A Class of Logarithmically Completely Monotonic Functions and the Best Bounds in the Second Kershaw's Double Inequality

A CLASS OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND THE BEST BOUNDS IN THE SECOND KERSHAW'S DOUBLE INEQUALITY

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Abstract. In the article, the sufficient and necessary conditions such that a class of functions which involve the psi function ψ and the ratio $\frac{\Gamma(x+t)}{\Gamma(x+s)}$ are logarithmically completely monotonic are established, the best bounds for the ratio $\frac{\Gamma(x+t)}{\Gamma(x+s)}$ are given, and some comparisons with known results are carried out, where s and t are two real numbers and $x>-\min\{s,t\}$.

1. Introduction

Recall [30, 32, 52, 54] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I such that $(-1)^k f^{(k)}(x)\ge 0$ for $x\in I$ and $k\ge 0$. Recall also [3, 30, 39, 41, 42, 43] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)}\ge 0$ for $k\in\mathbb{N}$ on I. For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I are denoted by $\mathcal{C}[I]$ and $\mathcal{C}_{\mathcal{L}}[I]$ respectively.

The famous Bernstein-Widder's Theorem [54, p. 161] states that $f\in\mathcal{C}[(0,\infty)]$ if and only if there exists a bounded and nondecreasing function $\mu(t)$ such that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\mu(t) \tag{1}$$

converges for $0<x<\infty$.

In [5, 29, 39, 41, 42, 43, 52] and many other references, the inclusions $\mathcal{C}_{\mathcal{L}}[I]\subset\mathcal{C}[I]$ and $\mathcal{S}\subset\mathcal{C}_{\mathcal{L}}[(0,\infty)]$ were revealed implicitly or explicitly, where $\mathcal{S}$ denotes the class of Stieltjes transforms [5, 54]. There are three different proofs in [5], [39, 41] and [29, 45] for the inclusion $\mathcal{C}_{\mathcal{L}}[I]\subset\mathcal{C}[I]$. The class $\mathcal{C}_{\mathcal{L}}[(0,\infty)]$ is characterized in [5, Theorem 1.1] implicitly and in [19, Theorem 4.4] explicitly: $f\in\mathcal{C}_{\mathcal{L}}[(0,\infty)]\iff f^\alpha\in\mathcal{C}$ for all $\alpha>0\iff\sqrt[n]{f}\in\mathcal{C}$ for all $n\in\mathbb{N}$. In other words, the functions in $\mathcal{C}_{\mathcal{L}}[(0,\infty)]$ are those completely monotonic functions for which the representing measure μ in (1) is infinitely divisible in the convolution sense: For each $n\in\mathbb{N}$ there exists a positive measure ν on $[0,\infty)$ with n-th convolution power equal to μ.

By the way, recall [30, 32, 48, 52, 54] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t)\ge 0$ for $t\in I$ and $k\in\mathbb{N}$. In [29, 45], it was defined that a positive function f is said

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to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \ge 0$ for $t \in I$ and $k \in \mathbb{N}$ and it was showed that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely.

In recent years, the logarithmically completely monotonic functions and their properties have been investigated extensively and explicitly in [3, 5, 9, 10, 11, 16, 17, 18, 23, 25, 26, 29, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 50] and the references therein.

Let Γ and $\psi = \frac{\Gamma'}{\Gamma}$ stand for the classical Euler's gamma function and the psi function respectively. The first and second Kershaw's inequalities [21] state that

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac12+\sqrt{s+\frac14}\right)^{1-s} \tag{2}$$

and

$$\exp\left[(1-s)\psi\big(x+\sqrt{s}\,\big)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{3}$$

for $s \in (0,1)$ and $x \ge 1$. There have been a lot of literature on these two double inequalities, for example, [4, 8, 12, 14, 15, 20, 21, 22, 23, 25, 26, 27, 28, 33, 34, 36, 38, 44, 53] and the references therein.

For real numbers a, b, c and $\rho = \min\{a,b,c\}$, let $H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$ in $(-\rho,\infty)$. Recently, the following sufficient and necessary conditions are established elegantly in [37]: $H_{a,b,c}(x) \in \mathcal{C}_{\mathcal{L}}[(-\rho,\infty)]$ if and only if $(a,b,c) \in \{(a,b,c) : (b-a)(1-a-b+2c) \ge 0\} \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \ge 0\} \setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\}$ and $H_{b,a,c}(x) \in \mathcal{C}_{\mathcal{L}}[(-\rho,\infty)]$ if and only if $(a,b,c) \in \{(a,b,c) : (b-a)(1-a-b+2c) \le 0\} \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \le 0\} \setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}$. These conclusions can be used to extend, generalize, refine and sharpen [25, Theorem 1], inequality (2) and some other known results.

