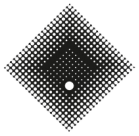
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ON SOME INEQUALITIES IN NORMED ALGEBRAS

S.S. DRAGOMIR

Abstract. Some inequalities in normed algebras that provides lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$ are obtained. Applications for estimating the quantities $\left\|\|x^{-1}\| x \pm \|y^{-1}\| y\right\|$ and $\left\|\|y^{-1}\| x \pm \|x^{-1}\| y\right\|$ for invertible elements x, y in unital normed algebras are also given.

1. Introduction

In [1], in order to provide a generalisation of a norm inequality for n vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

$$\begin{aligned}(1.1)\quad &\max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n} x_j\right\| - \sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\}\\ &\leq \left\|\sum_{j=1}^{n}\alpha_j x_j\right\| \leq \min_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\},\end{aligned}$$

where x_j, $j\in\{1,\dots,n\}$ are vectors in the normed linear space $(X,\|\cdot\|)$ over $\mathbb{K}$ while α_j, $j\in\{1,\dots,n\}$ are scalars in $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$.

For $\alpha_k = \frac{1}{\|x_k\|}$, with $x_k\neq 0$, $k\in\{1,\dots,n\}$ the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$\begin{aligned}(1.2)\quad &\max_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n} x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|\right]\right\}\\ &\leq \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \leq \min_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|\right]\right\},\end{aligned}$$

which implies the following refinement and reverse of the generalised triangle inequality due to M. Kato et al. [3]:

$$(1.3)\quad \min_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]$$

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$$\leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leq \max_{k\in\{1,\dots,n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right].$$

The other natural choice, $\alpha_k = \|x_k\|$, $k \in \{1,\dots,n\}$ in (1.1) produces the result

$$\begin{aligned} (1.4) \quad & \max_{k\in\{1,\dots,n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \, \|x_j\| \right\} \\ & \leq \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \, \|x_j\| \right\}, \end{aligned}$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

$$\begin{aligned} (1.5) \quad & (0 \leq) \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\|}{\max\limits_{k\in\{1,\dots,n\}} \{\|x_k\|\}} \\ & \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leq \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\|}{\min\limits_{k\in\{1,\dots,n\}} \{\|x_k\|\}}, \end{aligned}$$

provided $x_k \neq 0$, $k \in \{1,\dots,n\}$.

In [2], the authors have shown that the case $n = 2$ in (1.2) produces the *Maligranda-Mercer inequality*:

$$(1.6) \quad \frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}},$$

for any $x, y \in X \setminus \{0\}$.

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

$$\begin{aligned} (1.7) \quad & (0 \leq) \frac{\|x-y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \\ & \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x-y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}}, \end{aligned}$$

for any $x, y \in X \setminus \{0\}$.

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$, where $a_j, x_j, j \in \{1,\dots,n\}$ are elements in a normed algebra $(A, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$. In the case where $(A, \|\cdot\|)$ is a unital algebra and x, y are invertible, lower and upper bounds for the quantities

$$\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \quad \text{and} \quad \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$$

are provided as well.

2. Inequalities for n Pairs of Elements

Let $(A, \|\cdot\|)$ be a normed algebra over the real or complex number field $\mathbb{K}$.

Theorem 1. *If* $(a_j, x_j) \in A^2$, $j \in \{1, \dots, n\}$, *then*

$$
\begin{aligned}
(2.1) \qquad & \max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\} \\
& \leq \max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \right\} \\
& \leq \left\| \sum_{j=1}^{n} a_j x_j \right\| \\
& \leq \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \right\} \\
& \leq \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\}.
\end{aligned}
$$

