Universal Multiplication Table

UNIVERSAL MULTIPLICATION TABLE

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Abstract. In this paper, considering the concept of Universal Multiplication Table, we show that for every $n \geq 2$, the inequality:

$$M(n) = \#\{ij | 1 \leq i, j \leq n\} \geq \frac{n^2}{\mathfrak{N}(n^2)},$$

holds true with:

$$\mathfrak{N}(n) = n^{\frac{\log 2}{\log \log n}\left(1 + \frac{387}{200 \log \log n}\right)}.$$

1. Introduction

Consider the following $n \times n$ *Multiplication Table*, which we denote it by $MT_{n\times n}$:

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

Let $\mathfrak{M}(n; k)$ be the number of k's, which appear in $MT_{n\times n}$; i.e.

$$\mathfrak{M}(n; k) = \# \left\{(a, b) \in \mathbb{N}_n^2 \mid ab = k\right\}, \tag{1.1}$$

where $\mathbb{N}_n = \mathbb{N} \cap [1, n]$. For example, we have:

$$\mathfrak{M}(2; 2) = 2, \mathfrak{M}(7; 6) = 4, \mathfrak{M}(10; 9) = 3, \mathfrak{M}(100; 810) = 10, \mathfrak{M}(100; 9900) = 2.$$

In this paper first we study some elementary properties of the function $\mathfrak{M}(n; k)$, for a fixed $n \in \mathbb{N}$. Then we try to connect $\mathfrak{M}(n; k)$ by the famous *Multiplication Table Function*[1]; $M(n) = \#\{ij|(i, j) \in \mathbb{N}_n^2\}$ in order to get some lower bounds for it. To do this, we introduce the concept of *Universal Multiplication Table*, which is an infinite array generated by multiplying the components of points in the infinite lattice $\mathbb{N}^2$. Let $D(n) = \{d : d > 0,\ d|n\}$. To get above mentioned bounds for the function $M(n)$, we will need some upper bounds for the *Divisor Function* $d(n) = \#D(n)$, which we recall best known, due to J.L. Nicolas [5]:

$$\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log \log n}\left(1 + \frac{1.9349\cdots}{\log \log n}\right) \qquad (n \geq 3),$$

or

$$d(n) \leq \mathfrak{N}(n) \tag{1.2}$$

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[1]This sequence, has been indexed in "The On-Line Encyclopedia of Integer Sequences" data base with ID A027424. Web page of above data base is:
http://www.research.att.com/ njas/sequences/index.html

for $n \geq 3$, with

$$\mathfrak{N}(n) = n^{\frac{\log 2}{\log\log n}\left(1+\frac{387}{200\log\log n}\right)}.$$

2. Some Elementary Properties of the Function $\mathfrak{M}(n;k)$

Considering (1.1), for every $s \in \mathbb{C}$, we have:

$$\sum_{1\leq i,j\leq n} \frac{1}{(ij)^s} = \sum_{k=1}^{n^2} \frac{\mathfrak{M}(n;k)}{k^s} = \sum_{k=1}^{\infty} \frac{\mathfrak{M}(n;k)}{k^s}. \tag{2.1}$$

The left hand side of above identity is equal to $\zeta_n^2(s)$, in which $\zeta_n(s) = \sum_{i=1}^n \frac{1}{i^s}$, and the number of summands in the right hand side of above identity, is equal to $M(n)$. Also, summing and counting all numbers in $MT_{n\times n}$, we obtain respectively:

$$\sum_{k=1}^{n^2} k\mathfrak{M}(n;k) = \left(\frac{n(n+1)}{2}\right)^2,$$

and

$$\sum_{k=1}^{n^2} \mathfrak{M}(n;k) = n^2,$$

which both of them are special cases of (2.1) for $s=-1$ and $s=0$, respectively. To have some formulas for the function $\mathfrak{M}(n;k)$, we define *Incomplete Divisor Function* to be $d(k;x) = \#D(k) \cap [1,x]$. This function has some properties, which we list some of them:

1. It is trivial that for every $x \geq 1$ we have:

$$1 \leq d(k;x) \leq \min\{x, d(k)\}.$$

So, $d(k;x) = O(x)$ and naturally we ask: What is the exact order of $d(k;x)$? The next property, maybe useful to find answer.