It is easy to see that inequality (3) can be rewritten for $s \in (0,1)$ and $x \ge 1$ as

$$\exp\left[\psi\big(x+\sqrt{s}\,\big)\right] < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} < \exp\left[\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{4}$$

Now it is natural to ask: What are the best constants $\delta_1(s,t)$ and $\delta_2(s,t)$ such that

$$\exp[\psi(x+\delta_1(s,t))] \le \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \le \exp[\psi(x+\delta_2(s,t))] \tag{5}$$

holds for $x > -\min\{s,t,\delta_1(s,t),\delta_2(s,t)\}$, where s and t are two real numbers? In order to give an answer to this problem, we would like to establish the logarithmically complete monotonicity of the function

$$\nu_{s,t}(x) = \frac{1}{\exp\left[\psi\big(x+\theta(s,t)\big)\right]}\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \tag{6}$$

Our first main result is the following Theorem 1.

Theorem 1. *Let s and t be two real numbers with $s \ne t$ and $\theta(s,t)$ a constant depending on s and t.*

(1) *If $\theta(s,t) \le \min\{s,t\}$, then $\nu_{s,t}(x) \in \mathcal{C}_{\mathcal{L}}[(-\theta(s,t),\infty)]$.*

(2) $\frac{1}{\nu_{s,t}(x)}\in\mathcal{C}_{\mathcal{L}}[(-\min\{s,t\},\infty)]$ *if and only if* $\theta(s,t)\ge\frac{s+t}{2}$.

Our second main result, as a straightforward consequence of Theorem 1, is the following Theorem 2.

Theorem 2. *Let s and t be two real numbers with $s\ne t$.*

(1) *Inequality*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}<\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] \tag{7}$$

is valid in $(-\min\{s,t\},\infty)$. The constant $\frac{s+t}{2}$ in (7) is the best possible.

(2) *Inequality*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\ge\left[\frac{\Gamma(\delta+t)}{\Gamma(\delta+s)}\right]^{1/(t-s)} \tag{8}$$

validates for $x\ge\delta>-\min\{s,t\}$.

(3) *Inequality*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\ge\exp(-\psi(x+\theta(s,t))) \tag{9}$$

holds for $x>-\theta(s,t)>-\min\{s,t\}$.

(4) *Inequality*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}<\left[\frac{\Gamma(\tau+t)}{\Gamma(\tau+s)}\right]^{1/(t-s)}\exp[\psi(\tau+\theta(s,t))-\psi(x+\theta(s,t))] \tag{10}$$

sounds for $x>\tau\ge-\theta(s,t)>-\min\{s,t\}$.

Before proving Theorem 1 and Theorem 2 in Section 3, we would like to compare them with some recent known results and to give several remarks in Section 2.

2. Comparisons of theorems with some known results

2.1. In order to refine and extend the first Kershaw's double inequality (2), the logarithmically complete monotonicity of the function $(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$ for $x\in(-\rho,\infty)$ was studied in [25], where a, b and c are real numbers and $\rho=\min\{a,b,c\}$.

2.2. It is clear that inequality (7) extends the ranges of variables of the right hand side inequality in (4) which is a rearranged form of (3).

2.3. Taking $t=\delta=1$ and $s\in(0,1)$ in (8) gives

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)}\ge\frac{1}{[\Gamma(1+s)]^{1/(1-s)}}. \tag{11}$$

When

$$1\le x\le\psi^{-1}\big((s-1)\ln\Gamma(1+s)\big)-\sqrt{s} \tag{12}$$

inequality (11) is better than the left hand side inequality in (4), where ψ^{-1} stands for the inverse function of ψ. This can be realized since $\lim_{s\to0^+}[\psi^{-1}\big((s-1)\ln\Gamma(1+s)\big)-\sqrt{s}\,]$ equals the unique zero $1.4626\cdots$ of $\psi(x)$ in $(0,\infty)$ clearly.

2.4. Inequality (9) for the case of $t = -\theta(s,1) = 1$ and $s \in (0,1)$ is better than the lower bound in (4) when $\psi(x+\sqrt{s}\,) + \psi(x-1) \le 0$ which can be rewritten as $0 < s \le [\psi^{-1}(-\psi(x-1)) - x]^2 < 1$. This can be realized since $\lim_{x\to 1^+}[\psi(x+\sqrt{s}\,) + \psi(x-1)] = -\infty$ obviously.

2.5. Inequality (10) for the case of $\tau = t = 1$, $s \in (0,1)$ and $-1 < \theta(s,1) < s = \min\{s,t\}$ is better than the right hand side inequality in (4) when $x > 1$ and

$$\frac{\ln\Gamma(1+s)}{s-1} \le \psi\left(x + \frac{s+1}{2}\right) - \psi(1+\theta) + \psi(x+\theta). \tag{13}$$

This can be realized since $\lim_{x\to\infty}\left[\psi\left(x+\frac{s+1}{2}\right) - \psi(1+\theta) + \psi(x+\theta)\right] = \infty$ for any given s and $\theta(s,1)$ apparently.

2.6. Inequality (8) can also be deduced from a fact obtained in [44, Proposition 3]: The function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}$ is logarithmically completely monotonic in the interval $(-\min\{s,t\},\infty)$ with $s \ne t$.

