Semi-Inner Products and the Numerical Radius of Bounded Linear Operators in Hilbert Spaces

SEMI-INNER PRODUCTS AND THE NUMERICAL RADIUS OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. The main aim of this paper is to establish some connections that exist between the numerical radius $w(A)$, operator norm $\|A\|$ and the semi-inner products $\langle A, I\rangle_{p,n}$ and $\langle A, I\rangle_{p,w}$ with $p \in \{i, s\}$ that can be naturally defined on the Banach algebra $B(H)$ of all bounded linear operators defined on a Hilbert space H. Reverse inequalities that provide upper bounds for the nonnegative quantities $\|A\| - w(A)$ and $w(A) - \langle A, I\rangle_{p,n}$ under various assumptions for the operator A are also given.

1. Introduction

In any normed linear space $(E, \|\cdot\|)$, since the function $f : E \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$ is convex, one can introduce the following semi-inner products (see for instance [3]):

$$\langle x, y\rangle_i := \lim_{s\to 0-} \frac{\|y + sx\|^2 - \|y\|^2}{2s}, \qquad \langle x, y\rangle_s := \lim_{t\to 0-} \frac{\|y + tx\|^2 - \|y\|^2}{2t} \tag{1.1}$$

where x, y are vectors in E. The mappings $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_i$ are called the *superior* respectively the *inferior semi-inner product* associated with the norm $\|\cdot\|$.

In the Banach algebra $B(H)$ of all bounded linear operators defined on the real or complex Hilbert space H we can associate to both the operator norm $\|\cdot\|$ and the numerical radius $w(\cdot)$ the following semi-inner products:

$$\langle A, B\rangle_{s(i),n} := \lim_{t\to 0+(-)} \frac{\|B + tA\|^2 - \|B\|^2}{2t} \tag{1.2}$$

and

$$\langle A, B\rangle_{s(i),w} := \lim_{t\to 0+(-)} \frac{w^2(B + tA) - w^2(B)}{2t} \tag{1.3}$$

respectively, where $A, B \in B(H)$.

It is obvious that the semi-inner products $\langle\cdot,\cdot\rangle_{s(i),n(w)}$ defined above have the usual properties of such mappings defined on general normed spaces and some special properties that will be specified in the following.

For the sake of completeness we list here some properties of $\langle\cdot,\cdot\rangle_{s(i),n(w)}$ that will be used in the sequel.

We have:

(i) $\langle A, A\rangle_{s(i),n} = \|A\|^2$, $\langle A, A\rangle_{s(i),w} = w^2(A)$ for any $A \in B(H)$;

(ii) $\langle iA, A\rangle_{p,n(w)} = \langle A, iA\rangle_{p,n(w)} = 0$ for each $A \in B(H)$;

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(iii) $\langle \lambda A, B\rangle_{p,n(w)} = \lambda \langle A, B\rangle_{p,n(w)} = \langle A, \lambda B\rangle_{p,n(w)}$ for any $\lambda \geq 0$ and $A, B \in B(H)$;
(iv) $\langle -A, B\rangle_{p,n(w)} = \langle A, -B\rangle_{p,n(w)} = -\langle A, B\rangle_{q,n(w)}$ for any $A, B \in B(H)$;
(v) $\langle iA, B\rangle_{p,n(w)} = -\langle A, iB\rangle_{p,n(w)}$ for each $A, B \in B(H)$;
(vi) The following Schwarz type inequalities hold true:

$$\left|\langle A, B\rangle_{p,n}\right| \leq \|A\| \|B\| \tag{1.4}$$

and

$$\left|\langle A, B\rangle_{p,w}\right| \leq w(A) w(B) \tag{1.5}$$

for any $A, B \in B(H)$;
(vii) The following identities hold true

$$\langle \alpha A + B, A\rangle_{p,n} = \alpha \|A\|^2 + \langle B, A\rangle_{p,n} \tag{1.6}$$

and

$$\langle \alpha A + B, A\rangle_{p,w} = \alpha w^2(A) + \langle B, A\rangle_{p,w} \tag{1.7}$$

for each $\alpha \in \mathbb{R}$ and $A, B \in B(H)$;
(viii) The following sub(super)-additivity property holds:

$$\langle A + B, C\rangle_{s(i),p(w)} \leq (\geq) \langle A, C\rangle_{s(i),p(w)} + \langle B, C\rangle_{s(i),p(w)}, \tag{1.8}$$

where the sign "$\leq$" applies for the superior semi-inner product, while the sign "$\geq$" applies for the inferior one;
(ix) The following continuity properties are valid:

$$\left|\langle A + B, C\rangle_{p,n} - \langle B, C\rangle_{p,n}\right| \leq \|A\| \|C\| \tag{1.9}$$

and

$$\left|\langle A + B, C\rangle_{p,w} - \langle B, C\rangle_{p,w}\right| \leq w(A) w(C) \tag{1.10}$$

for each $A, B, C \in B(H)$;
(x) From the definition we have the inequality

$$\langle A, B\rangle_{i,n(w)} \leq \langle A, B\rangle_{s,n(w)} \tag{1.11}$$

for $A, B \in B(H)$;

where everywhere above $p, q \in \{i, s\}$ and $p \neq q$.

As a specific property that follows by the well known inequality between the norm and the numerical radius of an operator, i.e., $w(A) \leq \|A\|$ for each $A \in B(H)$, we have

$$\langle A, I\rangle_{i,n} \leq \langle A, I\rangle_{i,w} (\leq) \langle A, I\rangle_{s,w} \leq \langle A, I\rangle_{s,n} \tag{1.12}$$

for any $A \in B(H)$, where I is the identity operator on H. We also observe that

$$\langle A, I\rangle_{s(i),n} = \lim_{t \to 0+(-)} \frac{\|I + tA\| - 1}{t}$$

and

$$\langle A, I\rangle_{s(i),w} = \lim_{t \to 0+(-)} \frac{w(I + tA) - 1}{t}$$

for each $A \in B(H)$.