Proof. Observe that for any $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} (a_j - a_k)\, x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra properties, we have

$$
\begin{aligned}
\left\| \sum_{j=1}^{n} a_j x_j \right\| &\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \left\| \sum_{j=1}^{n} (a_j - a_k)\, x_j \right\| \\
&\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \\
&\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\|,
\end{aligned}
$$

for any $k \in \{1, \dots, n\}$, which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) - \sum_{j=1}^{n} (a_k - a_j)\, x_j$$

and utilising the continuity of the norm, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}a_jx_j\right\| &\geq \left|\left\|a_k\left(\sum_{j=1}^{n}x_j\right)-\sum_{j=1}^{n}\left(a_k-a_j\right)x_j\right\|\right| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\left\|\sum_{j=1}^{n}\left(a_k-a_j\right)x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|\left(a_k-a_j\right)x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|a_k-a_j\right\|\left\|x_j\right\|\end{aligned}$$

for any $k\in\{1,\dots,n\}$, which implies the first part in (2.1). ∎

Remark 1. *If there exists $r>0$ so that $\|a_j-a_k\|\leq r\|a_k\|$ for any $j,k\in\{1,\dots,n\}$, then, by the second part of (2.1), we have*

$$\left\|\sum_{j=1}^{n}a_jx_j\right\|\leq\min_{k\in\{1,\dots,n\}}\left\{\|a_k\|\right\}\left[\left\|\sum_{j=1}^{n}x_j\right\|+r\sum_{j=1}^{n}\|x_j\|\right]. \tag{2.2}$$

Corollary 1. *If $x_j\in A$, $j\in\{1,\dots,n\}$, then*

$$\begin{aligned}&\max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|x_j-x_k\right\|\left\|x_j\right\|\right\} \\ &\leq\max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|\left(x_j-x_k\right)x_j\right\|\right\}\leq\left\|\sum_{j=1}^{n}x_j^2\right\| \\ &\leq\min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|+\sum_{j=1}^{n}\left\|\left(x_j-x_k\right)x_j\right\|\right\} \\ &\leq\min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|+\sum_{j=1}^{n}\left\|x_j-x_k\right\|\left\|x_j\right\|\right\}.\end{aligned} \tag{2.3}$$

Corollary 2. *Assume that A is a unital normed algebra. If $x_j\in A$ are invertible for any $j\in\{1,\dots,n\}$, then*

$$\begin{aligned}&\min_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\right] \\ &\leq\sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\|-\left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\|x_j\right\| \\ &\leq\max_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\right].\end{aligned} \tag{2.4}$$

Proof. If $1 \in A$ is the unity, then on choosing $a_k = \left\|x_k^{-1}\right\| \cdot 1$ in (2.1) we get

$$\begin{aligned}(2.5)\qquad &\max_{k\in\{1,\dots,n\}} \left\{ \left\|x_k^{-1}\right\| \left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n \left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right| \|x_j\| \right\} \\ &\leq \left\|\sum_{j=1}^n \left\|x_j^{-1}\right\| x_j\right\| \\ &\leq \min_{k\in\{1,\dots,n\}} \left\{ \left\|x_k^{-1}\right\| \left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n \left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right| \|x_j\| \right\}.\end{aligned}$$

Now, assume that $\min\limits_{k\in\{1,\dots,n\}} \left\{\left\|x_k^{-1}\right\|\right\} = \left\|x_{k_0}^{-1}\right\|$. Then

$$\begin{aligned}&\left\|x_{k_0}^{-1}\right\| \left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n \left|\left\|x_j^{-1}\right\| - \left\|x_{k_0}^{-1}\right\|\right| \|x_j\| \\ &= -\left\|x_{k_0}^{-1}\right\| \left(\sum_{j=1}^n \|x_j\| - \left\|\sum_{j=1}^n x_j\right\|\right) + \sum_{j=1}^n \left\|x_j^{-1}\right\| \|x_j\|.\end{aligned}$$

Utilising the second inequality in (2.5), we deduce

$$\left\|x_{k_0}^{-1}\right\| \left(\sum_{j=1}^n \|x_j\| - \left\|\sum_{j=1}^n x_j\right\|\right) \leq \sum_{j=1}^n \left\|x_j^{-1}\right\| \|x_j\| - \left\|\sum_{j=1}^n \left\|x_j^{-1}\right\| x_j\right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. ∎

Remark 2. *An equivalent form of (2.4) is:*

$$\begin{aligned}(2.6)\qquad &\frac{\sum_{j=1}^n \left\|x_j^{-1}\right\| \|x_j\| - \left\|\sum_{j=1}^n \left\|x_j^{-1}\right\| x_j\right\|}{\max\limits_{k\in\{1,\dots,n\}} \left\|x_k^{-1}\right\|} \\ &\leq \sum_{j=1}^n \|x_j\| - \left\|\sum_{j=1}^n x_j\right\| \leq \frac{\sum_{j=1}^n \left\|x_j^{-1}\right\| \|x_j\| - \left\|\sum_{j=1}^n \left\|x_j^{-1}\right\| x_j\right\|}{\min\limits_{k\in\{1,\dots,n\}} \left\|x_k^{-1}\right\|},\end{aligned}$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.