2.7. Let a, b and c be real numbers and $\rho = \min\{a,b,c\}$. Define

$$F_{a,b,c}(x) = \begin{cases} \left[\dfrac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)} \exp[\psi(x+c)], & a \ne b \\ \exp[\psi(x+c) - \psi(x+a)], & a = b \ne c \end{cases} \tag{14}$$

for $x \in (-\rho,\infty)$. Furthermore, let $\theta(t)$ be an implicit function defined by equation

$$e^t - t = e^{\theta(t)} - \theta(t) \tag{15}$$

with $\theta(t) \ne t$ for $t \ne 0$ and let $p(t) = t - \theta(t-1)$ in $(-\infty,\infty)$, where p^{-1} stands for the inverse function of p. In [26], the following conclusions are proved:

(1) $F_{a,b,c}(x) \in \mathcal{C}_{\mathcal{L}}[(-\rho,\infty)]$ if $(a,b,c) \in D_1(a,b,c)$, where

$$\begin{aligned} D_1(a,b,c) = \{c \ge a, c \ge b\} &\cup \{c \ge a, 0 \ge c-b \ge \theta(c-a)\} \\ &\cup \{c \le a, c-b \ge \theta(c-a)\} \setminus \{a=b=c\}; \end{aligned} \tag{16}$$

(2) $[F_{a,b,c}(x)]^{-1} \in \mathcal{C}_{\mathcal{L}}[(-\rho,\infty)]$ if $(a,b,c) \in D_2(a,b,c)$, where

$$\begin{aligned} D_2(a,b,c) = \{c \le a, c \le b\} &\cup \{c \ge a, c-b \le \theta(c-a)\} \\ &\cup \{c \le a, 0 \le c-b \le \theta(c-a)\} \setminus \{a=b=c\}; \end{aligned} \tag{17}$$

(3) If $(a,b,c) \in D_1(a,b,c)$, then

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < \exp[\psi(x+c)] \tag{18}$$

for $x \in (-\rho,\infty)$ and

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} \ge \left[\frac{\Gamma(\delta+b)}{\Gamma(\delta+a)}\right]^{1/(b-a)} \exp[\psi(x+c) - \psi(\delta+c)] \tag{19}$$

for $x \in [\delta,\infty)$ are valid, where δ is a constant greater than $-\rho$;

(4) If $(a,b,c) \in D_2(a,b,c)$, inequalities (18) and (19) are reversed.

As special cases of inequalities (18) and (19), inequalities

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\bigl[(1-s)\psi\bigl(x+p^{-1}(s)\bigr)\bigr] \tag{20}$$

for $x \in (-s, \infty)$ and

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \geq \frac{\Gamma(\delta+1)}{\Gamma(\delta+s)} \exp\bigl[\psi\bigl(x+p^{-1}(s)\bigr) - \psi\bigl(\delta+p^{-1}(s)\bigr)\bigr] \tag{21}$$

for $x \in (\delta, \infty)$ are valid, where $s \in (0,1)$, $\delta > -s$ and $s \leq p^{-1}(s) \leq 1$.

Since the function $e^t - t$ is increasing in $(0,\infty)$ and decreasing in $(-\infty, 0)$, as showed by Figure 1, then $t\theta(t) < 0$ for $\theta(t) \neq t$. A ready differentiation on both

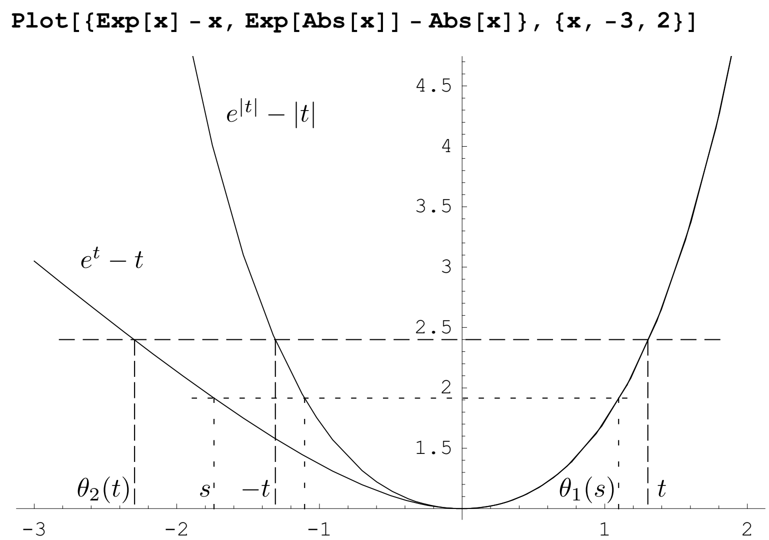

FIGURE 1. Graphs of the functions $e^t - t$ and $e^{|t|} - |t|$ by MATHEMATICA 5.2

sides of equation (15) yields $\theta'(t) = \frac{e^t - 1}{e^{\theta(t)} - 1} < 0$, and then $\theta(t)$ is decreasing and $p(t)$ is increasing for $t \in (-\infty, \infty)$.

