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A CLASS OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND THE BEST BOUNDS IN THE SECOND KERSHAW’S DOUBLE INEQUALITY

FENG QI AND BAI-NI GUO

Abstract. In the article, the sufficient and necessary conditions such that a class of functions which involve the psi function \( \psi \) and the ratio \( \frac{\Gamma(x+t)}{\Gamma(x+s)} \) are logarithmically completely monotonic are established, the best bounds for the ratio \( \frac{\Gamma(x+t)}{\Gamma(x+s)} \) are given, and some comparisons with known results are carried out, where \( s \) and \( t \) are two real numbers and \( x > -\min\{s, t\} \).

1. Introduction

Recall \([30, 32, 52, 54]\) that a function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) such that \((−1)^k f^{(k)}(x) \geq 0\) for \( x \in I \) and \( k \geq 0 \). Recall also \([3, 30, 39, 41, 42, 43]\) that a positive function \( f \) is said to be logarithmically completely monotonic on an interval \( I \) if its logarithm \( \ln f \) satisfies \((−1)^k \ln f^{(k)}(x) \geq 0\) for \( k \in \mathbb{N} \) on \( I \). For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on \( I \) are denoted by \( C[I] \) and \( C_L[I] \) respectively.

The famous Bernstein-Widder’s Theorem \([54, \text{p. 161}]\) states that \( f \in C((0, \infty)) \) if and only if there exists a bounded and nondecreasing function \( \mu(t) \) such that
\[
f(x) = \int_0^\infty e^{-xt} \, d\mu(t)
\] converges for \( 0 < x < \infty \).

In \([5, 29, 39, 41, 42, 43, 52]\) and many other references, the inclusions \( C_L[I] \subset C[I] \) and \( S \subset C_L[(0, \infty)] \) were revealed implicitly or explicitly, where \( S \) denotes the class of Stieltjes transforms \([5, 54]\). There are three different proofs in \([5, 39, 41]\) and \([29, 45]\) for the inclusion \( C_L[I] \subset C[I] \). The class \( C_L[(0, \infty)] \) is characterized in \([5, \text{Theorem 1.1}]\) implicitly and in \([19, \text{Theorem 4.4}]\) explicitly: \( f \in C_L[(0, \infty)] \iff f^\alpha \in C \) for all \( \alpha > 0 \iff \psi f \in C \) for all \( n \in \mathbb{N} \). In other words, the functions in \( C_L[(0, \infty)] \) are those completely monotonic functions for which the representing measure \( \mu \) in \((1)\) is infinitely divisible in the convolution sense: For each \( n \in \mathbb{N} \) there exists a positive measure \( \nu \) on \([0, \infty)\) with \( n\)-th convolution power equal to \( \mu \).

By the way, recall \([30, 32, 48, 52, 54]\) that a function \( f \) is said to be absolutely monotonic on an interval \( I \) if it has derivatives of all orders and \( f^{(k-1)}(t) \geq 0 \) for \( t \in I \) and \( k \in \mathbb{N} \). In \([29, 45]\), it was defined that a positive function \( f \) is said
to be logarithmically absolutely monotonic on an interval $I$ if it has derivatives of all orders and $[\ln f(t)]^{(k)} \geq 0$ for $t \in I$ and $k \in \mathbb{N}$ and it was showed that a logarithmically absolutely monotonic function on an interval $I$ is also absolutely monotonic on $I$, but not conversely.

In recent years, the logarithmically completely monotonic functions and their properties have been investigated extensively and explicitly in [3, 5, 9, 10, 11, 16, 17, 18, 23, 25, 26, 29, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 50] and the references therein.

Let $\Gamma$ and $\psi = \Gamma'$ stand for the classical Euler’s gamma function and the psi function respectively. The first and second Kershaw’s inequalities [21] state that

$$\left( x + \frac{s}{2} \right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left( x - \frac{1}{2} + \sqrt{s + \frac{1}{4}} \right)^{1-s}$$

(2)

and

$$\exp \left[ (1-s)\psi(x + \sqrt{s}) \right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp \left[ (1-s)\psi\left( x + \frac{s+1}{2} \right) \right]$$

(3)

for $s \in (0, 1)$ and $x \geq 1$. There have been a lot of literature on these two double inequalities, for example, [4, 5, 12, 13, 14, 20, 21, 22, 23, 25, 26, 27, 28, 33, 34, 36, 38, 44, 53] and the references therein.

