Monotonic Properties of Differences for Remainders of Psi Function

MONOTONIC PROPERTIES OF DIFFERENCES FOR REMAINDERS OF PSI FUNCTION

FENG QI, DA-WEI NIU, AND BAI-NI GUO

Abstract. Let $\Lambda_{p,q}(x) = \lambda(px) - q\lambda(x)$ and $\Phi_{p,q}(x) = \phi(px) - q\phi(x)$ in $x \in (0,\infty)$ for $p > 0$ and $q \in \mathbb{R}$, where $\lambda(x) = \int_0^\infty \frac{t\,\mathrm{d}t}{(t^2+x^2)(e^{2\pi t}-1)}$ and $\phi(x) = \int_0^\infty \frac{t\,\mathrm{d}t}{(t^2+4x^2)(e^{\pi t}+1)}$ are related to $\psi(x)$ and $\psi\bigl(x+\frac12\bigr)$. In this article, some sufficient conditions on $p > 0$ and $q \in \mathbb{R}$ such that $\Lambda_{p,q}(x)$ and $\Phi_{p,q}(x)$ are monotonic in $x \in (0,\infty)$ are obtained. Moreover, as by-product, an inequality involving the exponential function is established.

1. Introduction

Recall [7, 11] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders and $0 \le (-1)^k f^{(k)}(x) < \infty$ for all $k \ge 0$ on I. For our own convenience, the class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [11] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if $f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t)$, where $\mu(t)$ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. Note that a completely monotonic function in $(0,\infty)$ which is non-identically zero cannot vanish at any point in $(0,\infty)$, see [7, 8] and the references therein.

The noted Binet's formula (see [9] and [10, p. 103]) states that for $x > 0$,

$$\ln\Gamma(x) = \left(x - \frac12\right)\ln x - x + \ln\sqrt{2\pi} + \theta(x), \tag{1}$$

where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \tag{2}$$

is called the remainder of Binet's formula (1).

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Let $p>0$ and $q\in\mathbb{R}$ be real numbers and

$$h_{p,q}(x)=\theta(px)-q\theta(x) \tag{3}$$

in $(0,\infty)$.

It is clear that $h_{1,q}(x)\in\mathcal{C}[(0,\infty)]$ for $q\le 1$ and $-h_{1,q}(x)\in\mathcal{C}[(0,\infty)]$ for $q\ge 1$.

Among other things, the following was proved in [1, 2].

Theorem A ([1, 2])**.** *$h_{p,p}(x)\in\mathcal{C}[(0,\infty)]$ if $0<p<1$ and $-h_{p,p}(x)\in\mathcal{C}[(0,\infty)]$ if $p>1$.*

As a further generalization of Theorem A above, the following conclusion was obtained recently.

Theorem B ([4])**.** *$h_{p,q}(x)\in\mathcal{C}[(0,\infty)]$ if either $q\le\frac1p\le 1$ or $q\le 1\le\frac1p$ and $-h_{p,q}(x)\in\mathcal{C}[(0,\infty)]$ if either $q\ge\frac1p\ge 1$ or $q\ge 1\ge\frac1p$.*

In [3, p. 892] and [6, p. 17], it is given that for $x>0$,

$$\psi(x)=\ln x-\frac{1}{2x}-2\int_0^\infty\frac{t\,\mathrm{d}t}{(t^2+x^2)(e^{2\pi t}-1)} \tag{4}$$

and

$$\psi\Bigl(x+\frac12\Bigr)=\ln x+2\int_0^\infty\frac{t\,\mathrm{d}t}{(t^2+4x^2)(e^{\pi t}+1)}. \tag{5}$$

Let

$$\Lambda_{p,q}(x)=\lambda(px)-q\lambda(x) \tag{6}$$

and

$$\Phi_{p,q}(x)=\phi(px)-q\phi(x) \tag{7}$$

in $x\in(0,\infty)$ for $p>0$ and $q\in\mathbb{R}$, where

$$\lambda(x)=\int_0^\infty\frac{t\,\mathrm{d}t}{(t^2+x^2)(e^{2\pi t}-1)} \tag{8}$$

and

$$\phi(x)=\int_0^\infty\frac{t\,\mathrm{d}t}{(t^2+4x^2)(e^{\pi t}+1)}. \tag{9}$$

Our main results of this paper are given in the following theorems.

Theorem 1. *The function* $\Lambda_{p,q}(x)$ *is positive and decreasing in* $x \in (0,\infty)$ *if either* $p \ge 1$ *and* $q \le 0$ *or* $0 < p < 1$ *and* $pq \le 1$*; it is negative and increasing in* $x \in (0,\infty)$ *if* $p \ge 1$ *and* $pq \ge 1$.