It is claimed that $t + \theta(t) < 0$ for $\theta(t) \neq t$, as showed by Figure 1. This claim can be verified as follows. Let $\phi_1(t) = e^{|t|} - |t|$ and $\phi_2(t) = e^t - t$ in $(-\infty,\infty)$. If $t \in (-\infty, 0]$, then $\phi_1(t) = e^{-t} + t$; if $t \in [0,\infty)$, then $\phi_1(t) = \phi_2(t)$. It is clear that $\phi_1(0) = \phi_2(0) = 0$ and $\lim_{t\to-\infty}\phi_1(t) = \lim_{t\to-\infty}\phi_2(t) = \lim_{t\to\infty}\phi_1(t) = \lim_{t\to\infty}\phi_2(t) = \infty$. An easy calculation gives $\phi_1'(t) = -e^{-t} + 1$ and $\phi_2'(t) = e^t - 1$ in $(-\infty, 0]$. It is obvious that $\phi_1'(t) < \phi_2'(t) < 0$ in $(-\infty, 0)$. This implies that the functions $\phi_1(t)$ and $\phi_2(t)$ are decreasing with $0 < \phi_2(t) < \phi_1(t)$ in $(-\infty, 0)$. Accordingly, since the function $\phi_1(t)$ is even in $(-\infty,\infty)$, for any given negative number $s < 0$, there exists a unique point $\theta_1(s) > 0$ such that $s < -\theta_1(s) < 0$ and $\phi_2(s) = \phi_1(-\theta_1(s)) = \phi_1(\theta_1(s))$; for any given positive number $t > 0$, there exists a unique point $\theta_2(t) < 0$ such that $\theta_2(t) < -t < 0$ and $\phi_2(\theta_2(t)) = \phi_1(-t) = \phi_1(t)$. In conclusion, for any given $t \in (-\infty,\infty)\setminus\{0\}$, there exists a unique point $\theta(t) \neq t$ such that $t + \theta(t) < 0$ and $\phi_1(t) = \phi_2(\theta(t))$ which is equivalent to equation (15). In other words, if t and $\theta(t)$ with $t \neq \theta(t)$ satisfy equation (15), then $t + \theta(t) < 0$.

Now we can claim that, for $x\ge1$ and $s\in(0,1)$, inequality (20) is better than the right hand side inequality in (3), since

$$\psi\left(x+\frac{1+s}{2}\right)>\psi\left(x+p^{-1}(s)\right)\iff p\left(\frac{1+s}{2}\right)>s$$
$$\iff \frac{1+s}{2}-\theta\left(\frac{1+s}{2}-1\right)>s\iff \theta\left(\frac{s-1}{2}\right)<\frac{1-s}{2}$$

is valid, where the monotonicities of ψ and p and the fact that $t+\theta(t)<0$ for $t\theta(t)<0$ are used.

2.8. In [4, Theorem 2.4], the following double inequality was obtained:

$$\exp\left[(x-y)\psi\left(\frac{x-y}{\ln(x+1)-\ln(y+1)}-1\right)\right]\le\frac{\Gamma(x)}{\Gamma(y)}\le\exp\left[(x-y)\psi\left(\frac{x+y}{2}\right)\right],\quad(22)$$

where x and y are positive real numbers.

The right hand side inequality in (22) is the same as (7) essentially.

It is noted that a more strengthened conclusion than the right hand side inequality in (22) has been established in [12, p. 250] and [44, Proposition 4]: Let s and t be two real numbers and $\alpha=\min\{s,t\}$. Then the function

$$\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right]\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}\in\mathcal{C}_{\mathcal{L}}[(-\alpha,\infty)].\quad(23)$$

Consequently, inequality (7) follows.

In the left hand side inequality of (22), substituting x by $x+s$ and y by $x+t$ for two real numbers s and t and $x\in(-\min\{s,t\},\infty)$ leads to

$$\exp\left[\psi\left(\frac{s-t}{\ln(x+s+1)-\ln(x+t+1)}-1\right)\right]\le\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}.\quad(24)$$

It was proved in [12, p. 248] that

$$\exp\left(\psi\left(x+\psi^{-1}\left(\frac{1}{t-s}\int_s^t\psi(u)\,\mathrm{d}u\right)\right)\right)\le\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)},\quad(25)$$

where $x\ge0$, $s>0$, $t>0$, and ψ^{-1} denotes the inverse function of ψ. The lower bounds in (24) and (25) do not contain each other, since a simple numerical computation by the well known software MATHEMATICA 5.2 shows that

$$\psi\left(\frac{s-t}{\ln(x+s+1)-\ln(x+t+1)}-x-1\right)-\frac{1}{t-s}\int_s^t\psi(u)\,\mathrm{d}u$$

equals $0.21728\cdots$ if $(x,s,t)=(191,1,92)$ and $-0.10331\cdots$ if $(x,s,t)=(11,1,92)$.