Motivated by the natural connection that exists between the semi-inner products $\langle A,I\rangle_{p,n}$, $\langle A,I\rangle_{p,w}$ with $p\in\{i,s\}$, the numerical radius $w(A)$ and the operator norm $\|A\|$ outlined above, the aim of this paper is to establish deeper relationships between these concepts. Amongst others, we show, in fact, that the semi-inner product $\langle A,I\rangle_{p,n}$ is equal to $\langle A,I\rangle_{p,w}$ for $p\in\{i,s\}$ and as a consequence the numerical radius $w(A)$ is bounded below by the maximum of the quantities $\left|\langle A,I\rangle_{i,n}\right|$ and $\left|\langle A,I\rangle_{s,n}\right|$. Also, on utilising these quantities various reverse inequalities for the fundamental fact that

$$\frac{1}{2}\|A\|\le w(A)\le\|A\|,\quad A\in B(H),$$

are pointed out, improving the recent results of the author from [4], where some upper bounds for the nonnegative differences $\|A\|-w(A)$ and $\|A\|^2-w^2(A)$ have been established under appropriate conditions for the bounded linear operator A.

For recent results concerning inequalities between the operator norm and numerical radius see the papers [2], [5], [6], [7], [8], [9], the books [1], [10], [11] and the references therein.

2. The Functionals $v_{s(i)}$ on $B(H)$

The following representation result, which plays a crucial role in deriving various inequalities for numerical radius may be stated:

Theorem 1. *For any $A\in B(H)$, we have:*

$$\langle A,I\rangle_{s,n}=\langle A,I\rangle_{s,w}=\sup_{\|x\|=1}\operatorname{Re}\langle Ax,x\rangle \tag{2.1}$$

and

$$\langle A,I\rangle_{i,n}=\langle A,I\rangle_{i,w}=\inf_{\|x\|=1}\operatorname{Re}\langle Ax,x\rangle. \tag{2.2}$$

Proof. Let $x\in H$, $\|x\|=1$. Then for $t>0$ we obviously have:

$$\begin{aligned}\operatorname{Re}\langle Ax,x\rangle&=\frac{\operatorname{Re}\langle x+tAx,x\rangle-1}{t}\\&\le\frac{|\langle x+tAx,x\rangle|-1}{t}\le\frac{w(I+tA)-1}{t}.\end{aligned} \tag{2.3}$$

Taking the supremum over $x\in H$, $\|x\|=1$, we get

$$\sup_{\|x\|=1}\operatorname{Re}\langle Ax,x\rangle\le\frac{w(I+tA)-1}{t}$$

for each $t>0$, which implies, by letting $t\to0+$ that

$$\sup_{\|x\|=1}\operatorname{Re}\langle Ax,x\rangle\le\langle A,I\rangle_{s,w}, \tag{2.4}$$

for each $A\in B(H)$.

Now, let $\delta:=\sup_{\|x\|=1}\operatorname{Re}\langle Ax,x\rangle$. If $x\in H$, $\|x\|=1$, then for any $\alpha>0$ we have

$$\|(I-\alpha A)x\|\ge\operatorname{Re}\langle(I-\alpha A)x,x\rangle=1-\alpha\operatorname{Re}\langle Ax,x\rangle\ge1-\alpha\delta.$$

Therefore, by putting $x=\frac{y}{\|y\|}$ $(y\neq0)$ with $y\in H$, we have

$$\|(I-\alpha A)y\|\ge(1-\alpha\delta)\|y\| \tag{2.5}$$

for each $\alpha > 0$ and $y \in H$.

Now, if in (2.5) we replace y by $(I + \alpha A) z$ with $z \in H$, then we get

$$\left\| \left(I - \alpha^2 A^2 \right) z \right\| \geq (1 - \alpha\delta) \left\| (I + \alpha A) z \right\| \quad \text{for each} \quad z \in H. \tag{2.6}$$

Taking the supremum over $\|z\| = 1$ in (2.6), we get the operator norm inequality:

$$\left\| I - \alpha^2 A^2 \right\| \geq (1 - \alpha\delta) \left\| I + \alpha A \right\|$$

which is equivalent with

$$\left\| I - \alpha^2 A^2 \right\| + \alpha\delta \left\| I + \alpha A \right\| \geq \left\| I + \alpha A \right\|, \quad \alpha > 0. \tag{2.7}$$

If we subtract 1 from both sides of (2.7) and divide by $\alpha > 0$, we get

$$\frac{\left\| I - \alpha^2 A^2 \right\| - 1}{\alpha} + \delta \left\| I + \alpha A \right\| \geq \frac{\left\| I + \alpha A \right\| - 1}{\alpha}. \tag{2.8}$$

Taking the limit over $\alpha \to 0+$ in (2.8) and noticing that

$$\lim_{\alpha \to 0+} \left\| I + \alpha A \right\| = 1, \qquad \lim_{\alpha \to 0+} \frac{\left\| I + \alpha A \right\| - 1}{\alpha} = \langle A, I \rangle_{s,n}$$

and

$$\lim_{\alpha \to 0+} \frac{\left\| I - \alpha^2 A^2 \right\| - 1}{\alpha} = \lim_{\alpha \to 0+} \alpha \cdot \lim_{\alpha \to 0+} \frac{\left\| I - \alpha^2 A^2 \right\| - 1}{\alpha^2} = 0,$$

then, by (2.8), we have:

$$\delta \geq \langle A, I \rangle_{s,n}. \tag{2.9}$$

Since, by (1.12), we always have $\langle A, I \rangle_{s,n} \geq \langle A, I \rangle_{s,w}$, hence (2.4) and (2.9) imply the desired equality (2.1).