3. Inequalities for Two Pairs of Elements

The following particular case of Theorem 1 is of interest for applications.

Lemma 1. *If $(a,b), (x,y) \in A^2$, then*

$$\begin{aligned}(3.1)\qquad &\max\left\{\|a(x\pm y)\| - \|(b-a)y\|, \|b(x\pm y)\| - \|(b-a)x\|\right\} \\ &\leq \|ax \pm by\| \leq \min\left\{\|a(x\pm y)\| + \|(b-a)y\|, \|b(x\pm y)\| + \|(b-a)x\|\right\}\end{aligned}$$

or, equivalently,

$$
\begin{aligned}
(3.2)\qquad & \frac{1}{2}\left\{\|a\left(x\pm y\right)\|+\|b\left(x\pm y\right)\|-\left[\|\left(b-a\right)y\|+\|\left(b-a\right)x\|\right]\right\} \\
& \quad +\frac{1}{2}\left|\|a\left(x\pm y\right)\|-\|b\left(x\pm y\right)\|+\|\left(b-a\right)y\|-\|\left(b-a\right)x\|\right| \\
& \leq \|ax\pm by\| \\
& \leq \frac{1}{2}\left\{\|a\left(x\pm y\right)\|+\|b\left(x\pm y\right)\|+\left[\|\left(b-a\right)y\|+\|\left(b-a\right)x\|\right]\right\} \\
& \quad -\frac{1}{2}\left|\|a\left(x\pm y\right)\|+\|b\left(x\pm y\right)\|-\|\left(b-a\right)y\|-\|\left(b-a\right)x\|\right|.
\end{aligned}
$$

Proof. The inequality (3.1) follows from Theorem 1 for $n=2$, $a_1=a$, $a_2=b$, $x_1=x$ and $x_2=\pm y$.

Utilising the properties of real numbers,

$$\min\left\{\alpha,\beta\right\}=\frac{1}{2}\left[\alpha+\beta-\left|\alpha-\beta\right|\right],\quad \max\left\{\alpha,\beta\right\}=\frac{1}{2}\left[\alpha+\beta+\left|\alpha-\beta\right|\right];\quad \alpha,\beta\in\mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). ∎

The following result contains some upper bounds for $\|ax\pm by\|$ that are perhaps more useful for applications.

Theorem 2. *If* $(a,b),(x,y)\in A^2$, *then*

$$
\begin{aligned}
(3.3)\qquad \|ax\pm by\| & \leq \min\left\{\|a\left(x\pm y\right)\|,\|b\left(x\pm y\right)\|\right\}+\|b-a\|\max\left\{\|x\|,\|y\|\right\} \\
& \leq \|x\pm y\|\min\left\{\|a\|,\|b\|\right\}+\|b-a\|\max\left\{\|x\|,\|y\|\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\qquad \|ax\pm by\| & \leq \|x\pm y\|\max\left\{\|a\|,\|b\|\right\}+\min\left\{\|\left(b-a\right)x\|,\|\left(b-a\right)y\|\right\} \\
& \leq \|x\pm y\|\max\left\{\|a\|,\|b\|\right\}+\|b-a\|\min\left\{\|x\|,\|y\|\right\}.
\end{aligned}
$$

Proof. Observe that $\|\left(b-a\right)x\|\leq\|b-a\|\,\|x\|$ and $\|\left(b-a\right)y\|\leq\|b-a\|\,\|y\|$, and then

$$\|\left(b-a\right)x\|,\|\left(b-a\right)y\|\leq\|b-a\|\max\left\{\|x\|,\|y\|\right\},\tag{3.5}$$

which implies that

$$
\begin{aligned}
& \min\left\{\|a\left(x\pm y\right)\|+\|\left(b-a\right)y\|,\|b\left(x\pm y\right)\|+\|\left(b-a\right)x\|\right\} \\
& \leq \min\left\{\|a\left(x\pm y\right)\|,\|b\left(x\pm y\right)\|\right\}+\|b-a\|\max\left\{\|x\|,\|y\|\right\} \\
& \leq \|x\pm y\|\min\left\{\|a\|,\|b\|\right\}+\|b-a\|\max\left\{\|x\|,\|y\|\right\}.
\end{aligned}
$$