2.9. In [6] the following complete monotonicity were established:

(1) The functions

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]\quad\text{and}\quad\frac{\Gamma(x+1)}{\Gamma(x+s)}\left(x+\frac{s}{2}\right)^{s-1}\quad(26)$$

are completely monotonic on $(0,\infty)$ for $0\le s\le1$. When $0<s<1$, the functions in (26) satisfy $(-1)^nf^{(n)}(x)>0$ for $x>0$.

(2) Let $0 < s < 1$ and $x > 0$. Then both

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\left[(s-1)\psi\left(x+\sqrt{s}\right)\right] \quad \text{and} \quad \frac{\Gamma(x+s)}{\Gamma(x+1)}\left[x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right]^{1-s} \tag{27}$$

are strictly decreasing functions.

The complete monotonicities of the second functions in (26) and (27) are generalized in [25] to logarithmically complete monotonicities.

It is clear that the complete monotonicities of the first functions in (26) and (27) are included in Theorem 1 of this paper.

3. Proofs of theorems

In order to prove our main result, the following more general proposition than our need are presented.

Propsition 1. *Let ψ be the psi function defined by $\frac{\Gamma'}{\Gamma}$, and s and t two positive numbers.*

(1) *If $m > n \geq 0$ are two integers, then*

$$\left(\psi^{(m)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(m)}(v)\,\mathrm{d}v\right) \leq \left(\psi^{(n)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(n)}(v)\,\mathrm{d}v\right). \tag{28}$$

(2) *Inequality*

$$\psi^{(i)}\left(\frac{t-s}{\ln t-\ln s}\right) \leq \frac{1}{t-s}\int_s^t \psi^{(i)}(u)\,\mathrm{d}u \tag{29}$$

is valid for i being positive odd number or zero and reversed for i being nonnegative even number.

(3) *The function*

$$\left(\psi^{(\ell)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(\ell)}(x+v)\,\mathrm{d}v\right) - x \tag{30}$$

for $\ell \geq 0$ is increasing and concave in $x > -\min\{s,t\}$ and has a sharp upper bound $\frac{s+t}{2}$.

Proof. It was presented in [13, Theorem 3] that if the second derivative of f is continuous on an interval I such that f is increasingly concave and $\frac{f''}{f'}$ is increasing then

$$(f')^{-1}\left(\frac{1}{t-s}\int_s^t f'(u)\,\mathrm{d}u\right) \leq f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right) \tag{31}$$

holds for $s,t \in I$, where $(f')^{-1}$ and f^{-1} stand for the inverse functions of f' and f.

It was presented in [24, p. 366, Theorem 1] and [54, p. 167] that if $w(x) \in \mathcal{C}[I]$ then

$$w^{(k+1)}(x)w^{(k-1)}(x) \geq [w^{(k)}(x)]^2 \tag{32}$$

for $k \in \mathbb{N}$ and $x \in I$. This means that

$$\left[\frac{w^{(k)}(x)}{w^{(k-1)}(x)}\right]' = \frac{w^{(k+1)}(x)w^{(k-1)}(x) - [w^{(k)}(x)]^2}{[w^{(k-1)}(x)]^2} \geq 0 \tag{33}$$

and the function $\frac{w^{(k)}(x)}{w^{(k-1)}(x)}$ is increasing.

It is easy to see that an inverse function has the property that

$$(af(x))^{-1} = f^{-1}\left(\frac{x}{a}\right) \tag{34}$$

for $a \neq 0$, where $[af(x)]^{-1}$ denotes the inverse function of $af(x)$.

It is well known that $\psi'(x) \in \mathcal{C}[(0,\infty)]$ and $(-1)^i[\psi'(x)]^{(i)} \geq 0$ for nonnegative integer i. This implies $\psi^{(2k-1)}(x) \in \mathcal{C}[(0,\infty)]$, $-\psi^{(2k)}(x) \in \mathcal{C}[(0,\infty)]$ and

$$\psi^{(k+2)}(x)\psi^{(k)}(x) \geq \left[\psi^{(k+1)}(x)\right]^2 \tag{35}$$

for $k \in \mathbb{N}$. Hence, the functions $-\psi^{(2i+1)}(x)$ and $\psi^{(2i)}(x)$ are increasingly concave in $(0,\infty)$ and

$$\begin{aligned}\left\{\frac{[-\psi^{(2i+1)}(x)]''}{[-\psi^{(2i+1)}(x)]'}\right\}' &= \left[\frac{\psi^{(2i+3)}(x)}{\psi^{(2i+2)}(x)}\right]' \\ &= \frac{\psi^{(2i+4)}(x)\psi^{(2i+2)}(x) - \left[\psi^{(2i+3)}(x)\right]^2}{\left[\psi^{(2i+2)}(x)\right]^2} \geq 0,\end{aligned}$$