Now, on utilising (iv) and (2.1), we have for all $A \in B(H)$ that

$$\langle A, I \rangle_{i,n} = \langle -A, I \rangle_{s,n} = - \sup_{\|x\|=1} \operatorname{Re} \langle -Ax, x \rangle = \inf_{\|x\|=1} \operatorname{Re} \langle Ax, x \rangle$$

and the identity (2.2) is also obtained. ∎

It is well known that a lower bound for the numerical radius $w(A)$ is $\frac{1}{2}\|A\|$. The following corollary of the above theorem provides the following lower bounds as well:

Corollary 1. *For any $A \in B(H)$ we have*

$$\max \left\{ \left| \langle A, I \rangle_{s,n} \right|, \left| \langle A, I \rangle_{i,n} \right| \right\} \leq w(A). \tag{2.10}$$

Proof. By Schwarz's inequality for the semi-inner products $\langle \cdot, \cdot \rangle_{s,w}$ and $\langle \cdot, \cdot \rangle_{i,w}$ we have

$$\left| \langle A, I \rangle_{s,w} \right|, \left| \langle A, I \rangle_{i,w} \right| \leq w(A), \quad A \in B(H)$$

which, by (2.1) and (2.2), imply the desired inequality (2.10). ∎

Motivated by the representation Theorem 1, we can introduce the following functionals defined on $B(H)$:

$$v_s(A) := \sup_{\|x\|=1} \operatorname{Re} \langle Ax, x \rangle \quad \text{and} \quad v_i(A) := \inf_{\|x\|=1} \operatorname{Re} \langle Ax, x \rangle. \tag{2.11}$$

Now, on employing the properties of the semi-inner products outlined above, we can state that:

(a) $v_s(-A) = -v_i(A)$, $A \in B(H)$;

(aa) $v_i(A) \geq 0$ for accretive operators on H;
(aaa) $v_{s(i)}(A+B) \leq (\geq) v_{s(i)}(A) + v_{s(i)}(B)$ for each $A, B \in B(H)$;
(av) $v_{s(i)}(A) = \langle A, I\rangle_{s(i),n} = \langle A, I\rangle_{s(i),w}$ for any $A \in B(H)$;
(v) $\left|v_{s(i)}(A)\right| \leq w(A)$ for all $A \in B(H)$;
(va) $v_{s(i)}(A) = v_{s(i)}(\alpha I + A) - \alpha$ for any $\alpha \in \mathbb{R}$ and $A \in B(H)$;
(vaa) $\left|v_{s(i)}(A+B) - v_{s(i)}(B)\right| \leq w(A)$ for any $A, B \in B(H)$.

The following inequalities may be stated as well:

Proposition 1. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ we have*

$$\frac{1}{2}\left[\|A\|^2 + |\lambda|^2\right] \geq v_s(\bar{\lambda}A) \geq \begin{cases} \frac{1}{2}\left[\|A\|^2 + |\lambda|^2\right] - \frac{1}{2}\|A - \lambda I\|^2, \\ \\ \frac{1}{4}\left[\|A + \lambda I\|^2 - \|A - \lambda I\|^2\right]. \end{cases} \tag{2.12}$$

Proof. Let $x \in H$, $\|x\| = 1$. Then, obviously

$$0 \leq \|Ax\|^2 - 2\operatorname{Re}\left[\langle \bar{\lambda}Ax, x\rangle\right] + |\lambda|^2 = \|(A - \lambda I)x\|^2 \leq \|A - \lambda I\|^2,$$

which is equivalent with

$$\frac{1}{2}\left[\|Ax\|^2 + |\lambda|^2\right] - \frac{1}{2}\|A - \lambda I\|^2 \leq \operatorname{Re}\langle \bar{\lambda}Ax, x\rangle \leq \frac{1}{2}\left[\|Ax\|^2 + |\lambda|^2\right], \tag{2.13}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we get the first inequality in (2.12) and the one from the first branch in the second.

For $x \in H$, $\|x\| = 1$ we also have that

$$\|Ax + \lambda x\|^2 = \|Ax - \lambda x\|^2 + 4\operatorname{Re}\langle \bar{\lambda}Ax, x\rangle, \tag{2.14}$$

which, on taking the supremum over $\|x\| = 1$, will produce the second part of the second inequality in (2.12). ∎

It is well known, in general, that the semi-inner products $\langle \cdot, \cdot\rangle_{s(i)}$ are not commutative. However, for the Banach algebra $B(H)$ we can point out the following connection between $\langle A, I\rangle_{s(i),n(w)} = v_{s(i)}(A)$ and the quantities $\langle I, A\rangle_{s,n}$ and $\langle I, A\rangle_{i,n}$, where $A \in B(H)$.

Corollary 2. *For any $A \in B(H)$ we have*

$$v_i(A) \leq \frac{1}{2}\left[\langle I, A\rangle_{s,n} + \langle I, A\rangle_{i,n}\right] \leq v_s(A). \tag{2.15}$$

Proof. We have from the second part of the second inequality in (2.12) that

$$\frac{1}{2}\left[\frac{\|A + tI\|^2 - \|A\|^2}{2t} - \frac{\|A - tI\|^2 - \|A\|^2}{2t}\right] \leq v_s(A) \tag{2.16}$$

for any $t > 0$.

Taking the limit over $t \to 0+$ and noticing that

$$\lim_{t \to 0+} \frac{\|A - tI\|^2 - \|A\|^2}{2t} = \langle -I, A\rangle_{s,n} = -\langle I, A\rangle_{i,n},$$

we get the second inequality in (2.15).

Now, writing the second inequality in (2.15) for $-A$, we get

$$\begin{aligned} v_s(-A) &\geq \frac{1}{2}\left[\langle I, -A\rangle_{s,n} + \langle I, -A\rangle_{i,n}\right] \\ &= -\frac{1}{2}\left[\langle I, A\rangle_{s,n} + \langle I, A\rangle_{i,n}\right], \end{aligned}$$

which is equivalent with the first part of (2.15). ∎

Utilising a similar approach for the numerical radius instead of the operator norm, we can state the following result:

Proposition 2. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ we have the double inequality:*

$$\begin{aligned} \frac{1}{2}\left[w^2(A) + |\lambda|^2\right] &\geq v_s\left(\bar{\lambda}A\right) \\ &\geq \begin{cases} \frac{1}{2}\left[w^2(A) + |\lambda|^2\right] - \frac{1}{2}w^2(A - \lambda I), \\ \\ \frac{1}{4}\left[w^2(A + \lambda I) - w^2(A - \lambda I)\right]. \end{cases} \end{aligned} \tag{2.17}$$

The above result has the interesting consequence:

Corollary 3. *For any $A \in B(H)$ we have*

$$v_i(A) \leq \frac{1}{2}\left[\langle I, A\rangle_{s,w} + \langle I, A\rangle_{i,w}\right] \leq v_s(A). \tag{2.18}$$

Remark 1. *Since $w(A) \geq \|A\|$, hence the first inequality in (2.17) provides a better upper bound for $v_s\left(\bar{\lambda}A\right)$ than the first inequality in (2.12).*

3. Reverse Inequalities in Terms of Operator Norm

The following result concerning reverse inequalities for the numerical radius and operator norm may be stated:

Theorem 2. *For any $A \in B(H)\setminus\{0\}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ we have the inequality:*

$$(0 \leq)\, \|A\| - w(A) \leq \|A\| - v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \leq \frac{1}{2|\lambda|}\|A - \lambda I\|^2. \tag{3.1}$$

In addition, if $\|A - \lambda I\| \leq |\lambda|$, then we have:

$$\sqrt{1 - \left\|\frac{1}{\lambda}A - I\right\|^2} \leq \frac{v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)}{\|A\|} \leq \frac{w(A)}{\|A\|}\,(\leq 1) \tag{3.2}$$

and

$$\begin{aligned} (0 \leq)\, \|A\|^2 - w^2(A) &\leq \|A\|^2 - v_s^2\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \\ &\leq 2\left(|\lambda| - \sqrt{|\lambda|^2 - \|A - \lambda I\|^2}\right)v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \\ &\left(\leq 2\left(|\lambda| - \sqrt{|\lambda|^2 - \|A - \lambda I\|^2}\right)w(A)\right), \end{aligned} \tag{3.3}$$

respectively.

Proof. Utilising the property (v), we have

$$w(A) = w\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \geq \left|v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)\right| \geq v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right),$$

for each $\lambda \in \mathbb{C}\backslash\{0\}$ and the first inequality in (3.1) is proved.

By the arithmetic mean-geometric mean inequality we have

$$\frac{1}{2}\left[\|A\|^2 + |\lambda|^2\right] \geq |\lambda| \|A\|,$$

which, by (2.12) provides

$$v_s\left(\bar{\lambda}A\right) \geq |\lambda| \|A\| - \frac{1}{2}\|A - \lambda I\|^2$$

that is equivalent with the second inequality in (3.1).

Utilising the second part of the inequality (2.12) and under the assumption that $\|A - \lambda I\| \leq |\lambda|$ we can also state that

$$v_s\left(\bar{\lambda}A\right) \geq \frac{1}{2}\left[\|A\|^2 + \left(\sqrt{|\lambda|^2 - \|A - \lambda I\|^2}\right)^2\right]. \tag{3.4}$$

By the arithmetic mean-geometric mean inequality we have now:

$$\frac{1}{2}\left[\|A\|^2 + \left(\sqrt{|\lambda|^2 - \|A - \lambda I\|^2}\right)^2\right] \geq \|A\| \sqrt{|\lambda|^2 - \|A - \lambda I\|^2}, \tag{3.5}$$

which, together with (3.4) implies the first inequality in (3.2).

The second part of (3.2) follows from (v).

From the proof of Proposition 1 we can state that

$$\|Ax\|^2 + |\lambda|^2 \leq 2\operatorname{Re}\left\langle\bar{\lambda}Ax, x\right\rangle + r^2, \qquad \|x\| = 1 \tag{3.6}$$

where we denoted $r := \|A - \lambda I\| \leq |\lambda|$. We also observe, from (3.6), that $\operatorname{Re}\left\langle\bar{\lambda}Ax, x\right\rangle > 0$ for $x \in H$, $\|x\| = 1$.

Now, if we divide (3.6) by $\operatorname{Re}\left\langle\frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle > 0$, we get

$$\frac{\|Ax\|^2}{\operatorname{Re}\left\langle\frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} \leq 2|\lambda| + \frac{r^2}{\operatorname{Re}\left\langle\frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} - \frac{|\lambda|^2}{\operatorname{Re}\left\langle\frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} \quad \text{for} \quad \|x\| = 1. \tag{3.7}$$

If in this inequality we subtract from both sides the quantity $\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle$, then we get

$$\begin{aligned}
&\frac{\|Ax\|^2}{\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} - \operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle \\
&\leq 2|\lambda| + \frac{r^2 - |\lambda|^2}{\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} - \operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle \\
&= 2|\lambda| - \left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle}} - \sqrt{\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle} \right)^2 - 2\sqrt{|\lambda|^2 - r^2} \\
&\leq 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right),
\end{aligned}$$

which obviously implies that

$$\|Ax\|^2 \leq \left(\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle\right)^2 + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle \tag{3.8}$$

for any $x \in H$, $\|x\| = 1$.

Now, taking the supremum in (3.8) over $x \in H$, $\|x\| = 1$, we deduce the second inequality in (3.3). The other inequalities are obvious and the theorem is proved. ∎

The following lemma is of interest in itself.

Lemma 1. *For any $A \in B(H)$ and $\gamma, \Gamma \in \mathbb{K}$ we have:*

$$v_i\left[(A^* - \bar{\gamma}I)(\Gamma I - A)\right] = \frac{1}{4}|\Gamma - \gamma|^2 - \left\|A - \frac{\gamma + \Gamma}{2}I\right\|^2. \tag{3.9}$$

Proof. We observe that, for any $u, v, y \in H$ we have:

$$\operatorname{Re}\langle u - y, y - v\rangle = \frac{1}{4}\|u - v\|^2 - \left\|y - \frac{u + v}{2}\right\|^2. \tag{3.10}$$

Now, choosing $u = \Gamma x$, $y = Ax$, $v = \gamma x$ with $x \in H$, $\|x\| = 1$ we get

$$\operatorname{Re}\langle \Gamma x - Ax, Ax - \gamma x\rangle = \frac{1}{4}|\Gamma - \gamma|^2 - \left\|Ax - \frac{\gamma + \Gamma}{2}x\right\|^2,$$

giving

$$\inf_{\|x\|=1} \operatorname{Re}\langle (A^* - \bar{\gamma}I)(\Gamma I - A)x, x\rangle = \frac{1}{4}|\Gamma - \gamma|^2 - \sup_{\|x\|=1}\left\|Ax - \frac{\gamma + \Gamma}{2}x\right\|^2,$$

which is equivalent with (3.9). ∎

We recall that the bounded linear operator $B \in B(H)$ is called *strongly $m-$accretive* (with $m > 0$) if $\operatorname{Re}\langle By, y\rangle \geq m$ for any $y \in H$, $\|y\| = 1$. For $m = 0$ the operator is called *accretive.* In general, we then can call the operator *$m-$accretive* for $m \in [0, \infty)$.