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since $\|a\left(x\pm y\right)\|\leq\|a\|\,\|x\pm y\|$ and $\|b\left(x\pm y\right)\|\leq\|b\|\,\|x\pm y\|$, hence

$$\|a\left(x\pm y\right)\|,\|b\left(x\pm y\right)\|\leq\|x\pm y\|\max\left\{\|a\|,\|b\|\right\},$$

which implies that

$$
\begin{aligned}
& \min\left\{\|a\left(x\pm y\right)\|+\|\left(b-a\right)y\|,\|b\left(x\pm y\right)\|+\|\left(b-a\right)x\|\right\} \\
& \leq \|x\pm y\|\max\left\{\|a\|,\|b\|\right\}+\min\left\{\|\left(b-a\right)x\|,\|\left(b-a\right)y\|\right\} \\
& \leq \|x\pm y\|\max\left\{\|a\|,\|b\|\right\}+\|b-a\|\min\left\{\|x\|,\|y\|\right\},
\end{aligned}
$$

and the inequality (3.4) is also proved. ∎

The following corollary may be more useful for applications.

Corollary 3. *If* $(a,b),(x,y)\in A^2$, *then*

$$\|ax\pm by\|\leq\|x\pm y\|\cdot\frac{\|a\|+\|b\|}{2}+\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}. \tag{3.6}$$

Proof. Follows from Theorem 2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that $\min\{\alpha,\beta\}+\max\{\alpha,\beta\}=\alpha+\beta$, $\alpha,\beta\in\mathbb{R}$. ∎

The following lower bounds for $\|ax\pm by\|$ can be stated as well:

Theorem 3. *For any* (a,b) *and* $(x,y)\in A^2$, *we have:*

$$\begin{aligned}&\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\leq\max\{\|a(x\pm y)\|,\|b(x\pm y)\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\leq\|ax\pm by\|\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}\\&\leq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\&\leq\|ax\pm by\|.\end{aligned} \tag{3.8}$$

Proof. Observe that, by (3.5) we have that

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\&\geq\max\{\|ax\pm ay\|,\|bx\pm by\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\geq\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x\pm y)\|,\|b(x\pm y)\|\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\},$$

then

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\&\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\&\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}.\end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). ∎

Corollary 4. *For any* $(a,b),(x,y)\in A^2$, *we have*

$$\frac{1}{2}\cdot[|\|ax\|-\|ay\||+|\|bx\|-\|by\||]-\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}\leq\|ax\pm by\|. \tag{3.9}$$

The proof follows from Theorem 3 by adding (3.7) to (3.8). The details are omitted.

4. Applications for Two Invertible Elements

In this section we assume that A is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity $\left|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right|$, where x and y are invertible in A.

Proposition 1. *If $(x,y)\in A^2$ are invertible, then*

$$\begin{aligned}(4.1)\quad &\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\| \\ &\leq\|x\pm y\|\min\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\max\left\{\|x\|,\|y\|\right\}\end{aligned}$$

and

$$\begin{aligned}(4.2)\quad &\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\| \\ &\leq\|x\pm y\|\max\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\min\left\{\|x\|,\|y\|\right\}.\end{aligned}$$

Proof. Follows by Theorem 2 on choosing $a=\left\|x^{-1}\right\|\cdot 1$ and $b=\left\|y^{-1}\right\|\cdot 1$. ∎

Corollary 5. *With the above assumption for x and y, we have*

$$\begin{aligned}(4.3)\quad &\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\| \\ &\leq\|x\pm y\|\cdot\frac{\left\|x^{-1}\right\|+\left\|y^{-1}\right\|}{2}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\cdot\frac{\|x\|+\|y\|}{2}.\end{aligned}$$