The function $\Phi_{p,q}(x)$ *is positive and decreasing in* $x \in (0,\infty)$ *if either* $p \ge 1$ *and* $q \le 0$ *or* $0 < p < 1$ *and* $q \le 1$*; it is negative and increasing in* $x \in (0,\infty)$ *if* $p > 1$ *and* $q \ge 1$.

Theorem 2. *The function* $\Lambda_{p,q}(x)$ *is positive and decreasing in* $x \in (0,\infty)$ *for either* $q \le 0$ *or* $0 < q = \frac{1}{p^2} \le 1$, *it is negative and increasing in* $x \in (0,\infty)$ *for* $\frac{1}{p^2} = q \ge 1$.

The function $\Phi_{p,q}(x)$ *is positive and decreasing in* $x \in (0,\infty)$ *for either* $p^2q < 1$ *and* $q(p^2-1)[(1+3q)p^2-4] \le 0$ *or* $p^2q = 1$ *and* $0 < q \le 1$, *it is negative and increasing in* $x \in (0,\infty)$ *for either* $4 \le p^2(1+3q) \le 1+3q$ *or* $\frac{1}{p^2} = q \ge 1$.

As by-product, we obtain the following inequality.

Theorem 3. *Let* $\tau \in \mathbb{R}$ *be a nonzero constant. Then inequality*

$$e^{a+b} > \frac{b^\tau e^b - a^\tau e^a}{b^\tau - a^\tau} \tag{10}$$

for all $a > 0$ *and* $b > 0$ *with* $a \ne b$ *holds if and only if* $\tau \ge 1$ *and reverses if and only if* $\tau < 0$.

In particular, inequality

$$e^{a+b} > \frac{be^b - ae^a}{b-a} \tag{11}$$

is valid for all $a > 0$ *and* $b > 0$ *with* $a \ne b$, *which is equivalent to the following integral inequality*

$$e^{a+b} > \frac{1}{a-b}\int_b^a (1+u)e^u \,\mathrm{d}u. \tag{12}$$

2. Proofs of main results

Proof of Theorem 1. Direct calculation arrives at

$$\begin{aligned}\Lambda_{p,q}(x) &= \int_0^\infty \frac{t}{t^2+x^2}\left(\frac{1}{e^{2\pi pt}-1} - \frac{q}{e^{2\pi t}-1}\right)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{t^2+x^2}\frac{t}{e^{2\pi t}-1}\left(\frac{e^{2\pi t}-1}{e^{2\pi pt}-1} - q\right)\mathrm{d}t\end{aligned} \tag{13}$$

and

$$\begin{aligned}\Phi_{p,q}(x) &= \int_0^\infty \frac{t}{t^2+4x^2}\left(\frac{1}{e^{\pi pt}+1} - \frac{q}{e^{\pi t}+1}\right)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{t^2+4x^2}\frac{t}{e^{\pi t}+1}\left(\frac{e^{\pi t}+1}{e^{\pi pt}+1} - q\right)\mathrm{d}t.\end{aligned} \tag{14}$$

Let

$$\omega_{r,s}(t) = \frac{e^{rt}-1}{e^{st}-1} \quad \text{and} \quad \chi_{r,s}(t) = \frac{e^{rt}+1}{e^{st}+1} \tag{15}$$

in $t \in (0,\infty)$ for positive real numbers $r > 0$ and $s > 0$. The L'Hôspital rule yields

$$\lim_{t\to0+} \omega_{r,s}(t) = \frac{r}{s}, \quad \lim_{t\to0+} \chi_{r,s}(t) = 1, \tag{16}$$

and

$$\lim_{t\to\infty} \omega_{r,s}(t) = \lim_{t\to\infty} \chi_{r,s}(t) = \begin{cases} 0, & r < s, \\ \infty, & r > s. \end{cases} \tag{17}$$

Direct differentiation and standard argument gives that

$$\frac{\mathrm{d}\omega_{r,s}(t)}{\mathrm{d}t} = \frac{(r-s)e^{(r+s)t} - (re^{rt} - se^{st})}{(e^{st}-1)^2}$$

and

$$\frac{\mathrm{d}\chi_{r,s}(t)}{\mathrm{d}t} = \frac{(r-s)e^{(r+s)t} - (se^{st} - re^{rt})}{(e^{st}+1)^2}.$$

