



VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Density of n th-Power Frees

This is the Published version of the following publication

Razvan, Mohammad Reza, Hassani, Mehdi and Pirayesh, Gholam Ali (2004)
Density of n th-Power Frees. Research report collection, 7 (4).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/18058/>

DENSITY OF n^{th} -POWER FREES

Mohammad Reza Razvan,
Mehdi Hassani, Gholam Ali Pirayesh

Department of Mathematics, Institute for Advanced Studies in Basic Sciences

P.O. Box 45195-159

Zanjan, Iran.

<razvan, mhassani, pirayesh>@iasbs.ac.ir

Abstract

In this note we are going to analyze the density of n^{th} -power free integers.

1. Introduction

Let \mathbb{P} the set of all primes and suppose M is a positive integer, with the following prime factoring:

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad (p_1, p_2, \dots, p_k \in \mathbb{P}).$$

We call M , n^{th} -power free if for $1 \leq i \leq k$, $\alpha_i < n$. Let $f_n(x)$ = The number of n^{th} -power frees $\leq x$. By density we mean

$$\lim_{x \rightarrow \infty} \frac{f_n(x)}{x}.$$

It is well-know that [1],

$$f_2(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

So, the density of square frees is $\frac{6}{\pi^2}$ or approximately 61 percent!. Now, what about *cubic frees*? And generally the n^{th} -power frees?

2. Density Analysis

In this section we will show that the density of n^{th} -power free integers is $\frac{1}{\zeta(n)}$. Our main result is based on the following lemma.

Lemma 1 *Let $s > 1$ be a real number. We have*

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} = \frac{1}{\zeta(s)}.$$

PROOF:

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} = \sum_{k=1}^{\infty} \frac{(-1)^k}{p_1^s p_2^s \cdots p_k^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right) = \prod_{p \in \mathbb{P}} \frac{1}{\sum_{k=1}^{\infty} \frac{1}{p^{sk}}} = \frac{1}{\sum_{m=1}^{\infty} \frac{1}{m^s}} = \frac{1}{\zeta(s)}.$$

Theorem 1 For any integer $n \geq 2$ and any real $x \geq 1$, we have

$$\left| \frac{x}{\zeta(n)} - f_n(x) \right| < \frac{n}{n-1} \sqrt[n]{x} - 1. \quad (1)$$

PROOF: By a usual counting, we obtain

$$f_n(x) = x - \sum_{p \in \mathbb{P}} \lfloor \frac{x}{p^n} \rfloor + \sum_{p, q \in \mathbb{P}, p \neq q} \lfloor \frac{x}{(pq)^n} \rfloor - \cdots = \sum_{k \leq \sqrt[n]{x}} \mu(k) \lfloor \frac{x}{k^n} \rfloor.$$

So, we have

$$\begin{aligned} \left| \frac{x}{\zeta(n)} - f_n(x) \right| &= \left| \sum_{1 < k \leq \sqrt[n]{x}} \mu(k) \left(\frac{x}{k^n} - \lfloor \frac{x}{k^n} \rfloor \right) + \sum_{k > \sqrt[n]{x}} \mu(k) \frac{x}{k^n} \right| \\ &< (\sqrt[n]{x} - 1) + x \sum_{k > \sqrt[n]{x}} \frac{1}{k^n} < \sqrt[n]{x} - 1 + x \int_{\sqrt[n]{x}}^{\infty} \frac{ds}{s^n} = \frac{n}{n-1} \sqrt[n]{x} - 1. \end{aligned}$$

This completes the proof.

