0,h],$$

as proved in Lemma 2. It follows then that

$$K_J \geq J(||T(\cdot)x_0||) \geq J(q \cdot 1_{[0,h]})$$

which contradicts (2.2). ∎

Corollary 1. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a semigroup on a Banach space X as in Lemma 2 and $1 \leq p < \infty$. If (1.1) holds for all $x \in X$ then the semigroup $\mathbf{T}$ is exponentially stable.*

Proof. For each fixed positive h consider the bounded linear operator

$$x \mapsto T_h x : X \to L^p(\mathbb{R}_+, X)$$

defined by

$$(T_h x)(t) = \begin{cases} T(t)x, & \text{if } \quad 0 \leq t \leq h \\ 0, & \text{if } \quad t > h. \end{cases}$$

For each $x \in X$ we have:

$$||T_h x||_{L^p(\mathbb{R}_+,X)} = \left(\int_0^h ||T(t)x||^p dt\right)^{\frac{1}{p}} \leq M(p,x)^{\frac{1}{p}}.$$

From the Uniform Boundedness Theorem it follows that there exists a positive constant C_p such that

$$||T_h x||_{L^p(\mathbb{R}_+,X)} \leq C_p ||x|| \text{ for every } x \in X.$$

Now it is easy to derive the inequality

$$\sup_{||x||\leq 1} \int_0^\infty ||T(t)x||^p dt \leq K_p < \infty,$$

where K_p is a positive constant. Choose $J(f) := \int_0^\infty f(t)^p dt$, apply Theorem 1 and the proof is complete. ∎

Corollary 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a semigroup on a Banach space X as in the above Lemma 2. If there exists a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and (1.2) holds then the semigroup $\mathbf{T}$ is exponentially stable.*

Proof. Seemingly we could proceed as in the proof of Corollary 1, but, however, we cannot directly apply the Uniform Boundedness Theorem. First we prove that the semigroup $\mathbf{T}$ is uniformly bounded. In fact, this has been done in [2] in the general framework of the evolution families. For the sake of completeness we mention some steps of that proof for this particular case. We may assume that $\phi(0) = 0$, $\phi(1) = 1$ and that ϕ is strictly increasing on $\mathbb{R}_+$, if not, we replace ϕ by some multiple of the function

$$t \mapsto \bar{\phi}(t) := \begin{cases} \int_0^t \phi(u)du, & \text{if} \quad 0 \leq t \leq 1 \\ \frac{at}{at+1-a}, & \text{if} \quad t > 1, \end{cases}$$

where $a := \int_0^1 \phi(u)du$.

Let $x \in X$ be fixed, N be a positive integer such that $M_\phi(x) < N$ and let $t \geq N$. For each $\tau \in [t-N, t]$ and all $u \geq 0$ we have:

$$e^{-\omega N} 1_{[t-N,t]}(u)||T(t)x|| \leq e^{-\omega(t-\tau)} 1_{[t-N,t]}(u)||T(t-\tau)T(\tau)x|| \leq M||T(u)x||$$

and then

$$N\phi\left(\frac{||T(t)x||}{Me^{\omega N}}\right) \leq \int_{t-N}^t \phi\left(\frac{||T(t)x||}{Me^{\omega N}}\right) du \leq M_\phi(x).$$

Hence $||T(t)x|| \leq Me^{\omega N} M_\phi(x)$ for every $t \geq N$, and so the semigroup $\mathbf{T}$ is uniformly bounded.

From [11] Lemma 3.2.1 it follows that there exists an Orlicz's space E satisfying (1.3) such that for each $x \in X$ which satisfies (1.2), the map $t \mapsto T(t)x$ belongs to E. For each non-negative, bounded and measurable real-valued function f we put $J(f) := \sup\limits_{t \geq 0} |1_{[0,t]} f|_E$, giving,

$$J(||T(\cdot)x||) = \sup_{t \geq 0} |1_{[0,t]}|||T(\cdot)x|||_E \leq |||T(\cdot)x|||_E < \infty,$$

for every $x \in X$.

Arguing as in Corollary 1 it follows that there exists a positive constant K_ϕ, independent of x, such that

$$\sup_{||x|| \leq 1} J(||T(\cdot)x||) < K_\phi < \infty.$$

Application of Theorem 1 completes the proof. ∎

3. The Non-autonomous Case

We state and prove two lemmas that will be used in the sequel.

Lemma 3. *Let $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be an exponentially bounded evolution family on a Banach space X. If for each $x \in X$ there exists $t(x) > 0$ such that $U(s+t(x), s)x = 0$ for every $s \geq 0$ then the family $\mathcal{U}$ is uniformly exponentially stable.*

Proof. First we prove that there exists $M > 0$ such that

$$\sup_{s \geq 0} ||U(s+t, s)|| \leq M \text{ for all } t \geq 0.$$

Indeed, if we suppose the contrary then there exists a sequence (t_n) of positive real numbers with $t_n \to \infty$ such that $\lim_{n\to\infty} ||U(s+t_n, s)|| = \infty$. From the Uniform Boundedness Theorem it follows that there exists $x \in X$ such that $||U(s+t_n, s)x|| \to \infty$ when $n \to \infty$ which is in contradiction to the hypothesis. We now observe that the family $\{e^{\nu(t-s)}U(t,s)\}_{t\geq s\geq 0}$ verifies the hypothesis of the present lemma and then

$$||U(t,s)|| \leq Me^{-\nu(t-s)} \text{ for all } t \geq s,$$

i.e. the assertion holds. ∎

Lemma 4. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $y \in X$ and each $s \geq 0$ the map*

$$t \mapsto ||U(s+t,s)y|| : \mathbb{R}_+ \to \mathbb{R}_+$$

is continuous on $(0,\infty)$. If there exist positive real numbers h and $q < 1$ such that for every $x \in X$ there exists $t(x) \in (0,h]$ with the property that

$$\sup_{s\geq 0} ||U(s+t(x),s)x|| \leq q||x||,$$

then the family $\mathcal{U}$ is exponentially stable.

