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APPROXIMATION OF $\pi(x)$ BY $\Psi(x)$

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ABSTRACT. In this paper we find some lower and upper bounds of the form $\frac{n}{H_n - c}$ for the function $\pi(n)$, in which $H_n = \sum_{k=1}^n \frac{1}{k}$. Then, considering $H(x) = \Psi(x+1) + \gamma$ as generalization of H_n , in which $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ and γ is Euler constant, we find some lower and upper bounds of the form $\frac{x}{\Psi(x) - c}$ for the function $\pi(x)$.

1. INTRODUCTION

As usual, let \mathbb{P} be the set of all primes and $\pi(x) = \#\mathbb{P} \cap [2, x]$. If $H_n = \sum_{k=1}^n \frac{1}{k}$, then easily we have

$$(1.1) \quad \gamma + \log n < H_n < 1 + \log n \quad (n > 1),$$

in which γ is Euler constant. So, $H_n = \log n + O(1)$ and considering the prime number theorem [1], we obtain

$$\pi(n) = \frac{n}{H_n + O(1)} + o\left(\frac{n}{\log n}\right).$$

Thus, comparing $\frac{n}{H_n + O(1)}$ with $\pi(n)$ seems to be a nice problem. In 1959, L. Locker-Ernst [3] affirms that $\frac{n}{H_n - \frac{1}{2}}$ is very close to $\pi(n)$ and in 1999, L. Panaitopol [5] proved that for $n \geq 1429$ it is actually a lower bound for $\pi(n)$.

In this paper we improve Panaitopol's result by proving $\frac{n}{H_n - a} < \pi(n)$ for every $n \geq 3299$, in which $a \approx 1.546356705$. Also, we find same upper bound for $\pi(n)$. Then we consider generalization of H_n as a real value function which has been studied by J. Sándor [6] in 1988; for $x > 0$ let $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$, in which $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is well-known gamma function. Since $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = -\gamma$, we have $H_n = \Psi(n+1) + \gamma$, and this relation led him to define

$$(1.2) \quad \begin{cases} H : (0, 1) \longrightarrow \mathbb{R}, \\ H(x) = \Psi(x+1) + \gamma, \end{cases}$$

as a natural generalization of H_n , and more naturally, it motivated us to find some bounds for $\pi(x)$ concerning $\Psi(x)$. In our proofs, we use the obvious relation

$$(1.3) \quad \Psi(x+1) = \Psi(x) + \frac{1}{x}.$$

Also, we need some bounds of the form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$, which we yield them by using the following known sharp bounds [2] for $\pi(x)$

$$(1.4) \quad \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x) \quad (x \geq 32299),$$

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and

$$(1.5) \quad \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x} \right) \quad (x \geq 355991).$$

2. BOUNDS OF THE FORM $\frac{x}{\log x - 1 - \frac{c}{\log x}}$

Lower Bounds. We are going to find suitable values of a , in which $\frac{x}{\log x - 1 - \frac{a}{\log x}} \leq \pi(x)$. Considering (1.4) and letting $y = \log x$, we should study the inequality $\frac{1}{y-1-\frac{a}{y}} \leq \frac{1}{y} \left(1 + \frac{1}{y} + \frac{9}{5y^2} \right)$, which is equivalent with $\frac{y^4}{y^2-y-a} \leq y^2 + y + \frac{9}{5}$ and supposing $y^2 - y - a > 0$, it will be equivalent with $(\frac{4}{5} - a)y^2 - (a + \frac{9}{5})y - \frac{9a}{5} \geq 0$ and this force $\frac{4}{5} - a > 0$, or $a < \frac{4}{5}$. Let $a = \frac{4}{5} - \epsilon$ for some $\epsilon > 0$. Therefore we should study $\frac{1}{y-1-\frac{\frac{4}{5}-\epsilon}{y}} \leq \frac{1}{y} \left(1 + \frac{1}{y} + \frac{9}{5y^2} \right)$, which is equivalent with

$$(2.1) \quad \frac{25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36)}{5y^3(5y^2 - 5y + (5\epsilon - 4))} \geq 0.$$