For real numbers $a, b, c$ and $\rho = \min\{a, b, c\}$, let $H_{a,b,c}(x) = (x+\rho)^{b-a}\frac{\Gamma(x+\rho)}{\Gamma(x+a)}$ in $(-\rho, \infty)$. Recently, the following sufficient and necessary conditions are established elegantly in [37]: $H_{a,b,c}(x) \in C_{\mathbb{C}}[-\rho, \infty]$ if and only if $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \geq 0\} \cap \{(a, b, c) : (a-b)[a-b-a-b+2c] \geq 0\} \setminus \{(a, b, c) : c = b+1\}$ and $H_{b,c,a}(x) \in C_{\mathbb{C}}[-\rho, \infty]$ if and only if $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \leq 0\} \cap \{(a, b, c) : (a-b)[a-b-a-b+2c] \leq 0\} \setminus \{(a, b, c) : c = b+1\}$. These conclusions can be used to extend, generalize, refine and sharpen [25, Theorem 1], inequality (2) and some other known results.

It is easy to see that inequality (3) can be rewritten for $s \in (0, 1)$ and $x \geq 1$ as

$$\exp \left[ \psi\left( x + \sqrt{s} \right) \right] < \left[ \frac{\Gamma(x+1)}{\Gamma(x+s)} \right]^{1/(1-s)} < \exp \left[ \psi\left( x + \frac{s+1}{2} \right) \right].$$

(4)

Now it is natural to ask: What are the best constants $\delta_1(s, t)$ and $\delta_2(s, t)$ such that

$$\exp[\psi(x + \delta_1(s, t))] \leq \left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \leq \exp[\psi(x + \delta_2(s, t))]$$

(5)

holds for $x > -\min\{s, t, \delta_1(s, t), \delta_2(s, t)\}$, where $s$ and $t$ are two real numbers? In order to give an answer to this problem, we would like to establish the logarithmically complete monotonicity of the function

$$\nu_{s,t}(x) = \frac{1}{\exp[\psi(x + \Theta(s, t))]} \left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}.$$

(6)

Our first main result is the following Theorem 1

**Theorem 1.** Let $s$ and $t$ be two real numbers with $s \neq t$ and $\theta(s, t)$ a constant depending on $s$ and $t$.

1. If $\theta(s, t) \leq \min\{s, t\}$, then $\nu_{s,t}(x) \in C_{\mathbb{C}}[-\theta(s, t), \infty]$. 

In order to refine and extend the first Kershaw’s double inequality (2), the logarithmically complete monotonicity of the function \((x + c)^{b-a} \frac{\Gamma(x+s)}{\Gamma(x+a)}\) for \(x \in (-\rho, \infty)\) was studied in [25], where \(a, b\) and \(c\) are real numbers and \(\rho = \min\{a, b, c\}\).

It is clear that inequality (7) extends the ranges of variables of the right hand side inequality in (4) which is a rearranged form of (3).

Taking \(t = \delta = 1\) and \(s \in (0, 1)\) in (8) gives
\[
\frac{\Gamma(x+1)}{\Gamma(x+s)} \geq \frac{1}{\Gamma(1+s)^{1/(1-s)}}. \tag{11}
\]

When
\[
1 \leq x \leq \psi^{-1}\left((s-1) \ln \Gamma(1+s)\right) - \sqrt{s} \tag{12}
\]
inequality (11) is better than the left hand side inequality in (4), where \(\psi^{-1}\) stands for the inverse function of \(\psi\). This can be realized since \(\lim_{s \to +} \psi^{-1}\left((s-1) \ln \Gamma(1+s)\right) - \sqrt{s}\) equals the unique zero 1.4626 \cdots of \(\psi(x)\) in \((0, \infty)\) clearly.
2.4. Inequality (9) for the case of $t = -\theta(s, 1) = 1$ and $s \in (0, 1)$ is better than the lower bound in (4) when $\psi(x + \sqrt{s}) + \psi(x - 1) \leq 0$ which can be rewritten as $0 < s \leq |\psi^{-1}(-\psi(x-1))-x|^2 < 1$. This can be realized since $\lim_{x \to 1+} [\psi(x + \sqrt{s}) + \psi(x - 1)] = -\infty$ obviously.

2.5. Inequality (10) for the case of $\tau = t = 1$, $s \in (0, 1)$ and $-1 < \theta(s, 1) < s = \min\{s, t\}$ is better than the right hand side inequality in (4) when $x > 1$ and

$$\frac{\ln \Gamma(1 + s)}{s - 1} \leq \psi \left( x + \frac{s + 1}{2} \right) - \psi(1 + \theta) + \psi(x + \theta). \quad (13)$$

This can be realized since $\lim_{x \to \infty} [\psi \left( x + \frac{s + 1}{2} \right) - \psi(1 + \theta) + \psi(x + \theta)] = \infty$ for any given $s$ and $\theta(s, 1)$ apparently.

2.6. Inequality (8) can also be deduced from a fact obtained in [44, Proposition 3]: The function $\frac{\Gamma(x + t)}{\Gamma(x + \tau)}^{1/(s-t)}$ is logarithmically completely monotonic in the interval $(-\min\{s, t\}, \infty)$ with $s \neq t$.