$$\left\{\frac{[\psi^{(2i)}(x)]''}{[\psi^{(2i)}(x)]'}\right\}' = \left[\frac{\psi^{(2i+2)}(x)}{\psi^{(2i+1)}(x)}\right]' = \frac{\psi^{(2i+3)}(x)\psi^{(2i+1)}(x) - [\psi^{(2i+2)}(x)]^2}{[\psi^{(2i+1)}(x)]^2} \geq 0,$$

which are equivalent to the functions $\frac{[-\psi^{(2i+1)}(x)]''}{[-\psi^{(2i+1)}(x)]'}$ and $\frac{[\psi^{(2i)}(x)]''}{[\psi^{(2i)}(x)]'}$ are increasing in $(0,\infty)$ for given nonnegative integer $i \geq 0$. Accordingly, substituting $-\psi^{(2i+1)}(x)$ and $\psi^{(2i)}(x)$ into (31) and utilizing (34) yields

$$\begin{aligned}&\left(\psi^{(2i+2)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(2i+2)}(u)\,\mathrm{d}u\right) \\ &\qquad \leq \left(\psi^{(2i+1)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(2i+1)}(u)\,\mathrm{d}u\right)\end{aligned} \tag{36}$$

and

$$\left(\psi^{(2i+1)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(2i+1)}(u)\,\mathrm{d}u\right) \leq \left(\psi^{(2i)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t \psi^{(2i)}(u)\,\mathrm{d}u\right) \tag{37}$$

for positive real numbers s and t and nonnegative integer $i \geq 0$. As a result, by induction, inequality (28) follows.

By using Jensen's inequality, it was obtained in [7] that if g is strictly monotonic, f is strictly increasing and $f \circ g^{-1}$ is convex (or concave, respectively) on an interval I, then

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) \leq f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right) \tag{38}$$

holds (or reverses, respectively) for $s,t \in I$. It is apparent that $f(x) = (-1)^i\psi^{(i)}(x)$ for $i \geq 0$ is increasing strictly and $g(x) = \frac{1}{x}$ is decreasing strictly and $g^{-1}(x) = g(x)$. Direct computation gives

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) = \frac{t-s}{\ln t - \ln s}, \tag{39}$$

$$h(x) \triangleq f \circ g^{-1}(x) = (-1)^i\psi^{(i)}\left(\frac{1}{x}\right) \tag{40}$$

and

$$\begin{aligned}h''(x)&=\frac{(-1)^i\left[2x\psi^{(i+1)}\left(\frac1x\right)+\psi^{(i+2)}\left(\frac1x\right)\right]}{x^4}\\&=(-1)^iu^3\Big[2\psi^{(i+1)}(u)+u\psi^{(i+2)}(u)\Big].\end{aligned}$$

It was proved in [2] that the function $\frac{x\psi^{(k+1)}(x)}{\psi^{(k)}(x)}$ is strictly increasing from $[0,\infty)$ onto $[-(k+1),-k)$ for $k\in\mathbb{N}$. This means that

$$(-1)^k(k+1)\psi^{(k)}(x)\le(-1)^{k+1}x\psi^{(k+1)}(x)<(-1)^kk\psi^{(k)}(x)\tag{41}$$

holds in $(0,\infty)$ for $k\in\mathbb{N}$, which can be rewritten as

$$\begin{aligned}(-i)\big[(-1)^i\psi^{(i+1)}(x)\big]\le(-1)^i\big[2\psi^{(i+1)}(u)+x\psi^{(i+2)}(x)\big]\\<(1-i)\big[(-1)^i\psi^{(i+1)}(x)\big]\end{aligned}\tag{42}$$

in $(0,\infty)$ for given nonnegative integer i. Consequently, the function $h(x)$ is convex if $i=0$ or concave if $i\ge1$. So, the conditions of inequality (38) (or reversed inequality of (38), respectively) are satisfied by $f(x)=(-1)^i\psi^{(i)}(x)$ and $g(x)=\frac1x$ for $i=0$ (or for $i\ge1$, respectively). The case of $i=0$ in (38) is just inequality (29) for $i=0$. For $i\ge1$, this leads to

$$\begin{aligned}\frac{t-s}{\ln t-\ln s}&\ge\Big((-1)^i\psi^{(i)}\Big)^{-1}\left(\frac{1}{t-s}\int_s^t(-1)^i\psi^{(i)}(u)\,\mathrm{d}u\right)\\&=\Big(\psi^{(i)}\Big)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(i)}(u)\,\mathrm{d}u\right).\end{aligned}\tag{43}$$

Since $\psi^{(2i)}(x)$ is increasing and $\psi^{(2i-1)}(x)$ for $i\in\mathbb{N}$, inequality (29) or its reversed form is deduced from (43).