The requirement $\frac{\mathrm{d}\omega_{r,s}(t)}{\mathrm{d}t} \lesseqgtr 0$ is equivalent with

$$\begin{gathered}(r-s)e^{(r+s)t} \lesseqgtr re^{rt} - se^{st}, \\ (u-v)e^{u+v} \lesseqgtr ue^u - ve^v, \\ ue^u(e^v-1) \lesseqgtr ve^v(e^u-1), \\ \frac{ue^u}{e^u-1} \lesseqgtr \frac{ve^v}{e^v-1},\end{gathered}$$

where $u = rt > 0$ and $v = st > 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xe^x}{e^x-1}\right) = \frac{e^x(e^x-x-1)}{(e^x-1)^2} > 0$$

for $x > 0$, the function $\frac{xe^x}{e^x-1}$ is increasing in $x \in (0,\infty)$. This implies that $\frac{\mathrm{d}\omega_{r,s}(t)}{\mathrm{d}t} \lesseqgtr 0$ in $(0,\infty)$ if and only if $r \lesseqgtr s$, and then $\omega_{r,s}(t)$ is $\begin{cases}\text{decreasing} \\ \text{increasing}\end{cases}$ in $t \in (0,\infty)$ if and only if $r \lesseqgtr s$. Therefore, when $p > 1$,

$$-q < \frac{e^{2\pi t} - 1}{e^{2\pi p t} - 1} - q < \frac{1}{p} - q;$$

when $0 < p < 1$,

$$\frac{1}{p} - q < \frac{e^{2\pi t} - 1}{e^{2\pi p t} - 1} - q < \infty.$$

Thus, if $p > 1$ and $q \le 0$ or $0 < p < 1$ and $pq \le 1$, the function $\Lambda_{p,q}(x)$ is positive and decreasing; if $p > 1$ and $pq \ge 1$, it is negative and increasing in $(0, \infty)$.

The requirement $\frac{\mathrm{d}\chi_{r,s}(t)}{\mathrm{d}t} \lesseqgtr 0$ is equivalent with

$$\begin{aligned}
(r - s)e^{(r+s)t} &\lesseqgtr se^{st} - re^{rt},\\
(u - v)e^{u+v} &\lesseqgtr ve^{v} - ue^{u},\\
ue^{u}(e^{v} + 1) &\lesseqgtr ve^{v}(e^{u} + 1),\\
\frac{ue^{u}}{e^{u} + 1} &\lesseqgtr \frac{ve^{v}}{e^{v} + 1},
\end{aligned}$$

where $u = rt > 0$ and $v = st > 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xe^{x}}{e^{x} + 1}\right) = \frac{e^{x}(e^{x} + x + 1)}{(e^{x} + 1)^2} > 0$$

for $x > 0$, the function $\frac{xe^{x}}{e^{x}+1}$ is increasing in $x \in (0, \infty)$. This implies that $\frac{\mathrm{d}\chi_{r,s}(t)}{\mathrm{d}t} \lesseqgtr 0$ in $(0, \infty)$ if and only if $r \lesseqgtr s$, and then $\chi_{r,s}(t)$ is $\begin{cases}\text{decreasing}\\ \text{increasing}\end{cases}$ in $t \in (0, \infty)$ if and only if $r \lesseqgtr s$. Therefore, when $p > 1$,

$$-q < \frac{e^{\pi t} + 1}{e^{\pi p t} + 1} - q < 1 - q;$$

when $0 < p < 1$,

$$1 - q < \frac{e^{\pi t} + 1}{e^{\pi p t} + 1} - q < \infty.$$

Thus, if $p > 1$ and $q \le 0$ or $0 < p < 1$ and $q \le 1$, the function $\Phi_{p,q}(x)$ is positive and decreasing; if $p > 1$ and $q \ge 1$, it is negative and increasing in $(0, \infty)$. The proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Straightforward computation yields

$$\begin{aligned}\Lambda_{p,q}(x)&=\int_0^\infty\frac{t}{e^{2\pi t}-1}\biggl(\frac{1}{t^2+p^2x^2}-\frac{q}{t^2+x^2}\biggr)\,\mathrm{d}t\\&=\int_0^\infty\frac{t}{e^{2\pi t}-1}\frac{(1-q)t^2+(1-p^2q)x^2}{(t^2+p^2x^2)(t^2+x^2)}\,\mathrm{d}t\\&\triangleq\int_0^\infty\frac{t}{e^{2\pi t}-1}\rho_{p,q;t}(x)\,\mathrm{d}t,\end{aligned}\tag{18}$$

$$\begin{aligned}\Phi_{p,q}(x)&=\int_0^\infty\frac{t}{e^{\pi t}+1}\biggl(\frac{1}{t^2+4p^2x^2}-\frac{q}{t^2+4x^2}\biggr)\,\mathrm{d}t\\&=\int_0^\infty\frac{t}{e^{\pi t}+1}\frac{(1-q)t^2+4(1-p^2q)x^2}{(t^2+p^2x^2)(t^2+4x^2)}\,\mathrm{d}t\\&\triangleq\int_0^\infty\frac{t}{e^{\pi t}+1}\varrho_{p,q;t}(x)\,\mathrm{d}t.\end{aligned}\tag{19}$$