2.7. Let $a$, $b$ and $c$ be real numbers and $\rho = \min\{a, b, c\}$. Define

$$F_{a,b,c}(x) = \begin{cases} 
\left[ \frac{\Gamma(x + b)}{\Gamma(x + a)} \right]^{1/(a-b)} \exp[\psi(x + c)], & a \neq b \\
\exp[\psi(x + c) - \psi(x + a)], & a = b \neq c 
\end{cases} \quad (14)$$

for $x \in (-\rho, \infty)$. Furthermore, let $\theta(t)$ be an implicit function defined by equation

$$e^t - t = e^{\theta(t)} - \theta(t) \quad (15)$$

with $\theta(t) \neq t$ for $t \neq 0$ and let $p(t) = t - \theta(t - 1)$ in $(-\infty, \infty)$, where $p^{-1}$ stands for the inverse function of $p$. In [20], the following conclusions are proved:

1. $F_{a,b,c}(x) \in \mathcal{C}_C([-\rho, \infty])$ if $(a, b, c) \in D_1(a, b, c)$, where

$$D_1(a, b, c) = \{c \geq a, c \geq b\} \cup \{c \geq a, 0 \geq c - b \geq \theta(c - a)\}$$

$$\cup \{c \leq a, c - b \geq \theta(c - a)\} \setminus \{a = b = c\}; \quad (16)$$

2. $[F_{a,b,c}(x)]^{-1} \in \mathcal{C}_C([-\rho, \infty])$ if $(a, b, c) \in D_2(a, b, c)$, where

$$D_2(a, b, c) = \{c \leq a, c \leq b\} \cup \{c \geq a, c - b \leq \theta(c - a)\}$$

$$\cup \{c \leq a, 0 \leq c - b \leq \theta(c - a)\} \setminus \{a = b = c\}; \quad (17)$$

3. If $(a, b, c) \in D_1(a, b, c)$, then

$$\left[ \frac{\Gamma(x + b)}{\Gamma(x + a)} \right]^{1/(b-a)} \leq \exp[\psi(x + c)] \quad (18)$$

for $x \in (-\rho, \infty)$ and

$$\left[ \frac{\Gamma(x + b)}{\Gamma(x + a)} \right]^{1/(b-a)} \geq \left[ \frac{\Gamma(\delta + b)}{\Gamma(\delta + a)} \right]^{1/(b-a)} \exp[\psi(x + c) - \psi(\delta + c)] \quad (19)$$

for $x \in [\delta, \infty)$ are valid, where $\delta$ is a constant greater than $-\rho$;

4. If $(a, b, c) \in D_2(a, b, c)$, inequalities (18) and (19) are reversed.
As special cases of inequalities (18) and (19), inequalities
\[
\frac{\Gamma(x + 1)}{\Gamma(x + s)} \leq \exp[(1 - s)\psi(x + p^{-1}(s))]
\]
for \(x \in (-s, \infty)\) and
\[
\frac{\Gamma(x + 1)}{\Gamma(x + s)} \geq \frac{\Gamma(\delta + 1)}{\Gamma(\delta + s)} \exp[\psi(x + p^{-1}(s)) - \psi(\delta + p^{-1}(s))]
\]
for \(x \in (\delta, \infty)\) are valid, where \(s \in (0, 1)\), \(\delta > -s\) and \(s \leq p^{-1}(s) \leq 1\).

Since the function \(e^t - t\) is increasing in \((0, \infty)\) and decreasing in \((-\infty, 0)\), as showed by Figure 1, then \(t\theta(t) < 0\) for \(\theta(t) \neq t\). A ready differentiation on both sides of equation (15) yields \(\theta'(t) = \frac{e^t - 1}{e^t - |t|} < 0\), and then \(\theta(t)\) is decreasing and \(p(t)\) is increasing for \(t \in (-\infty, \infty)\).