Let $\phi_{s,t;\ell}(x)$ denote the function (30). It is said in [13, p. 194, Corollary 1] that if f is an increasing function such that f' is completely monotonic on an interval I, then the function $h_{f;s,t}(x)=f^{-1}\big(\frac{1}{t-s}\int_s^tf(x+v)\,\mathrm{d}v\big)-x$ is increasing and concave for $s,t\in I$ and $x>-\min\{s,t\}$. It is clear that the functions $\psi^{(2i)}(x)$ is increasing such that $\psi^{(2i+1)}(x)\in\mathcal{C}[(0,\infty)]$ for $i\ge0$, so do the functions $-\psi^{(2i+1)}(x)$ for $i\ge0$. From (34) it is easy to deduce that $h_{af;s,t}(x)=h_{f;s,t}(x)$ holds for any given nonzero constant a. Consequently, the increasing concavity of the functions $h_{\psi^{(\ell)};s,t}(x)=\phi_{s,t;\ell}(x)$ for $\ell\ge0$ is proved.

Since the function $(-1)^{\ell+1}\psi^{(\ell)}(x)$ for $\ell\ge0$ is decreasingly convex in $(0,\infty)$, by Hermite-Hadamard-Jensen's integral inequality [47, 49] and (34), it is deduced that

$$\begin{aligned}&\Big(\psi^{(\ell)}\Big)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(\ell)}(x+v)\,\mathrm{d}v\right)\\&\quad=\Big((-1)^{\ell+1}\psi^{(\ell)}\Big)^{-1}\left(\frac{1}{t-s}\int_s^t\Big[(-1)^{\ell+1}\psi^{(\ell)}(x+v)\Big]\,\mathrm{d}v\right)\\&\quad\le\Big((-1)^{\ell+1}\psi^{(\ell)}\Big)^{-1}\left((-1)^{\ell+1}\psi^{(\ell)}\left(x+\frac{s+t}{2}\right)\right)\\&\quad=x+\frac{s+t}{2}.\end{aligned}\tag{44}$$

Combining this with inequality (29) yields

$$\frac{t-s}{\ln(x+t)-\ln(x+s)}-x\le\phi_{s,t;\ell}(x)\le\frac{s+t}{2}. \tag{45}$$

Since

$$\lim_{x\to\infty}\left[\frac{t-s}{\ln(x+t)-\ln(x+s)}-x\right]=\frac{s+t}{2}$$

by L'Hôspital's rule, then the function $\phi_{s,t;\ell}(x)$ has a sharp upper bound $\frac{s+t}{2}$. The proof of Proposition 1 is complete. □

Now we are in a position to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. It is well known [1, 6.1.50 and 6.3.21] that

$$\ln\Gamma(x)=\int_0^\infty\frac1u\left[(x-1)e^{-u}-\frac{e^{-u}-e^{-xu}}{1-e^{-u}}\right]\,\mathrm{d}u, \tag{46}$$

$$\psi(x)=\int_0^\infty\left(\frac{e^{-u}}{u}-\frac{e^{-xu}}{1-e^{-u}}\right)\mathrm{d}u. \tag{47}$$

Straightforward calculation gives

$$\begin{aligned}\ln\nu_{s,t}(x)&=\frac{1}{t-s}[\ln\Gamma(x+t)-\ln\Gamma(x+s)]-\psi\big(x+\theta(s,t)\big)\\&=\int_0^\infty\frac{e^{-xu}}{1-e^{-u}}\left[\frac{e^{-tu}-e^{-su}}{(t-s)u}+e^{-u\theta(s,t)}\right]\mathrm{d}u\\&\triangleq\int_0^\infty\frac{e^{-[x+\theta(s,t)]u}}{1-e^{-u}}[q_{s,t}(u)+1]\,\mathrm{d}u,\end{aligned}$$

where

$$\begin{aligned}q_{s,t}(u)&=\frac{e^{-tu}-e^{-su}}{(t-s)u}e^{u\theta(s,t)}\\&=-e^{u\theta(s,t)}\left(\frac{1}{t-s}\int_s^t e^{-uv}\,\mathrm{d}v\right)\\&=-\exp\left\{u\left[\theta(s,t)+\ln\left(\frac{1}{t-s}\int_s^t e^{-uv}\,\mathrm{d}v\right)^{1/u}\right]\right\}\\&\triangleq-\exp\{u[\theta(s,t)+\ln p_{s,t}(u)]\}\end{aligned}$$

and, by using [31, p. 2], [32, Theorem 3.3] or [51, Theorem 1.1], see also [35], the function $p_{s,t}(u)$ is increasing in $u\ge0$ with

$$\lim_{u\to0}p_{s,t}(u)=e^{-(s+t)/2}\qquad\text{and}\qquad\lim_{u\to\infty}p_{s,t}(u)=e^{-\min\{s,t\}}.$$

Accordingly, if $\theta(s,t)\le\min\{s,t\}$ then $h_{s,t}(u)\ge0$, if $\theta(s,t)\ge\frac{s+t}{2}$ then $h_{s,t}(u)\le0$. This means $(-1)^k[\ln\nu_{s,t}(x)]^{(k)}\begin{cases}\ge0,&\theta(s,t)\le\min\{s,t\}\\\le0,&\theta(s,t)\ge\frac{s+t}{2}\end{cases}$ for $k\in\mathbb{N}$.