2. If we let $D(k) = \left\{1 = d_1, d_2, \cdots, d_{d(k)} = k\right\}$, then we have:

$$\begin{aligned}\int_1^k d(k;x)dx &= \sum_{i=1}^{d(k)-1} (d_{i+1}-d_i)i = \sum_{i=1}^{d(k)-1} (i+1)d_{i+1} - id_i - \sum_{i=1}^{d(k)-1} d_{i+1} \\ &= d(k)d_{d(k)} - 1d_1 - \sum_{d|k, d>1} d = kd(k) - \sigma(k),\end{aligned}$$

where $\sigma(k) = \sum_{a\in D(k)} a$, and we have the following bound due to G. Robin [7]:

$$\sigma(n) < \mathfrak{R}(n) \qquad (n \geq 3), \tag{2.2}$$

with

$$\mathfrak{R}(n) = e^\gamma n \log\log n + \frac{3241n}{5000\log\log n},$$

where $\gamma \approx 0.5772156649$ is Euler's constant. Considering (1.2) and (2.2), we obtain the following inequality for every $k \geq 3$:

$$2k - \mathfrak{R}(k) < \int_1^k d(k;x)dx < k\mathfrak{N}(k) - k - 1.$$

In general, every knowledge about $d(k;x)$ is useful, because:

Proposition 2.1. *For every positive integers k and n, we have:*

$$\mathfrak{M}(n;k) = d(k;n) - d\Big(k;\frac{k}{n}\Big) + R(n;k),$$

where

$$R(n;k) = \Big\lfloor \frac{k}{n} \Big\rfloor - \Big\lfloor \frac{k-1}{n} \Big\rfloor = \begin{cases} 1, & n \mid k, \\ 0, & \text{other wise.} \end{cases}$$

Proof. Considering (1.1), we have:

$$\mathfrak{M}(n;k) = \#\left\{(a,b) \in \mathbb{N}_n^2 \mid ab = k\right\} = \sum_{d|k, d\le n, \frac{k}{d}\le n} 1 = \sum_{d|k, \frac{k}{n}\le d\le n} 1.$$

Applying the definition of $d(k;x)$, completes the proof. □

3. Universal Multiplication Table Function

We define the *Universal Multiplication Table Function* $\mathfrak{M}(k)$ to be the number of k's, which appear in the universal multiplication table.

Proposition 3.1. *For every positive integer k, we have:*

$$\mathfrak{M}(k) = d(k).$$

Proof. Here we have two proofs:
Elementary Method. Considering the definition of universal multiplication table, we have:

$$\mathfrak{M}(k) = \lim_{n\to\infty} \mathfrak{M}(n;k) = \lim_{n\to\infty} \sum_{d|k, \frac{k}{n}\le d\le n} 1 = \sum_{d|k, 0<d<\infty} 1 = d(k).$$

Analytic Method. Considering (2.1) for $\Re(s) > 1$ and taking limit both sides of it, when n tends to infinity, we obtain:

$$\sum_{k=1}^{\infty} \frac{\mathfrak{M}(k)}{k^s} = \zeta^2(s), \tag{3.1}$$

in which $\zeta(s)$ is the Riemann zeta-function. According to the Theorem 11.17 of [1], we obtain:

$$\mathfrak{M}(k) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \zeta^2(\sigma+it) k^{\sigma+it} dt, \qquad (\sigma > 1).$$