It is claimed that \(t + \theta(t) < 0\) for \(\theta(t) \neq t\), as showed by Figure 1. This claim can be verified as follows. Let \(\phi_1(t) = e^{|t|} - |t|\) and \(\phi_2(t) = e^t - t\) in \((-\infty, \infty)\). If \(t \in (-\infty, 0]\), then \(\phi_1(t) = e^{-t} + t\); if \(t \in [0, \infty)\), then \(\phi_1(t) = \phi_2(t)\). It is clear that \(\phi_1(0) = \phi_2(0) = 0\) and \(\lim_{t \to -\infty} \phi_1(t) = \lim_{t \to -\infty} \phi_2(t) = \lim_{t \to -\infty} \phi_1(t) = \lim_{t \to -\infty} \phi_2(t) = \infty\). An easy calculation gives \(\phi_1'(t) = -e^{-t} + 1\) and \(\phi_2'(t) = e^t - 1\) in \((-\infty, 0]\). It is obvious that \(\phi_1'(t) < \phi_2'(t) < 0\) in \((-\infty, 0)\). This implies that the functions \(\phi_1(t)\) and \(\phi_2(t)\) are decreasing with \(0 < \phi_2(t) < \phi_1(t)\) in \((-\infty, 0)\). Accordingly, since the function \(\phi_1(t)\) is even in \((-\infty, \infty)\), for any given negative number \(s < 0\), there exists a unique point \(\theta_1(s) > 0\) such that \(s < -\theta_1(s) < 0\) and \(\phi_2(s) = \phi_1(-\theta_1(s)) = \phi_1(\theta_1(s))\); for any given positive number \(t > 0\), there exists a unique point \(\theta_2(t) < 0\) such that \(\theta_2(t) < -t < 0\) and \(\phi_2(\theta_2(t)) = \phi_1(-t) = \phi_1(t)\). In conclusion, for any given \(t \in (-\infty, \infty) \setminus \{0\}\), there exists a unique point \(\theta(t) \neq t\) such that \(t + \theta(t) < 0\) and \(\phi_1(t) = \phi_2(\theta(t))\) which is equivalent to equation (15). In other words, if \(t\) and \(\theta(t)\) with \(t \neq \theta(t)\) satisfy equation (15), then \(t + \theta(t) < 0\).
Now we can claim that, for \( x \geq 1 \) and \( s \in (0,1) \), inequality (20) is better than the right hand side inequality in (3), since
\[
\psi \left( x + \frac{1 + s}{2} \right) > \psi \left( x + p^{-1}(s) \right) \iff p \left( \frac{1 + s}{2} \right) > s \\
\iff \frac{1 + s}{2} - \theta \left( \frac{1 + s}{2} - 1 \right) > s \iff \theta \left( \frac{s - 1}{2} \right) < \frac{1 - s}{2}
\]
is valid, where the monotonicities of \( \psi \) and \( p \) and the fact that \( t + \theta(t) < 0 \) for \( t\theta(t) < 0 \) are used.

2.8. In [4] Theorem 2.4, the following double inequality was obtained:
\[
\exp \left[ (x-y)\psi \left( \frac{x - y}{\ln(x+1) - \ln(y+1) - 1} \right) \right] \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \exp \left[ (x-y)\psi \left( \frac{x + y}{2} \right) \right], \tag{22}
\]
where \( x \) and \( y \) are positive real numbers.

The right hand side inequality in (22) is the same as (7) essentially.

It is noted that a more strengthened conclusion than the right hand side inequality in (22) has been established in [12, p. 250] and [44 Proposition 4]: Let \( s \) and \( t \) be two real numbers and \( \alpha = \min\{s,t\} \). Then the function
\[
\exp \left[ \psi \left( x + \frac{s + t}{2} \right) \right] \left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(s-t)} \in C_{\mathcal{L}}[-\alpha, \infty]. \tag{23}
\]
Consequently, inequality (7) follows.

In the left hand side inequality of (22), substituting \( x \) by \( x + s \) and \( y \) by \( x + t \) for two real numbers \( s \) and \( t \) and \( x \in (-\min\{s,t\}, \infty) \) leads to
\[
\exp \left[ \psi \left( \frac{x - t}{\ln(x + s + 1) - \ln(x + t + 1) - 1} \right) \right] \leq \left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(t-s)}. \tag{24}
\]
It was proved in [12, p. 248] that
\[
\exp \left( \psi \left( x + \psi^{-1} \left( \frac{1}{t - s} \int_s^t \psi(u) \, du \right) \right) \right) \leq \left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(t-s)}, \tag{25}
\]
where \( x \geq 0, s > 0, t > 0 \), and \( \psi^{-1} \) denotes the inverse function of \( \psi \). The lower bounds in (24) and (25) do not contain each other, since a simple numerical computation by the well known software MATHEMATICA 5.2 shows that
\[
\psi \left( \frac{s - t}{\ln(x + s + 1) - \ln(x + t + 1) - x - 1} \right) - \frac{1}{t - s} \int_s^t \psi(u) \, du
\]
equals 0.21728 \cdots if \( (x, s, t) = (191, 1, 92) \) and \(-0.10331 \cdots \) if \( (x, s, t) = (11, 1, 92) \).