The following result providing a characterisation for a class of operators that will be used in the sequel is incorporated in:

Lemma 2. *For $A \in B(H)$, $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq \gamma$ and $q \in \mathbb{R}$, the following statements are equivalent:*

(i) *The operator $(A^* - \bar{\gamma} I)(\Gamma I - A)$ is q^2-accretive;*

(ii) *We have the norm inequality:*

$$\left\| A - \frac{\gamma + \Gamma}{2} I \right\|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 - q^2. \tag{3.11}$$

The proof is obvious by Lemma 1 and the details are omitted.

Remark 2. *Since the self-adjoint operators B satisfying the condition $B \geq mI$ in the operator partial under "$\geq$", are $m-$accretive, then, a sufficient condition for $C_{\gamma,\Gamma}(A) := (A^* - \bar{\gamma} I)(\Gamma I - A)$ to be q^2-accretive is that $C_{\gamma,\Gamma}(A)$ is self-adjoint and $C_{\gamma,\Gamma}(A) \geq q^2 I$.*

Corollary 4. *Let $A \in B(H)$, $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq \pm\gamma$ and $q \in \mathbb{R}$. If the operator $C_{\gamma,\Gamma}(A)$ is q^2-accretive, then*

$$\begin{aligned} (0 \leq) \|A\| - w(A) &\leq \|A\| - v_s \left(\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} A \right) \\ &\leq \frac{1}{|\gamma + \Gamma|} \left[\frac{1}{4} |\Gamma - \gamma|^2 - q^2 \right]. \end{aligned} \tag{3.12}$$

Remark 3. *If M, m are positive real numbers with $M > m$ and the operator $C_{m,M}(A) = (A^* - mI)(MI - A)$ is q^2-accretive, then*

$$\begin{aligned} (0 \leq) \|A\| - w(A) &\leq \|A\| - v_s(A) \\ &\leq \frac{1}{M + m} \left[\frac{1}{4} (M - m)^2 - q^2 \right]. \end{aligned} \tag{3.13}$$

Remark 4. *We observe that for $q = 0$, i.e., if $C_{\gamma,\Gamma}(A)$ respectively $C_{m,M}(A)$ are accretive, then we obtain from (3.12) and (3.13) the inequality:*

$$\begin{aligned} (0 \leq) \|A\| - w(A) &\leq \|A\| - v_s \left(\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} A \right) \\ &\leq \frac{|\Gamma - \gamma|^2}{4 |\Gamma + \gamma|} \end{aligned} \tag{3.14}$$

and

$$(0 \leq) \|A\| - w(A) \leq \|A\| - v_s(A) \leq \frac{(M - m)^2}{4(M + m)} \tag{3.15}$$

respectively, which provide refinements of the corresponding inequalities (2.7) and (2.34) in [4].

Remark 5. *For any bounded linear operator A we know that $\frac{w(A)}{\|A\|} \geq \frac{1}{2}$, therefore (3.2) would produce a useful result only if*

$$\frac{1}{2} \leq \sqrt{1 - \left\| \frac{1}{\lambda} A - I \right\|^2},$$

which is equivalent with

$$\|A - \lambda I\| \leq \frac{\sqrt{3}}{2} |\lambda|. \tag{3.16}$$

In conclusion, for $A\in B(H)\setminus\{0\}$ and $\lambda\in\mathbb{C}\setminus\{0\}$ satisfying the condition (3.16), the inequality (3.2) provides a refinement of the classical result:

$$\frac{1}{2}\leq\frac{w(A)}{\|A\|},\qquad A\in B(H). \tag{3.17}$$

Corollary 5. *If $\|A-\lambda I\|\leq|\lambda|$, then we have*

$$\begin{aligned}(0\leq)\,\|A\|^2-w^2(A)&\leq\|A\|^2-v_s^2\left(\frac{\bar{\lambda}}{|\lambda|}A\right)\\&\leq\frac{\|A\|^2\|A-\lambda I\|^2}{|\lambda|^2}.\end{aligned}\tag{3.18}$$

The proof follows by the inequality (3.2). The details are omitted.

The following corollary providing a sufficient condition in terms of q^2-accretive property may be stated as well:

Corollary 6. *Let $A\in B(H)\setminus\{0\}$ and $\gamma,\Gamma\in\mathbb{K}$, $\Gamma\neq-\gamma$, $q\in\mathbb{R}$ so that* $\operatorname{Re}(\Gamma\bar{\gamma})+q^2\geq0$. *If $C_{\gamma,\Gamma}(A)$ is q^2-accretive, then*

$$\frac{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})+q^2}}{|\Gamma+\gamma|}\leq\frac{v_s\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right)}{\|A\|}\leq\frac{w(A)}{\|A\|}\,(\leq1)\tag{3.19}$$

and

$$\begin{aligned}(0\leq)\,\|A\|^2-w^2(A)&\leq\|A\|^2-v_s^2\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right)\\&\leq\frac{2\|A\|^2}{|\Gamma+\gamma|}\left[\frac{1}{4}|\Gamma-\gamma|^2-q^2\right]\\&\left(\leq\frac{\|A\|^2|\Gamma-\gamma|^2}{2|\Gamma+\gamma|}\right).\end{aligned}\tag{3.20}$$

Proof. By Lemma 2, the fact that $C_{\gamma,\Gamma}(A)$ is q^2-accretive implies that

$$\left\|A-\frac{\gamma+\Gamma}{2}I\right\|^2\leq\frac{1}{4}|\Gamma-\gamma|^2-q^2.\tag{3.21}$$