The equation $25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36) = 0$ has discriminant $\Delta_1 = 169 + 14\epsilon - 155\epsilon^2$, which is non-negative for $-1 \leq \epsilon \leq \frac{169}{155}$ and the greater root of it, is $y_1 = \frac{13-5\epsilon+\sqrt{\Delta_1}}{10\epsilon}$. Also, the equation $5y^2 - 5y + (5\epsilon - 4) = 0$ has discriminant $\Delta_2 = 105 - 100\epsilon$, which is non-negative for $\epsilon \leq \frac{21}{20}$ and the greater root of it, is $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{10}$. Thus, (2.1) holds for every $0 < \epsilon \leq \min\{\frac{169}{155}, \frac{21}{20}\} = \frac{21}{20}$, with $y \geq \max_{0 < \epsilon \leq \frac{21}{20}} \{y_1, y_2\} = y_1$. Therefore, we have proved the following theorem:

Theorem 2.1. For every $0 < \epsilon \leq \frac{21}{20}$, the inequality

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5}-\epsilon}{\log x}} \leq \pi(x),$$

holds for all

$$x \geq \max \left\{ 32299, e^{\frac{13-5\epsilon+\sqrt{169+14\epsilon-155\epsilon^2}}{10\epsilon}} \right\}.$$

Corollary 2.2. For every $x \geq 32299$, we have

$$\frac{x}{\log x - 1 + \frac{1}{4\log x}} \leq \pi(x).$$

Proof. Taking $\epsilon = \frac{21}{20}$ in above theorem, we yield the result for $x \geq 32299$. For $3299 \leq x \leq 32298$ check it by computer. \square

Upper Bounds. Similar to lower bounds, we should search suitable values of b in which $\pi(x) \leq \frac{x}{\log x - 1 - \frac{b}{\log x}}$. Considering (1.5) and letting $y = \log x$, we should study $\frac{1}{y} \left(1 + \frac{1}{y} + \frac{251}{100y^2} \right) \leq \frac{1}{y-1-\frac{b}{y}}$. Assuming $y^2 - y - b > 0$, it will be equivalent with $(\frac{151}{100} - b)y^2 - (b + \frac{251}{100})y - \frac{251b}{100} \leq 0$, which force $b \geq \frac{151}{100}$. Let $b = \frac{151}{100} + \epsilon$ for some $\epsilon \geq 0$. Therefore we should study $\frac{1}{y} \left(1 + \frac{1}{y} + \frac{251}{100y^2} \right) \leq \frac{1}{y-1-\frac{\frac{151}{100}+\epsilon}{y}}$, which is equivalent with

$$(2.2) \quad \frac{10000\epsilon y^2 + (10000\epsilon + 40200)y + (25100\epsilon + 37901)}{100y^3(100y^2 - 100y - (100\epsilon + 151))} \geq 0.$$

The quadratic equation in numerator of (2.2) has discriminant $\Delta_1 = 40401 - 17801\epsilon - 22600\epsilon^2$, which is non-negative for $-\frac{40401}{22600} \leq \epsilon \leq 1$ and the greater root of it, is $y_1 = \frac{-201-50\epsilon+\sqrt{\Delta_1}}{100\epsilon}$. Also, the quadratic equation in denominator of it, has discriminant $\Delta_2 = 44 + 25\epsilon$, which is non-negative for $-\frac{44}{25} \leq \epsilon$ and the greater root of it, is $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{5}$. Thus, (2.2) holds for every $0 \leq \epsilon \leq \min\{1, +\infty\} = 1$, with $y \geq \max_{0 \leq \epsilon \leq 1} \{y_1, y_2\} = y_2$. Finally, we note that for $0 \leq \epsilon \leq 1$, $y_2(\epsilon)$ is strictly increasing and so, $6 < e^{\frac{1}{2} + \frac{\sqrt{44}}{5}} = e^{y_1(0)} \leq e^{y_1(\epsilon)} \leq e^{y_1(1)} = e^{\frac{1}{2} + \frac{\sqrt{69}}{5}} < 9$. Therefore, we obtain the following theorem:

Theorem 2.3. *For every $0 \leq \epsilon \leq 1$, we have*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{\frac{151}{100} + \epsilon}{\log x}} \quad (x \geq 355991).$$

Corollary 2.4. *For every $x \geq 7$, we have*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100 \log x}}.$$

Proof. Taking $\epsilon = 0$ in above theorem, we yield the result for $x \geq 355991$. For $7 \leq x \leq 35991$ it has been checked by computer [4]. \square

3. BOUNDS OF THE FORM $\frac{n}{H_n - c}$ AND $\frac{x}{\Psi(x) - c}$

Theorem 3.1. (i) *For every $n \geq 3299$, we have*

$$\frac{n}{H_n - a} < \pi(n),$$

in which $a = \gamma + 1 - \frac{1}{4 \log 3299} \approx 1.546356705$.