2.9. In [6] the following complete monotonicity were established:

(1) The functions
\[
\frac{\Gamma(x + s)}{\Gamma(x + 1)} \exp \left[ (1 - s)\psi \left( x + \frac{s + 1}{2} \right) \right] \quad \text{and} \quad \frac{\Gamma(x + 1)}{\Gamma(x + s)} \left( x + \frac{s}{2} \right)^{s-1} \tag{26}
\]
are completely monotonic on \((0, \infty)\) for \( 0 \leq s \leq 1 \). When \( 0 < s < 1 \), the functions in (26) satisfy \((-1)^n f^{(n)}(x) > 0 \) for \( x > 0 \).
(2) Let $0 < s < 1$ and $x > 0$. Then both
\[
\frac{\Gamma(x+1)}{\Gamma(x+s)} \exp \left[ (s-1)\psi \left( x + \sqrt{s} \right) \right] \quad \text{and} \quad \frac{\Gamma(x+s)}{\Gamma(x+1)} \left[ x - \frac{1}{2} + \sqrt{s + \frac{1}{4}} \right]^{1-s}
\]
are strictly decreasing functions.

The complete monotonicities of the second functions in (26) and (27) are generalized in [25] to logarithmically complete monotonicities.

It is clear that the complete monotonicities of the first functions in (26) and (27) are included in Theorem 1 of this paper.

3. Proofs of theorems

In order to prove our main result, the following more general proposition than our need are presented.

**Proposition 1.** Let $\psi$ be the psi function defined by $\frac{\Gamma'}{\Gamma}$, and $s$ and $t$ two positive numbers.

(1) If $m > n \geq 0$ are two integers, then
\[
\left( \psi^{(m)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(m)}(v) \, dv \right) \leq \left( \psi^{(n)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(n)}(v) \, dv \right).
\]

(2) Inequality
\[
\psi^{(i)} \left( \frac{t-s}{\ln t - \ln s} \right) \leq \frac{1}{t-s} \int_s^t \psi^{(i)}(u) \, du
\]
is valid for $i$ being positive odd number or zero and reversed for $i$ being nonnegative even number.

(3) The function
\[
\left( \psi^{(\ell)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(\ell)}(x+v) \, dv \right) - x
\]
for $\ell \geq 0$ is increasing and concave in $x > -\min\{s,t\}$ and has a sharp upper bound $\frac{s+t}{2}$.

**Proof.** It was presented in [13, Theorem 3] that if the second derivative of $f$ is continuous on an interval $I$ such that $f$ is increasingly concave and $f^{(-)}$ is increasing then
\[
\left( f^{(-)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t f'(u) \, du \right) \leq f^{-1} \left( \frac{1}{t-s} \int_s^t f(u) \, du \right)
\]
holds for $s, t \in I$, where $(f')^{-1}$ and $f^{-1}$ stand for the inverse functions of $f'$ and $f$.

It was presented in [24, p. 366, Theorem 1] and [54, p. 167] that if $w(x) \in C[I]$ then
\[
w^{(k+1)}(x)w^{(k-1)}(x) \geq [w^{(k)}(x)]^2
\]
for $k \in \mathbb{N}$ and $x \in I$. This means that
\[
\left[ \frac{w^{(k)}(x)}{w^{(k-1)}(x)} \right]' = \frac{w^{(k+1)}(x)w^{(k-1)}(x) - [w^{(k)}(x)]^2}{[w^{(k-1)}(x)]^2} \geq 0
\]
and the function $\frac{w^{(k)}(x)}{w^{(k-1)}(x)}$ is increasing.
It is easy to see that an inverse function has the property that
\[(af(x))^{-1} = f^{-1}\left(\frac{x}{a}\right)\]  \hspace{1cm} (34)
for \(a \neq 0\), where \([af(x)]^{-1}\) denotes the inverse function of \(af(x)\).

It is well known that \(\psi'(x) \in \mathcal{C}[[0, \infty]]\) and \((-1)^i[\psi'(x)]^{(i)} \geq 0\) for nonnegative integer \(i\). This implies \(\psi^{(2k-1)}(x) \in \mathcal{C}[[0, \infty]], \ -\psi^{(2k)}(x) \in \mathcal{C}[[0, \infty]]\) and
\[\psi^{(k+2)}(x)\psi^{(k)}(x) \geq \left[\psi^{(k+1)}(x)\right]^2\]  \hspace{1cm} (35)
for \(k \in \mathbb{N}\). Hence, the functions \(-\psi^{(2i+1)}(x)\) and \(\psi^{(2i)}(x)\) are increasingly concave in \((0, \infty)\) and
\[
\begin{align*}
\left\{ \begin{array}{l}
\left[\psi^{(2i+1)}(x)\right]' = \psi^{(2i+3)}(x)' \\
-\psi^{(2i+1)}(x)' = \psi^{(2i+2)}(x)' \end{array} \right.
&= \left[\frac{\psi^{(2i+4)}(x)\psi^{(2i+2)}(x) - \left[\psi^{(2i+3)}(x)\right]^2}{\left[\psi^{(2i+2)}(x)\right]^2}\right] \geq 0,