Conversely, if $\frac{1}{\nu_{s,t}(x)}$ is logarithmically completely monotonic, then $[\ln\nu_{s,t}(x)]'\ge0$ which can be rearranged as

$$\frac{\psi(x+t)-\psi(x+s)}{t-s}\ge\psi'(x+\theta(s,t)). \tag{48}$$

Since ψ' is decreasing, thus

$$\theta(s,t)\ge(\psi')^{-1}\left(\frac{\psi(x+t)-\psi(x+s)}{t-s}\right)-x$$
$$=(\psi')^{-1}\left(\frac{1}{t-s}\int_s^t\psi'(x+v)\,\mathrm{d}v\right)-x=\phi_{s,t;1}(x),\quad(49)$$

where $(\psi')^{-1}$ denotes the inverse function of ψ' and $\phi_{s,t;\ell}(x)$ is defined by (30). Proposition 1 tells us that the function $\phi_{s,t;1}(x)$ has a sharp upper bound $\frac{s+t}{2}$, thus, it holds that $\theta(s,t)\ge\frac{s+t}{2}$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. If $\theta(s,t)\ge\frac{s+t}{2}$, then the function $\nu_{s,t}(x)$ defined by (6) is increasing by Theorem 1. Hence, for any given $\delta>-\min\{s,t\}$ and $\theta(s,t)\ge\frac{s+t}{2}$, inequality

$$\nu_{s,t}(\delta)\le\nu_{s,t}(x)\quad(50)$$

holds in $[\delta,\infty)$ and

$$\nu_{s,t}(x)<\lim_{x\to\infty}\nu_{s,t}(x)\quad(51)$$

is valid in $(-\min\{s,t\},\infty)$.

For a and b being two constants, as $x\to\infty$, the following asymptotic formula is given in [1, p. 261, 6.1.47]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}=1+\frac{(a-b)(a+b-1)}{2x}$$
$$+\frac{1}{12}\binom{a-b}{2}\frac{3(a+b-1)^2-a+b-1}{x^2}+O\left(\frac{1}{x^3}\right)=1+O\left(\frac{1}{x}\right).\quad(52)$$

In [36], it was proved that $\psi(x)-\ln x+\frac{\alpha}{x}\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\ge1$ and $\ln x-\frac{\alpha}{x}-\psi(x)\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\le\frac12$. From this, it is deduced that

$$\ln x-\frac{1}{x}<\psi(x)<\ln x-\frac{1}{2x}\quad(53)$$

in $(0,\infty)$. Utilization of (52) and (53) leads to

$$\lim_{x\to\infty}\frac{1}{\nu_{s,t}(x)}=\lim_{x\to\infty}\left\{\frac{\exp[\psi(x+\theta(s,t))]}{x}\left[1+O\left(\frac{1}{x}\right)\right]^{1/(t-s)}\right\}$$
$$=\lim_{x\to\infty}\frac{\exp[\psi(x+\theta(s,t))]}{x}\le\lim_{x\to\infty}\left\{\frac{x+\theta(s,t)}{x}\exp\left[-\frac{1}{2(x+\theta(s,t))}\right]\right\}=1$$

and

$$\lim_{x\to\infty}\frac{1}{\nu_{s,t}(x)}\ge\lim_{x\to\infty}\left\{\frac{x+\theta(s,t)}{x}\exp\left[-\frac{1}{x+\theta(s,t)}\right]\right\}=1,$$

thus $\lim_{x\to\infty}\nu_{s,t}(x)=1$ and inequality (51) is reduced to

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}<\exp\bigl[\psi\bigl(x+\theta(s,t)\bigr)\bigr]\quad(54)$$

for $x>-\min\{s,t\}$ and $\theta(s,t)\ge\frac{s+t}{2}$. From the increasing monotonicity of ψ, inequality (7) is proved.

By standard calculation, inequality (50) can be rearranged as

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\ge\left[\frac{\Gamma(\delta+t)}{\Gamma(\delta+s)}\right]^{1/(t-s)}\exp[\psi(x+\theta(s,t))-\psi(\delta+\theta(s,t))]\quad(55)$$

for $x \in [\delta, \infty)$ and $\theta(s,t) \geq \frac{s+t}{2}$. From the decreasing monotonicity in y of the function $\psi(x+y) - \psi(\delta+y)$ and $\lim_{y\to\infty}[\psi(x+y) - \psi(\delta+y)] = 0$ for $x \geq \delta$, inequality (8) is concluded.

Combination of the conclusion $\nu_{s,t}(x) \in \mathcal{C}_{\mathcal{L}}[(-\theta(s,t),\infty)]$ for $\theta(s,t) \leq \min\{s,t\}$ in Theorem 1 with $\lim_{x\to\infty} \nu_{s,t}(x) = 1$ and discussion by standard argument yields inequalities (9) and (10). The proof of Theorem 2 is complete. □

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