(ii) *For every $n \geq 9$, we have*

$$\pi(n) < \frac{n}{H_n - b},$$

in which $b = 2 + \frac{151}{100 \log 7} \approx 2.775986497$.

Proof. For $n \geq 3299$, we have $\gamma + \log n > a + \log n - 1 + \frac{1}{4 \log n}$ and considering this with left hand side of (1.1), we obtain $\frac{n}{H_n - a} < \frac{x}{\log x - 1 + \frac{1}{4 \log x}}$ and this inequality with corollary 2.2, yield the first part of theorem.

For $n \geq 9$, we have $b + \log n - 1 - \frac{151}{100 \log n} \geq 1 + \log n$ and considering this with right hand side of (1.1), we obtain $\frac{x}{\log x - 1 - \frac{151}{100 \log x}} < \frac{n}{H_n - b}$. Considering this, with corollary 2.4, complete the proof. \square

Theorem 3.2. (i) *For every $x \geq 3299$, we have*

$$\frac{x}{\Psi(x) - A} < \pi(x),$$

in which $A = \frac{3298}{3299} - \frac{1}{4 \log 3299} \approx 0.9688379174$.

(ii) *For every $x \geq 9$, we have*

$$\pi(x) < \frac{x}{\Psi(x) - B},$$

in which $B = 2 + \frac{151}{100 \log 7} - \gamma \approx 2.198770832$.

Proof. Let H_x be the step function defined by $H_x = H_n$ for $n \leq x < n + 1$. Considering (1.2) we have $H(x-1) < H_x \leq H(x)$.

For $x \geq 3299$ by considering part (i) of previous theorem, we have

$$\pi(x) > \frac{x}{H_x - a} \geq \frac{x}{H(x) - a} = \frac{x}{\Psi(x+1) + \gamma - a}.$$

Thus, by considering (1.3) we obtain

$$\pi(x) > \frac{x}{\Psi(x) + \frac{1}{x} + \gamma - a} \geq \frac{x}{\Psi(x) + \frac{1}{3299} + \gamma - a} = \frac{x}{\Psi(x) - A},$$

in which $A = a - \gamma - \frac{1}{3299} = \frac{3298}{3299} - \frac{1}{4 \log 3299}$.

For $x \geq 9$ by considering second part of previous theorem, we obtain

$$\pi(x) < \frac{x}{H_x - b} < \frac{x}{H(x-1) - b} = \frac{x}{\Psi(x) + \gamma - b} = \frac{x}{\Psi(x) - B},$$

in which $B = b - \gamma = 2 + \frac{151}{100 \log 7} - \gamma$, and this completes the proof. \square

Inverse of this theorem seems to be nice; because using it, for every $x \geq 3299$ we obtain

$$\frac{x}{\pi(x)} + A < \Psi(x) < \frac{x}{\pi(x)} + B.$$

Moreover, considering this inequality with (1.4) and (1.5), we yield the following bounds for $x \geq 355991$

$$\frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}} + A < \Psi(x) < \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}} + B.$$

REFERENCES

- [1] H. Davenport, *Multiplicative Number Theory* (Second Edition), Springer-Verlag, 1980.
- [2] P. Dusart, Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, *C. R. Math. Acad. Sci. Soc. R. Can.* **21** (1999), no. 2, 53–59.
- [3] L. Locker-Ernst, Bemerkungen über die Verteilung der Primzahlen, *Elemente der Mathematik* **XIV**, nr. **1**, 1-5, Basel (1959).
- [4] L. Panaitopol, A SPECIAL CASE OF THE HARDY-LITTLEWOOD CONJECTURE, *Math. Reports*, 4(54)(2002), 3, 265-258.
- [5] L. Panaitopol, SEVERAL APPROXIMATION OF $\pi(x)$, *Mathematical Inequalities & Applications*, Volume 2, Number 3 (1999), 317-324.
- [6] József Sándor, Remark on a function which generalizes the harmonic series, *C. R. Acad. Bulgare Sci.*, **41**(1988), no. 5, 19-21.

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