\left\{ \begin{array}{l}
\left[\psi^{(2i)}(x)\right]' = \psi^{(2i+2)}(x)' \\
\psi^{(2i+1)}(x)' = \psi^{(2i+3)}(x)' \end{array} \right.
&= \left[\frac{\psi^{(2i+3)}(x)\psi^{(2i+1)}(x) - \left[\psi^{(2i+2)}(x)\right]^2}{\left[\psi^{(2i+1)}(x)\right]^2}\right] \geq 0,
\end{align*}
\]
which are equivalent to the functions \(-\psi^{(2i+1)}(x)\) and \(\psi^{(2i)}(x)\) are increasing in \((0, \infty)\) for given nonnegative integer \(i \geq 0\). Accordingly, substituting \(-\psi^{(2i+1)}(x)\) and \(\psi^{(2i)}(x)\) into (31) and utilizing (34) yields
\[
\left(\psi^{(2i+2)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i+2)}(u) \, du\right) \leq \left(\psi^{(2i+1)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i+1)}(u) \, du\right) \]  \hspace{1cm} (36)
and
\[
\left(\psi^{(2i+1)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i+1)}(u) \, du\right) \leq \left(\psi^{(2i)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i)}(u) \, du\right) \]  \hspace{1cm} (37)
for positive real numbers \(s, t\) and nonnegative integer \(i \geq 0\). As a result, by induction, inequality (28) follows.

By using Jensen’s inequality, it was obtained in [7] that if \(g\) is strictly monotonic, \(f\) is strictly increasing and \(f \circ g^{-1}\) is convex (or concave, respectively) on an interval \(I\), then
\[
\left(\psi^{(2i+1)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i+1)}(u) \, du\right) \leq \left(\psi^{(2i)}\right)^{-1}\left(\frac{1}{t-s} \int_s^t \psi^{(2i)}(u) \, du\right) \]  \hspace{1cm} (38)
holds (or reverses, respectively) for \(s, t \in I\). It is apparent that \(f(x) = (-1)^i[\psi^{(i)}(x) \, du] \) for \(i \geq 0\) is increasing strictly and \(g(x) = \frac{1}{2} x^2\) is decreasing strictly and \(g^{-1}(x) = g(x)\). Direct computation gives
\[
g^{-1}\left(\frac{1}{t-s} \int_s^t g(u) \, du\right) = \left(\frac{t-s}{\ln t - \ln s}\right) \]  \hspace{1cm} (39)
and
\[
h(x) \triangleq f \circ g^{-1}(x) = (-1)^i[\psi^{(i)}(x) \, \frac{1}{2} x^2] \]  \hspace{1cm} (40)
and
\[
h''(x) = \frac{(-1)^i \left[2x\psi^{(i+1)}(x) + \psi^{(i+2)}(x)\right]}{x^4} = (-1)^i u^3 \left[2\psi^{(i+1)}(u) + u\psi^{(i+2)}(u)\right].
\]

It was proved in [2] that the function \( \frac{x\psi^{(k+1)}(x)}{\psi^{(k)}(x)} \) is strictly increasing from \([0, \infty)\) onto \([-k+1, -k)\) for \(k \in \mathbb{N}\). This means that
\[
(-1)^k (k+1) \psi^{(k)}(x) \leq (-1)^{k+1} x \psi^{(k+1)}(x) < (-1)^k k \psi^{(k)}(x)
\]
holds in \((0, \infty)\) for \(k \in \mathbb{N}\), which can be rewritten as
\[
(-i) \left[(-1)^i \psi^{(i+1)}(x)\right] \leq (-1)^i \left[2\psi^{(i+1)}(u) + x\psi^{(i+2)}(x)\right] < (1 - i) \left[(-1)^i \psi^{(i+1)}(x)\right]
\]
in \((0, \infty)\) for given nonnegative integer \(i\). Consequently, the function \(h(x)\) is convex if \(i = 0\) or concave if \(i \geq 1\). So, the conditions of inequality \((33)\) (or reversed inequality of \((33)\), respectively) are satisfied by \(f(x) = (-1)^i \psi^{(i)}(x)\) and \(g(x) = \frac{1}{x}\) for \(i = 0\) (or for \(i \geq 1\), respectively). The case of \(i = 0\) in \((33)\) is just inequality \((29)\) for \(i = 0\). For \(i \geq 1\), this leads to
\[
\frac{t - s}{\ln t - \ln s} \geq \left((-1)^i \psi^{(i)}(x)\right)^{-1} \left(\frac{1}{t - s} \int_{s}^{t} (-1)^i \psi^{(i)}(u) \, du\right) = \left(\psi^{(i)}(x)\right)^{-1} \left(\frac{1}{t - s} \int_{s}^{t} \psi^{(i)}(u) \, du\right).
\]

Since \(\psi^{(2i)}(x)\) is increasing and \(\psi^{(2i-1)}(x)\) for \(i \in \mathbb{N}\), inequality \((29)\) or its reversed form is deduced from \((43)\).