A weak but nice form of the above theorem is

Corollary 1 For any integer $n \geq 2$ and any real

$$f_n(x) = \frac{x}{\zeta(n)} + O(\sqrt[n]{x}).$$

Corollary 2 For any integer $n \geq 2$, the density of n^{th} -power free integers is

$$\frac{1}{\zeta(n)}.$$

According the definition of $f_n(x)$ we obtain $0 \leq \frac{f_n(x)}{x} < 1$. We desire to find better lower bounds:

Lemma 2 Let $n \geq 2$ is an integer. For any real $0 \leq \alpha < \frac{1}{\zeta(n)}$ and any real $x > \left(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))} \right)^{\frac{n}{n-1}}$, we have

$$\alpha < \frac{f_n(x)}{x}.$$

PROOF: From (1), we have

$$\frac{1}{\zeta(n)} - \frac{n}{(n-1)x^{1-\frac{1}{n}}} < \frac{f_n(x)}{x}.$$

Let $LB(n, x)$ denote the left hand side of the above inequality. If $0 \leq \alpha < \frac{1}{\zeta(n)}$ and $x > \left(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))} \right)^{\frac{n}{n-1}}$, then $\alpha < LB(n, x)$. This completes the proof.

The obtained results are useful in study of distribution of n^{th} -power free integers. In the next section we do this.

3. Computational Results

The sequence $\frac{f_n(x)}{x}$ for any fixed n is convergent, and it may affairs minimum for some $x \in \mathbb{N}$. The lemma 2 led us to the following algorithm to find the minimum of $\frac{f_n(x)}{x}$ on \mathbb{N} .

Step(1). Find $x_0 \in \mathbb{N}$ such that

$$\frac{f_n(x_0)}{x_0} < \frac{1}{\zeta(n)}.$$

Step(2). For $\alpha = \frac{f_n(x_0)}{x_0}$, take

$$N = \left\lfloor \left(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))} \right)^{\frac{n}{n-1}} \right\rfloor.$$

Step(3). Find

$$\min_{1 \leq x \leq N} \left\{ \frac{f_n(x)}{x} \right\}.$$

We note that there is no guarantee for the existence of x_0 in **step(1)**, but there are some evidences for the following question.

Question Is there exists an x_0 in the interval $[5^n, 6^n]$ with $\frac{f_n(x_0)}{x_0} < \frac{1}{\zeta(n)}$?

Our computer program gave an affirmative answer to above question for $n = 2, \dots, 10$. It is based on the following recursive relation:

$$f_n(x) = f_n(x-1) + \begin{cases} 1 & x \text{ is } n^{\text{th}}\text{-power free} \\ 0 & x \text{ other wise} \end{cases}$$

Since $f_n(2^n) = 2^n - 1$, we start from $x = 2^n$. Then divide our interval into sub intervals $[2^n, 3^n], [3^n, 5^n], [5^n, 7^n], \dots$. The following table includes the value of x_0 , exact value of the minimum of $\frac{f_n(x)}{x}$ and the value of x at which the minimum occur.

n	x_0	N	x	$f_n(x)$	$\min_{x \in \mathbb{N}} \frac{f_n(x)}{x}$
2	28	6503647	176	106	0.602273
3	136	55980	378	314	0.830688
4	656	171931	2512	2320	0.923567
5	3168	269627	3168	3055	0.964331
6	16064	1346593	31360	30825	0.982940
7	78732	10552627	236288	234331	0.991718
8	393728	25381201	1174528	1169758	0.995939
9	1968640	146390429	7814151	7798488	0.997996
10	9802752	816756521	48833536	48785015	0.999006

Now, we use our numerical results to get the following corollaries.

Corollary 3 *Let $n > 1$ is an integer. The number of cases that we can write n as sum of two square frees is greater than $\frac{n}{10}$.*

PROOF: More than 60 percent of integers between 1 and n are square free. The number of pairs $\{i, j\}$ such that $i + j = n$ is not greater than $\frac{n}{2}$, so, there are more than $\frac{n}{10}$ of this pairs with square free members. This complete the proof.

Corollary 4 *The probabilty that two successive positive integer numbers both be square free is more than 20%.*

PROOF: Obvious.

References

[H-W] G. H. HARDY AND E. M. WRIGHT, An Introduction to THE THEORY OF NUMBERS, fifth edition, Oxford University Press, London, 1979.