Since, obviously

$$\frac{1}{4}|\Gamma+\gamma|^2-\left[\frac{1}{4}|\Gamma-\gamma|^2-q^2\right]=\operatorname{Re}(\Gamma\bar{\gamma})+q^2\geq0,$$

hence $\left\|A-\frac{\gamma+\Gamma}{2}I\right\|\leq\frac{1}{2}|\Gamma+\gamma|$ and we can apply the inequality (3.2) for $\lambda=\frac{\gamma+\Gamma}{2}$ to get:

$$\frac{\sqrt{\left|\frac{\gamma+\Gamma}{2}\right|^2-\left\|A-\frac{\gamma+\Gamma}{2}I\right\|^2}}{\left|\frac{\gamma+\Gamma}{2}\right|}\leq\frac{v_s\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right)}{\|A\|},\tag{3.22}$$

and since, by (3.21),

$$\begin{aligned}\left|\frac{\gamma+\Gamma}{2}\right|^2-\left\|A-\frac{\gamma+\Gamma}{2}I\right\|^2&\geq\left|\frac{\gamma+\Gamma}{2}\right|^2-\frac{1}{4}|\Gamma-\gamma|^2+q^2\\&=\operatorname{Re}(\Gamma\bar{\gamma})+q^2,\end{aligned}$$

hence by (3.22) we deduce the desired inequality in (3.19).

The inequality (3.20) follows from (3.19). We omit the details. ∎

Remark 6. *If γ, Γ and q are such that $|\Gamma+\gamma| \leq 4\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})+q^2}$, then (3.19) will provide a refinement of the classical result (3.17).*

Remark 7. *If $M > m \geq 0$ and the operator $C_{m,M}(A)$ is q^2-accretive, then*

$$\frac{2\sqrt{Mm+q^2}}{m+M} \leq \frac{v_s(A)}{\|A\|} \leq \frac{w(A)}{\|A\|} \; (\leq 1) \tag{3.23}$$

and

$$\begin{aligned}(0 \leq) \|A\|^2 - w^2(A) &\leq \|A\|^2 - v_s^2(A) \\ &\leq \frac{2\|A\|^2}{m+M}\left[\frac{1}{4}(M-m)^2 - q^2\right].\end{aligned} \tag{3.24}$$

Remark 8. *We also observe that, for $q = 0$, i.e., if $C_{\gamma,\Gamma}(A)$ respectively $C_{m,M}(A)$ are accretive, then we obtain:*

$$\frac{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}}{|\Gamma+\gamma|} \leq \frac{v_s\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right)}{\|A\|} \leq \frac{w(A)}{\|A\|}, \tag{3.25}$$

$$\frac{2\sqrt{Mm}}{m+M} \leq \frac{v_s(A)}{\|A\|} \leq \frac{w(A)}{\|A\|}, \tag{3.26}$$

$$\begin{aligned}(0 \leq) \|A\|^2 - w^2(A) &\leq \|A\|^2 - v_s^2\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right) \\ &\leq \frac{\|A\|^2 |\Gamma-\gamma|^2}{2|\Gamma+\gamma|}\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}(0 \leq) \|A\|^2 - w^2(A) &\leq \|A\|^2 - v_s^2(A) \\ &\leq \frac{\|A\|^2 (M-m)^2}{2(m+M)}\end{aligned} \tag{3.28}$$

respectively, which provides refinements of the inequalities (2.17), (2.31) and (2.20) in [4], *respectively. The inequality between the first and the last term in (3.28) was not stated in* [4].

Corollary 7. *Let $A \in B(H)$, $\gamma, \Gamma \in \mathbb{K}$, $\Gamma \neq -\gamma$, $q \in \mathbb{R}$ so that $\operatorname{Re}(\Gamma\bar{\gamma}) + q^2 \geq 0$. If $C_{\gamma,\Gamma}(H)$ is q^2-accretive, then*

$$\begin{aligned}(0 \leq) \|A\|^2 - w^2(A) &\leq \|A\|^2 - v_s^2\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right) \\ &\leq \left(|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})+q^2}\right) v_s\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}A\right) \\ &\leq \left(|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})+q^2}\right) w(A).\end{aligned} \tag{3.29}$$

The proof follows by the last part of Theorem 2 on utilising a similar argument to the one employed in Corollary 6. The details are omitted.

Remark 9. *If $M > m \geq 0$ and the operator $C_{m,M}(A)$ is q^2-accretive, then*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \|A\|^2 - v_s^2(A) \leq \left(M + m - 2\sqrt{Mm + q^2}\right) v_s(A) \leq \left(M + m - 2\sqrt{Mm + q^2}\right) w(A). \tag{3.30}$$

Remark 10. *Finally, for $q = 0$, i.e., if $C_{\gamma,\Gamma}(A)$ respectively $C_{m,M}(A)$ are accretive, then we obtain from (3.29) and (3.30) some refinements of the inequalities (2.29) and (2.33) from* [4].

4. Reverse Inequalities in Terms of Numerical Radius

In Section 2 we established amongst others the following lower bound for the numerical radius $w(A)$

$$\left|v_{s(i)}(A)\right| \leq w(A) \tag{4.1}$$

for any A a bounded linear operator, where

$$v_{s(i)}(A) = \langle A, I\rangle_{s(i)} = \sup_{\|x\|=1}\left(\inf_{\|x\|=1}\right) \operatorname{Re}\langle Ax, x\rangle. \tag{4.2}$$

It is then a natural problem to investigate how far the left side of (4.1) is from the numerical radius $w(A)$.