Let \(\phi_{s,t,t}(x)\) denote the function \((30)\). It is said in [13, p. 194, Corollary 1] that if \(f\) is an increasing function such that \(f'\) is completely monotonic on an interval \(I\), then the function \(h_{f,s,t}(x) = f^{-1}\left(\frac{1}{t - s} \int_{s}^{t} f(x + v) \, dv\right) - x\) is increasing and concave for \(s, t \in I\) and \(x > -\min\{s, t\}\). It is clear that the functions \(\psi^{(2i)}(x)\) is increasing such that \(\psi^{(2i+1)}(x) \in C([0, \infty])\) for \(i \geq 0\), so do the functions \(-\psi^{(2i+1)}(x)\) for \(i \geq 0\). From \((44)\) it is easy to deduce that \(h_{f,s,t}(x) = h_{f,s,t}(x)\) holds for any given nonzero constant \(a\). Consequently, the increasing concavity of the functions \(\phi_{s,t,t}(x) = \phi_{s,t,t}(x)\) for \(t \geq 0\) is proved.

Since the function \((-1)^{t+1} \psi^{(t)}(x)\) for \(t \geq 0\) is decreasingly convex in \((0, \infty)\), by Hermite-Hadamard-Jensen’s integral inequality \([47, 49]\) and \((34)\), it is deduced that
\[
\left(\psi^{(t)}\right)^{-1} \left(\frac{1}{t - s} \int_{s}^{t} \psi^{(t)}(x + v) \, dv\right) = \left((-1)^{t+1} \psi^{(t)}\right)^{-1} \left(\frac{1}{t - s} \int_{s}^{t} \left((-1)^{t+1} \psi^{(t)}(x + v)\right) \, dv\right) \leq \left((-1)^{t+1} \psi^{(t)}\right)^{-1} \left((-1)^{t+1} \psi^{(t)} \left(x + \frac{s + t}{2}\right)\right)
\]
Combining this with inequality (29) yields
\[
\frac{t-s}{\ln(x+t) - \ln(x+s)} - x \leq \phi_{s,t}(x) \leq \frac{s+t}{2}.
\] (45)

Since
\[
\lim_{x \to -\infty} \left[ \frac{t-s}{\ln(x+t) - \ln(x+s)} - x \right] = \frac{s+t}{2}
\]
by L'Hôpital's rule, then the function \(\phi_{s,t}(x)\) has a sharp upper bound \(\frac{s+t}{2}\). The proof of Proposition \([1]\) is complete. \(\square\)

Now we are in a position to prove Theorem \([1]\) and Theorem \([2]\).

**Proof of Theorem \([7]\)** It is well known \([1]\), 6.1.50 and 6.3.21 that
\[
\ln \Gamma(x) = \int_0^\infty \frac{1}{u} [(x-1)e^{-u} - \frac{e^{-u} - e^{-xu}}{1-e^{-u}}] \, du,
\] (46)
\[
\psi(x) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-xu}}{1-e^{-u}} \right) \, du.
\] (47)

Straightforward calculation gives
\[
\ln \nu_{s,t}(x) = \frac{1}{t-s} [\ln \Gamma(x+t) - \ln \Gamma(x+s)] - \psi(x + \theta(s,t))
\]
\[
\quad = \int_0^\infty \frac{e^{-xu}}{1-e^{-u}} \left[ \frac{e^{-tu} - e^{-su}}{(t-s)u} + e^{-u\theta(s,t)} \right] \, du
\]
\[
\quad \triangleq \int_0^\infty \frac{e^{-[x+\theta(s,t)]u}}{1-e^{-u}} [q_{s,t}(u) + 1] \, du,
\]
where
\[
q_{s,t}(u) = \frac{e^{-tu} - e^{-su}}{(t-s)u} e^{u\theta(s,t)}
\]
\[
\quad = -e^{u\theta(s,t)} \left( \frac{1}{t-s} \int_s^t e^{-uv} \, dv \right)
\]
\[
\quad = -\exp \left\{ u [\theta(s,t) + \ln \left( \frac{1}{t-s} \int_s^t e^{-uv} \, dv \right)^{1/u}] \right\}
\]
\[
\triangleq -\exp \{ u [\theta(s,t) + \ln p_{s,t}(u)] \}
\]
and, by using \([31]\) p. 2, \([32]\) Theorem 3.3 or \([51]\) Theorem 1.1, see also \([35]\), the function \(p_{s,t}(u)\) is increasing in \(u \geq 0\) with
\[
\lim_{u \to 0} p_{s,t}(u) = e^{-(s+t)/2} \quad \text{and} \quad \lim_{u \to \infty} p_{s,t}(u) = e^{-\min\{s,t\}}.
\]

Accordingly, if \(\theta(s,t) \leq \min\{s,t\}\) then \(h_{s,t}(u) \geq 0\), if \(\theta(s,t) \geq \frac{s+t}{2}\) then \(h_{s,t}(u) \leq 0\).

