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This is the Published version of the following publication

Hassani, Mehdi (2005) A Refinement of Mandl's Inequality. Research report collection, 8 (2).

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A REFINEMENT OF MANDL'S INEQUALITY

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ABSTRACT. In this short note, we prove that $\frac{n}{2}p_n - \sum_{i=1}^n p_i > 0.01659n^2$ holds for every $n \geq 9$. This is a refinement of Mandl's inequality which asserts $\frac{n}{2}p_n - \sum_{i=1}^n p_i > 0$, for those values of n .

1. INTRODUCTION AND REFINEMENT

As usual, let p_n be the n^{th} prime. The Mandl's conjecture (see [1] and [2]) asserts that for every $n \geq 9$, we have:

$$\frac{n}{2}p_n - \sum_{i=1}^n p_i > 0.$$

To prove Mandl's inequality, Dusart ([1], page 50) uses the following inequality

$$(1.1) \quad \int_2^{p_n} \pi(t)dt \geq c + \frac{p_n^2}{2 \log p_n} \left(1 + \frac{3}{2 \log p_n} \right) \quad (n \geq 109),$$

in which

$$c = 35995 - 3Li(599^2) + \frac{599^2}{\log 599} \approx -47.06746,$$

and

$$Li(x) = \lim_{\epsilon \rightarrow 0} \left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right),$$

is logarithmic function, and base of all logarithms are e . Note that one can get (1.1), using the following known bound [1]:

$$\pi(t) \geq \frac{t}{\log t} \left(1 + \frac{1}{\log t} \right) \quad (t \geq 599).$$

Also, for using (1.1) to prove Mandl's inequality, we note that:

$$\int_2^{p_n} \pi(t)dt = \sum_{i=2}^n (p_i - p_{i-1})(i-1) = \sum_{i=2}^n (ip_i - (i-1)p_{i-1}) - \sum_{i=2}^n p_i = np_n - \sum_{i=1}^n p_i.$$

Therefore, we have:

$$(1.2) \quad np_n - \sum_{i=1}^n p_i \geq c + \frac{p_n^2}{2 \log p_n} \left(1 + \frac{3}{2 \log p_n} \right) \quad (n \geq 109).$$

Now, we use the following bound ([1], page 36):

$$\frac{x}{\log x} \left(1 + \frac{1.2762}{\log x} \right) \geq \pi(x) \quad (x \geq 2).$$

1991 *Mathematics Subject Classification.* 11A41.
Key words and phrases. Primes.

Considering this bound and (1.2), for every $n \geq 109$, we obtain:

$$\begin{aligned} np_n - \sum_{i=1}^n p_i &\geq c + \frac{p_n^2}{2 \log p_n} \left(\frac{0.2238}{\log p_n} \right) + \frac{p_n^2}{2 \log p_n} \left(1 + \frac{1.2762}{\log p_n} \right) \\ &\geq c + 0.1119 \frac{p_n^2}{\log^2 p_n} + \frac{p_n}{2} \pi(p_n) = c + 0.1119 \frac{p_n^2}{\log^2 p_n} + \frac{n}{2} p_n. \end{aligned}$$

So, for every $n \geq 109$, we have:

$$(1.3) \quad \frac{n}{2} p_n - \sum_{i=1}^n p_i \geq c + 0.1119 \frac{p_n^2}{\log^2 p_n}.$$

In other hand, we have the following bounds for p_n ([3], page 69):

$$n \log n \leq p_n \leq n(\log n + \log \log n) \quad (n \geq 6).$$

Combining these bounds with (1.3), for every $n \geq 109$, we yield that:

$$\frac{n}{2} p_n - \sum_{i=1}^n p_i \geq c + \frac{0.1119(n \log n)^2}{\log^2 (n(\log n + \log \log n))}.$$

But, for every $n \geq 89$, we have $c + \frac{0.1119(n \log n)^2}{\log^2 (n(\log n + \log \log n))} > 0$, and so, we obtain the following inequality for every $n \geq 89$:

$$\frac{n}{2} p_n - \sum_{i=1}^n p_i \geq \frac{(n \log n)^2}{10 \log^2 (n(\log n + \log \log n))}.$$

This holds also for $10 \leq n \leq 88$. Thus, considering $\log(n(\log n + \log \log n)) < 2 \log n + 1$, we yield that:

$$\frac{n}{2} p_n - \sum_{i=1}^n p_i \geq \frac{n^2}{10} \left(\frac{\log n}{2 \log n + 1} \right)^2,$$

which holds for every $n \geq 9$. This is a refinement of Mandl's inequality, with quadratic and logarithms terms. Now, for every $n \geq 9$, we note that:

$$\frac{n}{2} p_n - \sum_{i=1}^n p_i \geq \frac{n^2}{10} \left(\frac{\log n}{2 \log n + 1} \right)^2 \geq \frac{n^2}{10} \left(\frac{\log 9}{2 \log 9 + 1} \right)^2 > 0.01659n^2.$$

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