By standard argument, we have

$$\begin{aligned}\frac{\mathrm{d}\rho_{p,q;t}(x)}{\mathrm{d}x}&=\frac{2xt^4[p^2(p^2q-1)u^2+2p^2(q-1)u+(q-p^2)]}{(t^2+p^2x^2)^2(t^2+x^2)^2},\\\frac{\mathrm{d}\varrho_{p,q;t}(x)}{\mathrm{d}x}&=\frac{2xt^4[16p^2(p^2q-1)u^2+8p^2(q-1)u+(4q-p^2-3p^2q)]}{(t^2+p^2x^2)^2(t^2+4x^2)^2},\end{aligned}$$

where $u=\left(\frac{x}{t}\right)^2>0$. Hence, if either

$$\begin{cases}p^2q-1\lesseqgtr 0\\q\le 0\end{cases}\quad\text{or}\quad\begin{cases}p^2q-1=0\\q-1\lesseqgtr 0\\q-p^2\lesseqgtr 0,\end{cases}$$

then the derivative $\frac{\mathrm{d}\rho_{p,q;t}(x)}{\mathrm{d}x}\lesseqgtr 0$; if either

$$\begin{cases}p^2q-1\lesseqgtr 0\\q(p^2-1)[(1+3q)p^2-4]\le 0\end{cases}\quad\text{or}\quad\begin{cases}p^2q-1=0\\q-1\lesseqgtr 0\\4q-p^2-3p^2q\lesseqgtr 0,\end{cases}$$

then the derivative $\frac{\mathrm{d}\varrho_{p,q;t}(x)}{\mathrm{d}x}\lesseqgtr 0$.

Consequently, if either $q\le 0$ or $0<q=\frac{1}{p^2}\le 1$ and $p^4\ge 1$ then $\frac{\mathrm{d}\rho_{p,q;t}(x)}{\mathrm{d}x}\le 0$ and the function $\Lambda_{p,q}(x)$ is decreasing in $x\in(0,\infty)$; if $q=\frac{1}{p^2}\ge 1$ and $p^4\le 1$ then $\frac{\mathrm{d}\rho_{p,q;t}(x)}{\mathrm{d}x}\ge 0$ and the function $\Lambda_{p,q}(x)$ is increasing in $x\in(0,\infty)$; if either $p^2q<1$ and $q(p^2-1)[(1+3q)p^2-4]\le 0$ or $p^2q=1$ and $0<q\le 1$ then $\frac{\mathrm{d}\varrho_{p,q;t}(x)}{\mathrm{d}x}\le 0$ and the function $\Phi_{p,q}(x)$ is decreasing in $x\in(0,\infty)$; if either $p^2q>1$ and

$(p^2-1)[(1+3q)p^2-4] \le 0$ or $p^2q = 1$ and $q \ge 1$ then $\frac{\mathrm{d}\varrho_{p,q;t}(x)}{\mathrm{d}x} \ge 0$ and the function $\Phi_{p,q}(x)$ is increasing in $x \in (0,\infty)$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Without loss of generality, assume $b > a > 0$ in (10). Then it can be rearranged as

$$\frac{b^\tau e^b}{e^b - 1} > \frac{a^\tau e^a}{e^a - 1}.$$

Direct calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^\tau e^x}{e^x - 1}\right) = \frac{x^{\tau-1}e^x\left(\tau - \frac{x}{e^x-1}\right)}{e^x - 1}.$$

It is easy to see that the function $\frac{x}{e^x-1}$ is decreasing in $(0,\infty)$, with

$$\lim_{x\to 0+}\frac{x}{e^x-1} = 1 \quad \text{and} \quad \lim_{x\to\infty}\frac{x}{e^x-1} = 0.$$

Hence, the function $\frac{x^\tau e^x}{e^x-1}$ is increasing (or decreasing, respectively) in $x \in (0,\infty)$ if and only if $\tau \ge 1$ (or $\tau < 0$, respectively). Inequality (10) follows. □

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(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com

URL: http://rgmia.vu.edu.au/qi.html

(D.-W. Niu) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: nnddww@tom.com

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: guobaini@hpu.edu.cn