This means \((-1)^k [\ln \nu_{s,t}(x)]^{(k)} \geq 0\), \(\theta(s,t) \leq \min\{s,t\}\) \(\leq 0\), \(\theta(s,t) \geq \frac{s+t}{2}\) \(\in \mathbb{N}\).

Conversely, if \(\frac{1}{\nu_{s,t}(x)}\) is logarithmically completely monotonic, then \([\ln \nu_{s,t}(x)]' \geq 0\) which can be rearranged as
\[
\frac{\psi(x+t) - \psi(x+s)}{t-s} \geq \psi'(x + \theta(s,t)).
\] (48)

Since \(\psi'\) is decreasing, thus...
\[ \theta(s, t) \geq (\psi')^{-1} \left( \frac{\psi(x + t) - \psi(x + s)}{t - s} \right) - x \]
\[ = (\psi')^{-1} \left( \frac{1}{t - s} \int_s^t \psi'(x + v) \, dv \right) - x = \phi_{s,t;1}(x), \quad (49) \]

where \((\psi')^{-1}\) denotes the inverse function of \(\psi\) and \(\phi_{s,t;1}(x)\) is defined by (30). Proposition [1] tells us that the function \(\phi_{s,t;1}(x)\) has a sharp upper bound \(\frac{s + t}{2}\). The proof of Theorem [1] is complete. \(\square\)

**Proof of Theorem [2]** If \(\theta(s, t) \geq \frac{s + t}{2}\), then the function \(\nu_{s,t}(x)\) defined by (6) is increasing by Theorem [1]. Hence, for any given \(\delta > -\min\{s, t\}\) and \(\theta(s, t) \geq \frac{s + t}{2}\), inequality

\[ \nu_{s,t}(\delta) \leq \nu_{s,t}(x) \]

holds in \([\delta, \infty)\) and

\[ \nu_{s,t}(x) < \lim_{x \to \infty} \nu_{s,t}(x) \]

is valid in \((-\min\{s, t\}, \infty)\).

For \(a\) and \(b\) being two constants, as \(x \to \infty\), the following asymptotic formula is given in [1] p. 261, 6.1.47:

\[ x^{b-a} \frac{\Gamma(x + a)}{\Gamma(x + b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + \frac{1}{12} \left( \frac{a-b}{2} \right) \frac{3(a+b-1)^2 - a + b - 1}{x^2} + O \left( \frac{1}{x^3} \right) = 1 + O \left( \frac{1}{x} \right). \quad (52) \]

In [36], it was proved that \(\psi(x) = \ln x + \frac{\alpha}{x} \in \mathcal{C}([0, \infty])\) if and only if \(\alpha \geq 1\) and \(\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}([0, \infty])\) if and only if \(\alpha \leq \frac{1}{2}\). From this, it is deduced that

\[ \ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \]

in \((0, \infty)\). Utilization of (52) and (53) leads to

\[ \lim_{x \to \infty} \frac{1}{\nu_{s,t}(x)} = \lim_{x \to \infty} \left\{ \frac{\exp[\psi(x + \theta(s, t))] x}{x} \left[ 1 + O \left( \frac{1}{x} \right) \right]^{1/(t-s)} \right\} \]

\[ = \lim_{x \to \infty} \exp[\psi(x + \theta(s, t))] \leq \lim_{x \to \infty} \left\{ \frac{x + \theta(s, t)}{x} \exp \left[ \frac{1}{2(x + \theta(s, t))} \right] \right\} \]

In (51) is valid in \((-\min\{s, t\}, \infty)\). Thus, from the increasing monotonicity of \(\psi\), inequality (7) is proved.

By standard calculation, inequality (50) can be rearranged as

\[ \left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(t-s)} \geq \left[ \frac{\Gamma(\delta + t)}{\Gamma(\delta + s)} \right]^{1/(t-s)} \exp[\psi(x + \theta(s, t)) - \psi(\delta + \theta(s, t))] \]

\[ (55) \]
for \( x \in [\delta, \infty) \) and \( \theta(s, t) \geq \frac{\psi''(y)}{\psi'(y)} \). From the decreasing monotonicity in \( y \) of the function \( \psi(x + y) - \psi(\delta + y) \) and \( \lim_{y \to \infty} [\psi(x + y) - \psi(\delta + y)] = 0 \) for \( x \geq \delta \), inequality (8) is concluded.

Combination of the conclusion \( \nu_{s,t}(x) \in C^2(\theta(s, t), \infty) \) for \( \theta(s, t) \leq \min\{s, t\} \) in Theorem 1 with \( \lim_{x \to \infty} \nu_{s,t}(x) = 1 \) and discussion by standard argument yields inequalities (9) and (10). The proof of Theorem 2 is complete. \( \square \)

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