We start with the following result:

Theorem 3. *For any $A \in B(H) \setminus \{0\}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ we have*

$$(0 \leq) w(A) - \left|v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)\right| \leq w(A) - v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \leq \frac{1}{2|\lambda|} w^2(A - \lambda I) \left(\leq \frac{1}{2|\lambda|}\|A - \lambda I\|^2\right). \tag{4.3}$$

Moreover, if $w(A - AI) \leq |\lambda|$, then we have:

$$\sqrt{1 - w^2\left(\frac{1}{\lambda}A - I\right)} \leq \frac{v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)}{w(A)} \leq \frac{\left|v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)\right|}{w(A)} (\leq 1) \tag{4.4}$$

and

$$(0 \leq) w^2(A) - v_s^2\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \leq 2\left(|\lambda| - \sqrt{|\lambda|^2 - w^2(A - \lambda I)}\right) v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \left(\leq 2\left(|\lambda| - \sqrt{|\lambda|^2 - w^2(A - \lambda I)}\right) w(A)\right), \tag{4.5}$$

respectively

Proof. From (4.1), we obviously have that

$$w(A) = w\left(\frac{\bar{\lambda}}{|\lambda|}A\right) \geq \left|v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)\right| \geq v_s\left(\frac{\bar{\lambda}}{|\lambda|}A\right)$$

for $A \in B(H)$ and $\lambda \in \mathbb{C}\setminus\{0\}$.

Now, utilising the inequality (2.17) and the elementary arithmetic-geometric mean inequality, we have

$$v_s\left(\bar{\lambda}A\right) \geq \frac{1}{2}\left[w^2(A) + |\lambda|^2\right] - \frac{1}{2}w^2(A-\lambda I)$$
$$\geq |\lambda|\, w(A) - \frac{1}{2}w^2(A-\lambda I)$$

which is clearly equivalent with the third inequality in (4.3).

Under the assumption that $w(A - AI) \leq |\lambda|$, on making use of (2.17) and the arithmetic mean-geometric mean inequality, we can also state that

$$v_s\left(\bar{\lambda}A\right) \geq \frac{1}{2}\left[w^2(A) + \left(\sqrt{|\lambda|^2 - w^2(A-\lambda I)}\right)^2\right]$$
$$\geq w(A)\sqrt{|\lambda|^2 - w^2(A-\lambda I)}\ ,$$

which is clearly equivalent to (4.4).

Let us denote $\rho := w(A-\lambda I) \leq |\lambda|$. Then for any $x \in H$, $\|x\| = 1$ we have

$$\rho^2 \geq |\langle Ax, x\rangle - \lambda|^2 = |\langle Ax, x\rangle|^2 - 2\operatorname{Re}\left[\bar{\lambda}\langle Ax, x\rangle\right] + |\lambda|^2$$

which yields that

$$|\langle Ax, x\rangle|^2 + |\lambda|^2 \leq 2\operatorname{Re}\left[\bar{\lambda}\langle Ax, x\rangle\right] + \rho^2 \tag{4.6}$$

for any $x \in H$, $\|x\| = 1$.

Making use of an argument similar to that in the proof of Theorem 2, we can get out of (4.6) the following inequality:

$$|\langle Ax, x\rangle|^2 \leq \left(\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle\right)^2 + 2\left(|\lambda| - \sqrt{|\lambda|^2 - \rho^2}\right)\operatorname{Re}\left\langle \frac{\bar{\lambda}}{|\lambda|}Ax, x\right\rangle \tag{4.7}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce the desired inequality in (4.5). ∎

Then following lemma is of interest in itself.

Lemma 3. *For any $A \in B(H)$ and $\gamma, \Gamma \in \mathbb{K}$ we have*

$$\inf_{\|x\|=1} \operatorname{Re}\left[\langle(\Gamma I - A)x, x\rangle\langle x, (A-\gamma I)x\rangle\right] = \frac{1}{4}|\Gamma-\gamma|^2 - w^2\left(A - \frac{\gamma+\Gamma}{2}I\right). \tag{4.8}$$

Proof. We observe that for any u, v, y complex numbers, we have the elementary identity:

$$\operatorname{Re}\left[(u-y)(\bar{y}-\bar{v})\right] = \frac{1}{4}|u-v|^2 - \left|y - \frac{u+v}{2}\right|^2. \tag{4.9}$$

If we choose in (4.9) $u = \Gamma$, $y = \langle Hx, x\rangle$ and $v = \gamma$ with $x \in H$, $\|x\| = 1$, then by (4.9) we have:

$$\operatorname{Re}\left[\langle(\Gamma I - A)x, x\rangle\langle x, (A-\gamma I)x\rangle\right] = \frac{1}{4}|\Gamma-\gamma|^2 - \left|\left\langle\left(A - \frac{\gamma+\Gamma}{2}I\right)x, x\right\rangle\right|^2 \tag{4.10}$$

for each $x \in H$, $\|x\| = 1$.

Now, taking the infimum over $\|x\| = 1$ in (4.10) we deduce the desired identity (4.8). ∎

Remark 11. *We observe that for any $x \in H$, $\|x\| = 1$ we have*

$$\begin{aligned}\mu(A;\gamma,\Gamma)(x) &:= \operatorname{Re}\left[\langle(\Gamma I - A)x, x\rangle \langle x, (A-\gamma I)x\rangle\right] \\ &= (\operatorname{Re}\Gamma - \operatorname{Re}\langle Ax,x\rangle)(\operatorname{Re}\langle Ax,x\rangle - \operatorname{Re}\gamma) \\ &\quad + (\operatorname{Im}\Gamma - \operatorname{Im}\langle Ax,x\rangle)(\operatorname{Im}\langle Ax,x\rangle - \operatorname{Im}\gamma)\end{aligned}$$

and therefore a sufficient condition for $\mu(A;\gamma,\Gamma)(x)$ to be nonnegative for each $x \in H$, $\|x\| = 1$ is that:

$$\left\{\begin{array}{l}\operatorname{Re}\Gamma \geq \operatorname{Re}\langle Ax,x\rangle \geq \operatorname{Re}\gamma \\ \\ \operatorname{Im}\Gamma \geq \operatorname{Im}\langle Ax,x\rangle \geq \operatorname{Im}\gamma\end{array}\right., \quad x \in H,\ \|x\| = 1. \tag{4.11}$$

Now, if we denote by $\mu_i(A;\gamma,\Gamma) := \inf_{\|x\|=1} \mu(A;\gamma,\Gamma)(x)$, then we can state the following lemma.