Lower bounds for $\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|$ are provided below:

Proposition 2. *If $(x,y)\in A^2$ are invertible, then*

$$\begin{aligned}(4.4)\quad &\|x\pm y\|\max\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\max\left\{\|x\|,\|y\|\right\} \\ &\leq\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|\end{aligned}$$

and

$$\begin{aligned}(4.5)\quad &\|x\pm y\|\min\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\min\left\{\|x\|,\|y\|\right\} \\ &\leq\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|.\end{aligned}$$

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing $a=\left\|x^{-1}\right\|\cdot 1$ and $b=\left\|y^{-1}\right\|\cdot 1$.

We know from the proof of Theorem 3 that

$$(4.6)\quad \max\left\{\|a\left(x\pm y\right)\|-\|\left(b-a\right)y\|,\|b\left(x\pm y\right)\|-\|\left(b-a\right)x\|\right\}\leq\|ax\pm by\|.$$

If in this inequality we choose $a=\left\|x^{-1}\right\|\cdot 1$ and $b=\left\|y^{-1}\right\|\cdot 1$, then we get

$$\begin{aligned}&\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\| \\ &\geq\max\left\{\left\|x^{-1}\right\|\|x\pm y\|-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\|y\|,\left\|y^{-1}\right\|\|x\pm y\|-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\|x\|\right\} \\ &\geq\|x\pm y\|\min\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\min\left\{\|x\|,\|y\|\right\}\end{aligned}$$

and the inequality (4.5) is obtained. ∎

Corollary 6. *If $(x,y)\in A^2$ are invertible, then*

$$\begin{aligned}(4.7)\quad &\|x\pm y\|\cdot\frac{\left\|x^{-1}\right\|+\left\|y^{-1}\right\|}{2}-\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\cdot\frac{\|x\|+\|y\|}{2} \\ &\leq\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|.\end{aligned}$$

Remark 3. *We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:*

$$\left|\left\|\left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y\right\| - \|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2}\right| \leq \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.8}$$

Now we consider the dual expansion $\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|$, for which the following upper bounds can be stated.

Proposition 3. *If (x, y) are invertible in A, then*

$$\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| \leq \|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} + \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \max\left\{\|x\|, \|y\|\right\} \tag{4.9}$$

and

$$\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| \leq \|x \pm y\| \max\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} + \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \min\left\{\|x\|, \|y\|\right\}. \tag{4.10}$$

In particular,

$$\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| \leq \|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2} + \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.11}$$

The proof follows from Theorem 2 on choosing $a = \left\|y^{-1}\right\| \cdot 1$ and $b = \left\|x^{-1}\right\| \cdot 1$.

The lower bounds for the quantity $\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|$ are incorporated in:

Proposition 4. *If (x, y) are invertible in A, then*

$$\|x \pm y\| \max\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \max\left\{\|x\|, \|y\|\right\} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| \tag{4.12}$$

and

$$\|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \min\left\{\|x\|, \|y\|\right\} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|. \tag{4.13}$$

In particular,

$$\|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|. \tag{4.14}$$

Remark 4. *We observe that the inequalities (4.11) and (4.14) are equivalent with*

$$\left|\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| - \|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2}\right| \leq \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.15}$$

References

[1] S.S. DRAGOMIR, A generalisation of the Pečarić-Rajić inequality in normed linear spaces, Preprint. *RGMIA Res. Rep. Coll.*, **10**(3) (2007), Art. 3. [ONLINE: http://rgmia.vu.edu.au/v10n3.html].

[2] J. PEČARIĆ and R. RAJIĆ, The Dunkl-Williams inequality with n elements in normed linear spaces, *Math. Ineq. & Appl.,* **10**(2) (2007), 461-470.

[3] M. KATO, K.-S. SAITO and T. TAMURA, Sharp triangle inequality and its reverses in Banach spaces, *Math. Ineq. & Appl.,* **10**(3) (2007).

[4] P.R. MERCER, The Dunkl-Williams inequality in an inner product space, *Math. Ineq. & Appl.,* **10**(2) (2007), 447-450.

[5] L. MALIGRANDA, Simple norm inequalities, *Amer. Math. Monthly*, **113** (2006), 256-260.

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