Proof. Is similar to that of Lemma 2 and so we omit the details. ∎

Theorem 2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on a Banach space X as in the above Lemma 4 and let J be a functional as in Theorem 1. If there exists $r > 0$ such that*

$$\sup_{s\geq 0} \sup_{||x||\leq r} J(||U(s+\cdot,s)x||) := L(J,r) < \infty, \tag{3.1}$$

then the evolution family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Suppose that the family $\mathcal{U}$ is not uniformly exponentially stable. Under such circumstances as proved in Lemma 4, for every positive real number h and every $q \in (0,1)$ there exist $x_0 \in X$ of norm one and $s_0 \geq 0$ such that

$$||U(s_0+t,s_0)x_0|| > q \text{ for all } t \in [0,h].$$

Thus

$$L(J,r) \geq J(||U(s_0+t,s_0)rx_0||) \geq J(rq \cdot 1_{[0,h]})$$

for each $h > 0$, which contradicts (3.1). ∎

Theorem 3. *Let J be as in the above Theorem 1. We suppose, in addition, that J is lower semi-continuous and convex in the sense of Jensen (or sub-additive, that is, $J(f+g) \leq J(f) + J(g)$ for every f and g in $\mathcal{M}_{loc}(\mathbb{R}_+)$). Let $\mathcal{U}$ be an evolution family as in the Lemma 4. If the set $\mathcal{X}$ of all $x \in X$ for which*

$$\sup_{s\geq 0} J(||U(s+\cdot,s)x||) < \infty$$

is of the second category in X, then the family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Let $s \geq 0$, be fixed. The map $x \mapsto ||U(s+\cdot,s)x|| : X \to \mathcal{M}_{loc}(\mathbb{R}_+)$ is continuous. As a consequence, the map

$$x \mapsto \Phi_s(x) := J(||U(s+\cdot,s)x||) : X \to [0,\infty]$$

is lower semi-continuous as well. For each positive integer k, the set

$$X_k(s) := \{x \in X : J(||U(s+\cdot,s)x||) \leq k\}$$

is closed, because it is the reverse image of the real closed interval $[0, k]$ by the map Φ_s. It is clear that the set

$$X_k := \left\{ x \in X : \sup_{s \geq 0} J(||U(s+\cdot, s)x||) \leq k \right\} = \cap_{s\geq 0} X_k(s)$$

is also closed and moreover that $\mathcal{X}$ is the union of all sets X_k. Because $\mathcal{X}$ is of the second category in X, there exists a set X_{k_0} whose interior is non empty. Let $x_0 \in X$ and $r_0 > 0$ such that $B(x_0, r_0)$ belongs to X_{k_0}. It is easy to see that $B\left(0, \frac{1}{2}r_0\right)$ belongs to X_{k_0}, that is,

$$\sup_{s\geq 0} \sup_{||x|| \leq \frac{1}{2}r_0} J(||U(s+\cdot, s)x||) \leq k_0.$$

Indeed for every $x \in X$ with $||x|| \leq r_0$ we have:

$$\begin{aligned} J\left(\left\| U(s+\cdot, s)\left(\frac{1}{2}x\right)\right\|\right) &= J\left(\frac{1}{2}||U(s+\cdot, s)[(x+x_0) - x_0]||\right) \\ &\leq J\left(\frac{1}{2}\left[||U(s+\cdot, s)(x+x_0) + ||U(s+\cdot, s)x_0||\right]\right) \\ &\leq \frac{1}{2}J(||U(s+\cdot, s)(x+x_0)||) + \frac{1}{2}J(||U(s+\cdot, s)x_0||) \\ &\leq k_0. \end{aligned}$$

Application of Theorem 2 completes the proof. ∎

Corollary 3. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $x \in X$ the map $t \mapsto ||U(s+t, s)x||$ is continuous on $(0, \infty)$ for every $s \geq 0$. Consider the following three inequalities:*

1. *There exists $p \in [1, \infty)$ such that*

$$\sup_{s\geq 0} \int_0^\infty ||U(s+t, s)x||^p dt < \infty$$

for every $x \in X$.

2. *There exists a Banach function space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $U(s+\cdot, s)x$ belongs to E and for every $x \in X$ we have*

$$\sup_{s\geq 0} |||U(s+\cdot, s)x|||_E < \infty.$$

3. *There exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for each $t > 0$ such that*

$$\sup_{s\geq o} \int_0^\infty \phi(||U(s+t, s)x||)dt < \infty$$

for every $x \in X$.

If any one of these statements is true then the family $\mathcal{U}$ is exponentially stable.

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