Lemma 4. *For $A \in B(H)$, $\phi, \Phi \in \mathbb{K}$, the following statements are equivalent:*

(i) $\mu_i(A;\phi;\Phi) \geq 0$;
(ii) $w\left(A - \frac{\phi+\Phi}{2} I\right) \leq \frac{1}{2}|\Phi - \phi|$.

Utilising the above results we can provide now some particular reverse inequalities that are of interest.

Corollary 8. *Let $A \in B(H)$ and $\phi, \Phi \in \mathbb{K}$ with $\Phi \neq \pm\phi$ such that either (i) or (ii) of Lemma 4 holds true. Then*

$$\begin{aligned}(0 \leq)\, w(A) - \left|v_s\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right)\right| &\leq w(A) - v_s\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right) \\ &\leq \frac{1}{4}\cdot\frac{|\Phi-\phi|^2}{|\Phi+\phi|}.\end{aligned} \tag{4.12}$$

Remark 12. *If $N > n > 0$ are such that either $\mu_i(A;n,N) \geq 0$ or $w\left(A - \frac{n+N}{2}I\right) \leq \frac{1}{2}(N-n)$ for a given operator $A \in B(A)$, then*

$$(0 \leq) w(A) - |v_s(A)| \leq w(A) - v_s(A) \leq \frac{1}{4}\cdot\frac{(N-n)^2}{N+n}. \tag{4.13}$$

From a different perspective, we can state the following multiplicative reverse of the inequality (4.1).

An equivalent additive version of (4.4) is incorporated in the following:

Corollary 9. *If $w(A - \lambda I) \leq |\lambda|$, then we have*

$$\begin{aligned}(0 \leq) w^2(A) - w_s^2\left(\frac{\bar{\lambda}}{|\lambda|}A\right) &\leq \frac{w^2(A)\, w^2(A-\lambda I)}{|\lambda|^2} \\ &\left(\leq \left\{\begin{array}{c}\frac{\|A\|^2 w^2(A-\lambda I)}{|\lambda|^2} \\ \\ \frac{w^2(A)\|A-\lambda I\|^2}{|\lambda|^2}\end{array}\right. \leq \frac{\|A\|^2\|A-\lambda I\|^2}{|\lambda|^2}\right).\end{aligned} \tag{4.14}$$

In applications, the following variant of (4.4) can be perhaps more convenient:

Corollary 10. *Let $A \in B(H) \setminus \{0\}$ and $\phi, \Phi \in \mathbb{K}$ with $\Phi \neq -\phi$. If $\operatorname{Re}\left(\Phi\bar{\phi}\right) > 0$ and either the statement (i) or equivalently (ii) from Lemma 4 holds true, then:*

$$\frac{2\sqrt{\operatorname{Re}\left(\Phi\bar{\phi}\right)}}{|\phi+\Phi|} \leq \frac{v_s\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right)}{w(A)} \leq \frac{\left|v_s\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right)\right|}{w(A)} (\leq 1) \tag{4.15}$$

and

$$\begin{aligned}(0 \leq) & w^2(A) - v_s^2\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right) \\ & \leq \frac{1}{4}\cdot\frac{|\Phi-\phi|^2}{\operatorname{Re}\left(\Phi\bar{\phi}\right)} v_s^2\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right) \\ & \left(\leq \frac{1}{4}\cdot\frac{|\Phi-\phi|^2}{\operatorname{Re}\left(\Phi\bar{\phi}\right)} w^2(A) \leq \frac{1}{4}\cdot\frac{|\Phi-\phi|^2}{\operatorname{Re}\left(\Phi\bar{\phi}\right)} \|A\|^2\right).\end{aligned} \tag{4.16}$$

The proof follows by Theorem 3 on utilising a similar argument to the one incorporated in the proof of Corollary 6 and the details are omitted.

Remark 13. *If $N > n > 0$ are such that either $\mu_i(A; n, N) \geq 0$ or, equivalently*

$$w\left(A - \frac{n+N}{2}\right) \leq \frac{1}{2}(N-n), \tag{4.17}$$

then

$$\frac{2\sqrt{nN}}{n+N} \leq \frac{v_s(A)}{w(A)} (\leq 1), \tag{4.18}$$

$$(0 \leq) w(A) - v_s(A) \leq \frac{\left(\sqrt{N}-\sqrt{n}\right)^2}{2\sqrt{nN}} v_s(A) \left(\leq \frac{\left(\sqrt{N}-\sqrt{n}\right)^2}{2\sqrt{nN}} w(A)\right) \tag{4.19}$$

and

$$(0 \leq) w^2(A) - v_s^2(A) \leq \frac{(N-n)^2}{4nN} v_s^2(A) \left(\leq \frac{(N-n)^2}{4nN} w^2(A)\right). \tag{4.20}$$

Finally, we can state the following result as well:

Corollary 11. *Let $A \in B(H)$, $\phi, \Phi \in \mathbb{K}$ such that $\operatorname{Re}\left(\Phi\bar{\phi}\right) > 0$. If either $\mu_i(A; \phi, \Phi) \geq 0$ or, equivalently*

$$w\left(A - \frac{\Phi+\phi}{2}I\right) \leq \frac{1}{2}|\Phi-\phi|,$$

then

$$\begin{aligned}(0 \leq) & w^2(A) - v_s^2\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right) \\ & \leq \left[|\phi+\Phi| - 2\sqrt{\operatorname{Re}\left(\Phi\bar{\phi}\right)}\right] v_s\left(\frac{\bar{\phi}+\bar{\Phi}}{|\phi+\Phi|}A\right) \\ & \left(\leq \left[|\phi+\Phi| - 2\sqrt{\operatorname{Re}\left(\Phi\bar{\phi}\right)}\right] w(A)\right).\end{aligned} \tag{4.21}$$

Remark 14. *If $N > n > 0$ are as in Remark 13, then we have the inequality:*

$$(0 \leq) w^2(A) - v_s^2(A) \leq \left(\sqrt{N} - \sqrt{n}\right)^2 v_s(A) \left(\leq \left(\sqrt{N} - \sqrt{n}\right)^2 w(A)\right). \tag{4.22}$$

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