Since $\zeta^2(s) = \sum_{m=1}^{\infty} d(m) m^{-s}$, we have:

$$\begin{aligned} \mathfrak{M}(k) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \zeta^2(\sigma+it) k^{\sigma+it} dt \\ &= \sum_{m=1}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{k}{m}\right)^{it} dt \\ &= \sum_{m=1, m\neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{k}{m}\right)^{it} dt + d(k) \\ &= \sum_{m=1, m\neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T\to\infty} \frac{1}{T} \sin\left(T \log\left(\frac{k}{m}\right)\right) + d(k) = d(k). \end{aligned}$$

This completes the proof. □

Now, fix positive integer k and consider $\mathfrak{M}(n;k)$, as an arithmetic function of the variable n. Clearly, $\mathfrak{M}(n;k)$ is increasing, and for $n > k$, we have $\mathfrak{M}(n;k) = \mathfrak{M}(k)$. Thus considering Proposition 3.1, we obtain:

$$\mathfrak{M}(n;k) \leq d(k), \tag{3.2}$$

and if $k \geq 3$, considering (1.2) yields that:

$$\mathfrak{M}(n;k) \leq \mathfrak{N}(k).$$

4. Statistical Study of $\mathfrak{M}(n;k)$'s

Consider $S = \left[\mathfrak{M}(n;k) \mid 1 \leq k \leq n^2\right]$ as a list of statistical data and suppose $\overline{\mathfrak{M}}(n)$ is the average of above list, then we have:

$$\overline{\mathfrak{M}}(n) = \frac{\sum_{k=1}^{n^2} \mathfrak{M}(n;k)}{\#\{ij|(i,j) \in \mathbb{N}_n^2\}} = \frac{n^2}{M(n)}.$$

Thus, we have:

$$M(n) = \frac{n^2}{\overline{\mathfrak{M}}(n)}. \tag{4.1}$$

Considering (3.2), it is clear that:

$$\overline{\mathfrak{M}}(n) \leq \max\{\mathfrak{M}(n;k)\}_{k=1}^{n^2} \leq \max\{d(k)\}_{k=1}^{n^2}.$$

To use (1.2), we observe that the function $\mathfrak{N}(n)$ is increasing for $n \geq 114$. So, we have:

$$\overline{\mathfrak{M}}(n) \leq \max\{d(1), d(2), \cdots, d(114), d(n^2)\} \leq \max\{12, \mathfrak{N}(n^2)\} \qquad (n \geq \sqrt{3}),$$

and since $\mathfrak{N}(n) > 114.1$ holds for every $n > 0$, we obtain:

$$\overline{\mathfrak{M}}(n) \leq \mathfrak{N}(n^2) \qquad (n \geq 2).$$

Therefore, we have proved the following result.

Theorem 4.1. *For every $n \geq 2$, we have:*

$$M(n) \geq \frac{n^2}{\mathfrak{N}(n^2)}.$$

Remark 4.2. One of the wonderful results about $MT_{n\times n}$ is *Erdös Multiplication Table Theorem* [6], which asserts:

$$\lim_{n\to\infty} \frac{M(n)}{n^2} = 0.$$

Above theorem yields that in the Erdös's theorem, however the ratio $\frac{M(n)}{n^2}$ tends to zero, but it doesn't faster than $\frac{1}{\mathfrak{N}(n^2)}$. More precisely, Erdös showed that $M(n) = n^2(\log n)^{-c+o(1)}$ for $c = 1 + \frac{\log\log 2}{\log 2}$ [2, 3]. The following table includes some computational results about $M(n)$ by the Maple software.

n	$M(n)$	$M(n)/n^2 \approx$	n	$M(n)$	$M(n)/n^2 \approx$
10	42	0.4200000000	2000	959759	0.2399397500
50	800	0.3200000000	3000	2121063	0.2356736667
100	2906	0.2906000000	4000	3723723	0.2327326875
1000	248083	0.2480830000	5000	5770205	0.2308082000

Note that, the true order of $M(n)$ is $n^2(\log n)^{-c}(\log\log n)^{-3/2}$ [3].

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